SEQUENCE-COVERING CS-IMAGES OF METRIC SPACES *

Jinjin Li† and Shou Lin††

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Abstract. In this paper,we establish the characterizations of metric spaces under some sequence-covering and cs-mappings by means of certain kinds of compact-countable networks.

1 Introduction

Quitely recently, Z.B.Qu and Z.M. Gao introdued the concept of cs-mappings in order to study the relation between spaces with certain compact-countable k-networks and certain images of metric spaces. They proved that a space with compact-countable closed k-network is a compact-covering cs-image of a metric space[1]. It initially shows an efficiency by cs-mappings to realize Alexandroff's idea that by means of various mappings the relations between various topological spaces and metric spaces are established. The purpose of this paper is to further discuss the images of metrics by certain cs-mappings. In section2,we establish a relation between spaces with a compact-countable network and metric spaces. In section 3, we give some characterizations for spaces with a compact-countable cs*-network. In section 4, we characterize the sequence-covering and cs-images of metric spaces. In section 5, we study the relation between spaces with certain compact-countable weak bases and certain images of metric spaces.

In this paper, all spaces are considered to be T_2, and all maps continuous and onto, $\omega = \{0\} \cup N$.

2 Spaces with a compact-countable network

A map $f : X \longrightarrow Y$ is called a cs-map[1], if for any compact subset C of Y, $f^{-1}(C)$ is separable.

Let X be a space ,and let $\mathcal{P}$ be a cover of X. $\mathcal{P}$ is called compact -countable, if for any compact subset K of X, only countable many members of $\mathcal{P}$ intersect K. $\mathcal{P}$ is a network if whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

Theorem 2.1 A space X has a compact-countable network if and only if X is an image of a metric space under a cs-mapping.

Proof. Suppose X is a cs-image of metric space .Then there exists a cs-mapping f from M onto X. Since M is a metric space, by Nagata-Smirnov metrization theorem (see[3] or[4]), M has a σ-locally finite base. Let $\mathcal{B}$ be a σ-locally finite base for M, put $\mathcal{P} = \{f(B) : B \in \mathcal{B}\}$, then $\mathcal{P}$ is a network for X because a mapping preserves a network. Since f is a cs-mapping, for any compact subset K of X, $f^{-1}(K)$ is separable in M, and so $f^{-1}(K)$ is a Lindelöf subspace. Hence $\mathcal{P}$ a is compact-countable network for X.

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Conversely, suppose X has a compact-countable network $\mathcal{P} = \{P_\alpha : \alpha \in A\}$. For each $n \in N$, let A_n be a copy of A, and it is endowed with discrect topology.Put
$M = \{\alpha = (\alpha_n) \in \prod_{n\in N} A_n : \{P_{\alpha_n} : n \in N\}$ is a network of some point x_α in $X\}$,
and give M the subspace topology induced from the usual product topology of the product space $\prod_{n\in N} A_n$, x_α is unique in X because X is T_2. We define $f : M \longrightarrow X$ by $f(\alpha) = x_\alpha$. Obviously, M is a metric space, and it is easy to check that f is continuous and onto. For any compact subset C of X, by the compact-countable property of $\mathcal{P}$, we have that $\{\alpha \in A : C \bigcap P_\alpha \neq \emptyset\}$ is countable. Put

$$L = (\prod_{n\in N} \{\alpha \in A_n : C \bigcap P_\alpha \neq \emptyset\} \bigcap M$$

,
then L is a hereditarily separable subspace of M and $f^{-1}(C) \subset L$. Thus $f^{-1}(C)$ is separable in M. Hence f is a cs-mapping.

3 Spaces with a compact-countable cs^*-network

In [1], Z.Qu and Z.Gao got the following result:

Theorem 3.1[1] For a space X, the following are equivalent:
(1) X is a sequence-covering quotient cs-image of a metric space.
(2) X is a quotient cs-image of a metric space.
(3) X is a k-space with a compact-countable cs^*-network.

Remark 1 The sequence-covering mapping above is used in the sense of the following Def.3.4(2).

The purpose of this section is to improve the result above. Recall some basic definitions.

Definition 3.2 Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1)$\mathcal{P}$ is a cs^*-network [5] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x\} \bigcup \{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

(2)$\mathcal{P}$ is a CFP cover of a compact subset K in X if there are a finite collection $\{K_n : n \leq k\}$ of closed subsets of K and $\{P_n : n \leq k\} \subset \mathcal{P}$ such that $K = \bigcup\{K_n : n \leq k\}$ and each $K_n \subset P_n$(see[6]).

(3)$\mathcal{P}$ is an sfp-network(i.e., sequence-finite-partition network)[6] if whenever $K \subset V$ with K convergent sequence including its limit point and V open in X, there exists a finite $\mathcal{P}' \subset \mathcal{P}$ such that $\mathcal{P}'$ is a CFP cover of K and $\bigcup \mathcal{P}' \subset V$.

It is easy to check the following Lemma by Definition 3.2

Lemma 3.3 Let $\mathcal{P}$ be a point-countable cover of X, then $\mathcal{P}$ is a cs^*-network if and only if $\mathcal{P}$ is an sfp-network.

Definition 3.4 Let $f : X \longrightarrow Y$ be a map.

(1)f is a sequentially quotient map[7](resp. subsequence-covering map[8]) if for each convergent sequence S of Y, there is a convergent sequence L (resp. compact subset L) of X such that $f(L)$ is a subsequence of S.

(2)f is a pseudo-sequence-covering if for each convergent sequence L of Y, there is a compact subset K of X such that $f(K) = L$. While in [9] or [1], such a pseudo-sequence-covering map is called sequence-covering.

It is easy to show the following Lemma by Definition 3.4

Lemma 3.5 Let $f : X \longrightarrow Y$ be a subsequence-covering map, and let X be a sequential space. Then f is sequentially quotient.

The following Lemma is due to [10].

Lemma 3.6 Let $f : X \longrightarrow Y$ be a pseudo-sequence-covering map, and let X be a sequential space. Then f is sequentially quotient.

Theorem 3.7 The following are equivalent for a space X:

(1)X is a pseudo-sequence-covering and cs-image of a metric space.

(2)X is a sequentially quotient and cs-image of a metric space.

(3)X is a subsequence-covering and cs-image of a metric space.

(4)X has a compact-countable cs^*-network.

(5)X has a compact-countable sfp-network.

Proof. (1) $\Longrightarrow$ (2). By Lemma 3.6.

(2) $\Longrightarrow$ (3). By Definition 3.4.

(3) $\Longrightarrow$ (2). By Lemma 3.5.

(4) $\Leftrightarrow$ (5). By Lemma 3.3.

(2) $\Longrightarrow$ (4). Suppose $f : M \longrightarrow X$ is a sequentially quotient mapping, where M is a metric space. Let $\mathcal{B}$ be a σ-locally-finite base for M. From the proof of Theorem 2.1, $f(\mathcal{B})$ is compact-countable in X. Since f is sequentially quotient, $f(\mathcal{B})$ is a cs^*-network for X because a sequentially quotient mapping preserves a cs^*-network. Hence $f(\mathcal{B})$ is a compact-countable cs^*-network.

(4) $\Longrightarrow$ (1). From the proof of Theorem 2.5 in [1].

Corollary 3.8 The following are equivalent for a space X:

(1) X is a pseudo-sequence-covering and quotient cs-image of a metric space.

(2) X is a sequentially quotient and quotient cs-image of a metric space.

(3) X is a subsequence-covering and quotient cs-image of a metric space.

(4) X is a quotient cs-image of a metric space.

(5) X is a sequential space with a compact-countable cs^*-network.

(6) X is a sequential space with a compact-countable sfp-network.

Remark 2 (1) $\Leftrightarrow$(4) $\Leftrightarrow$ (5) is shown in [1].

4 Space with a compact-countable cs-network

Recall that a cover $\mathcal{P}$ of X is a cs-network[11] if,whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then $\{x\} \bigcup \{x_n : n \geq m\} \subset P \subset U$ for some $m \in N$ and some $P \in \mathcal{P}$. A mapping $f : X \longrightarrow Y$ is sequence-covering [12] if each convergent sequence of Y is the image of some convergent sequence of X.

Theorem 4.1 A space X has a compact-countable cs-network if and only if X is a sequence-covering and cs-image of a metric space.

Proof. Let X be a space with a compact-countable cs-network $\mathcal{P}$. Suppose $\mathcal{P}$ is closed under finite intersections. Denote $\mathcal{P}$ by $\{P_\alpha : \alpha \in A\}$. Let A_i denote the set A with discrete topology each $i \in N$. Put

$M = \{\beta = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\}$ is a network at some point $x(\beta)$ in $X\}$, then M is a metric space, and $f : M \longrightarrow X$ defined by $f(\beta) = x(\beta)$ is a cs-mapping in view of the proof of Theorem 2.1. We shall show that f is sequence-covering. For a sequence $\{x_n\}$ of X converging to a point x_0 in X, we assume that all x_n's are distinct. Let $K = \{x_m : m \in \omega\}$, and let $K \subset U$ with U open in X, a subset $\mathcal{F}$ of $\mathcal{P}$ is said to have the property $F(K,U)$ if $\mathcal{F}$ satisfies that

(1)$\mathcal{F}$ is finite;

(2)$\emptyset \neq P \bigcap K \subset P \subset U$ for each $P \in \mathcal{F}$;

(3)for each $x \in K$ there is a unique $P_x \in \mathcal{F}$ with $x \in P_x$;

(4)if $x_0 \in P \in \mathcal{F}$, then $K - P$ is finite.

Put $\{\mathcal{F} \subset \mathcal{P} : \mathcal{F}$ has the property $F(K,X)\}$=$\{\mathcal{F}_i : i \in N\}$,
for each $i \in N$ and each $m \in \omega$,there is $\alpha_{im} \in A_i$ with $x_m \in P_{\alpha_{im}} \in \mathcal{F}_i$. It can be check that $\{P_{\alpha_{im}} : i \in N\}$ is a network at the point x_m. Let $\beta_m = (\alpha_{im})$ for each $m \in \omega$, then $\beta_m \in M$ and $f(\beta_m) = x_m$. For each $i \in N$, there is $n(i) \in N$ such that $\alpha_{in} = \alpha_{i0}$ if $n \geq n(i)$. Thus the sequence $\{\alpha_{in}\}$ converges to α_{i0} in A_i, and the sequence $\{\beta_n\}$ converges to β_0 in M. This shows that f is a sequence-covering mapping.

Conversely,suppose that $f : M \longrightarrow X$ is a sequence-covering and cs-mapping, where M is a metric space. Let $\mathcal{B}$ be a σ-locally-finite base for M, then $\{f(B) : B \in \mathcal{B}\}$ is a compact-countable cs-network for the space X.

Corollary 4.2 A space X is a sequential space with a compact-countable cs-network if and only if X is a sequence-covering and quotient cs-image of a metric space.

5 Space with a compact-countable weak base

Let $f : X \longrightarrow Y$ be a map,then f is a 1-sequence-covering map[13] if for each $y \in Y$,there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

Obviously, a 1-sequence-covering map is a sequentially quotient map.

The following lemma is due to [7].

Lemma 5.1 Let $f : X \longrightarrow Y$ be a map.If X is a sequential space, then f is quotient if and only if Y is a sequential space and f is sequentially quotient.

Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that for each $x \in X$,

(1)$\mathcal{P}_x$ is a network at $x \in X$.

(2) If $U, V \in \mathcal{P}_x$, $W \subset U \bigcap V$ for some $W \in \mathcal{P}_x$. $\mathcal{P}$ is a weak base [14] for X if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in X. $\mathcal{P}$ is called an sn-network[13,15] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$(i.e., P is a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in N$.), here $\mathcal{P}_x$ is an sn-network of x in X.

The following Lemma is due to [13] or [15].

Lemma 5.2 (1) For a space, weak base $\Longrightarrow$ sn-network $\Longrightarrow$ cs-network.

(2) An sn-network for a sequential space is a weak base.

Theorem 5.3 The following are equivalent for a space X:

(1) X has a compact-countable sn-network.

(2) X is a 1-sequenence-covering and cs-image of a metric space.

Proof. (1) $\Longrightarrow$ (2). Assume $\mathcal{P}$ is a compact-countable sn-network for X. Denote $\mathcal{P} = \{P_\alpha : \alpha \in A\}$. For each $i \in N$, let A_i be a copy of A, and it is endowed with discrete

topology. Put
$M = \{\alpha = (\alpha_n) \in \prod_{n\in N} A_n : \{P_{\alpha_n} : n \in N\}$ is a network of some point x_α in $X\}$,
and give M the subspace topology induced from the product topology of the product space $\prod_{n\in N} A_n$. We define $f : M \longrightarrow X$ by $f(\alpha) = x_\alpha$. From the proof of Theorem 2.1, M is a metric space,and f is a cs-mapping. We shall show that f is a 1-sequence-covering mapping. In fact, For each $x \in X$, we can assume $\mathcal{P}_x = \{P_{\alpha_i} : i \in N\}$, then $\{P_{\alpha_i} : i \in N\}$ is an sn-network of $x \in X$. Put $\beta = (\alpha_i)$, then $\beta \in f^{-1}(x)$. Denote
$B_n = \{(\gamma_i) \in M$:if $i \leq n$,then $\gamma_i = \alpha_i\}$. Then $\{B_n : n \in N\}$ is a monotonic decreasing neighborhood base of β in M. For each $n \in N$, it is easy to check that $f(B_n) = \bigcap_{i\leq n} P_{\alpha_i}$. For a convergent sequence x_j of X with $x_j \longrightarrow x$, since $\{f(B_n) : n \in N\}$ is an sn-network of x in X, there is $i(n) \in N$ such that if $i \geq i(n)$, then $x_i \in f(B_n)$, and so $f^{-1}(x_i) \bigcap B_n \neq \emptyset$. We may assume that $1 < i(n) < i(n+1)$. For each $j \in N$, let

$$\beta_j \in \begin{cases} f^{-1}(x_j), & \text{if} \quad j < i(1), \\ f^{-1}(x_j) \bigcap B_n, & \text{if} \quad i(n) \leq j < i(n+1), n \in N. \end{cases}$$

Then, we have that the sequence $\{\beta_j\}$ converges to β in M. Therefore f is 1-sequence-covering.

(2)$\Longrightarrow$(1) Suppose $f : M \longrightarrow X$ is a 1-sequence-covering and cs-map, where M is a metric space. Let $\mathcal{B}$ be a σ-locally-finite base for M. For each $x \in X$, there is $\beta_x \in f^{-1}(x)$ satisfying that for each convergent sequence $\{x_n\}$ in X with $x_n \longrightarrow x$, there is $\beta_n \in f^{-1}(x_n)$ with $\beta_n \longrightarrow \beta_x$. Put
$\mathcal{P}_x = \{f(B) : \beta_x \in B \in \mathcal{B}\}$, $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$,
then, it is easy to chect that $\mathcal{P}$ is a compact-countable sn-network for X.

Theorem 5.4 The following are equivalent for a space X:
(1)X has a compact-countable weak base.
(2)X is a 1-sequence-covering and quotient cs-image of a metric space.

Proof. (1) $\Longrightarrow$ (2). Suppose X has a compact-countable weak, then X is a sequential space, and has a compact-countable sn-network. According to Theorem 5.3, X is a 1-sequence-covering and cs-image of a metric space. By Lemma 5.1, the 1-sequence-covering mapping is a quotient mapping.

(2) $\Longrightarrow$ (1) Suppose X is a 1-sequence-covering and quotient cs-image of a metric space. By Lemma 5.1, X is a sequential space. In view of Theorem 5.3, X has a compact-countable sn-network $\mathcal{P}$, and so $\mathcal{P}$ is a compact-countable weak base for X by Lemma 5.2.

Remark 3 A map $f : X \longrightarrow Y$ is an s-map(resp. ss-map[10]) if each $f^{-1}(y)$ is separable(resp. for each $y \in Y$, there is an open neighborhood V of y in Y such that $f^{-1}(V)$ is separable). Obviously, if X is a metric space, then an ss-map $\Longrightarrow$ a cs-map $\Longrightarrow$ an s-map. However, we have the following facts.

Example 1 Let $X = \beta N$, the Stone-Čech compactification of the integers. Then X has a point-countable sn-network. By Proposition 2.3 in [13], X is a 1-sequence-covering and s-image of a metric space. But X is not a pseudo-sequence-covering and cs-image of a metric space because X has no compact-countable cs^*-network.

Example 2 Let X be a paracompact space with a point-countable base, and let X be not metrizable. Then X has a compact-countale base, and so X has a compact-countable sn-network. By Theorem 5.3, X is a 1-sequence-covering and cs-image of a metric space.

But X is not a 1-sequence-covering and ss-image of a metric space because X is not a metric space.

Example 3 Let X be a sequential fan, that is $X = S_\omega$. Then X is a sequence-covering and quotient cs-image of a metric space, but X is not a 1-sequence-covering and quotient cs-image of a metric space because it has no point-countable weak base.

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† Dept. of Math., Zhangzhou Teachers' College, Zhangzhou, Fujian 363000, P.R.China.

†† Dept. of Math. Fujian Normal University, Fuzhou 350000, P.R.China.