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A note on the Arens' space and sequential fan ☆

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Abstract

In this paper we discuss the spaces containing a subspace having the Arens' space or sequential fan as its sequential coreflection. A sequential coreflection of a space which is weakly first-countable is characterized, and some generalized metric spaces which contain no Arens' space or sequential fan as its sequential coreflection are studied. © 1997 Elsevier Science B.V.

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1. Introduction

g-metrizable spaces and $\aleph$-spaces play an important role in metrization theory. We know that every metric space is a g-metrizable space, and every g-metrizable space is an $\aleph$-space. Further relationships among these spaces can be characterized by the canonical quotient spaces which are Arens' space S_2 and sequential fan $S(\omega)$. For example,

Theorem 1.1 [20]. *A space is a metrizable space if and only if it is a g-metrizable space containing no (closed) copy of S_2.*

Theorem 1.2 [11]. *A space is a g-metrizable space if and only if it is a k and $\aleph$-space containing no (closed) copy of $S(\omega)$.*

Using these concrete spaces S_2 and $S(\omega)$ we can analyze the gaps among some generalized metric spaces. Spaces containing a copy of S_2 or $S(\omega)$ and their applications

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have been studied in [11,14–17,20]. Since S_2 and $S(\omega)$ are all sequential spaces, this encourages us to discuss the spaces containing a subspace having S_2 or $S(\omega)$ as its sequential coreflection and those which do not. We obtain that

Theorem 1.3. *A regular space has a σ-locally finite sequentially open network if and only if it has a σ-locally finite universal cs-network contains no (closed) subspace having S_2 as its sequential coreflection* (*Corollary* 2.9).

Theorem 1.4. *A regular space has a σ-locally finite universal cs-network if and only if it has a σ-locally finite cs-network and contains no (closed) subspace having $S(\omega)$ as its sequential coreflection* (*Theorem* 3.15).

Theorem 1.5. *Suppose X is a quotient s-image of a metric space. X has a point-countable base if and only if X contains no (closed) copy of S_2 and $S(\omega)$* (*Corollary* 3.10).

In this paper all spaces are T_2, $\mathcal{N}$ denotes the set of all natural numbers. The *Arens' space* S_2 [1] and *sequential fan* $S(\omega)$ [5] are defined as follows. Let $T_0 = \{a_n\colon n \in \mathcal{N}\}$ be a sequence converging to $x \notin T_0$ and let each T_n $(n \in \mathcal{N})$ be a sequence converging to $a_n \notin T_n$. Let T be the topological sum of $\{T_n \cup \{a_n\}\colon n \in \mathcal{N}\}$. Thus $S_2 = \{x\} \cup (\bigcup\{T_n\colon n \geqslant 0\})$ is a quotient space obtained from the topological sum of T_0 and T by identifying each $a_n \in T_0$ with $a_n \in T$. Also, $S(\omega) = \{x\} \cup (\bigcup\{T_n\colon n \in \mathcal{N}\})$ is a quotient space obtained from T by identifying all the points $a_n \in T$ to the point x.

2. On the Arens' space S_2

For a space X and $x \in P \subset X$, P is a *sequential barrier* at x if, whenever $\{x_n\}$ is a sequence converging to x in X, then $x_n \in P$ for all but finitely many $n \in \mathcal{N}$; equivalently, $x_n \in P$ for infinitely many $n \in \mathcal{N}$. P is *sequentially open* in X if P is a sequential barrier at each of its points, and is *sequentially closed* in X if its complement is sequential open.

A space X is called a *sequential space* [7] if each sequentially open subset of X is open in X. Thus the topology is naturally definable using convergent sequences, and two sequential topologies on the same set X are the same if and only if they have the same convergent sequences. Each space (X, τ) has a *sequential coreflection*, which we denote (X, σ_τ) or σX if there is no danger of confusion. As is well known, σX is a sequential space, and $B \in \sigma_\tau$ if and only if B is sequentially open in X; also, X and σX have the same convergent sequences.

Definition 2.1. Call a subspace of a space a *comb* (at a point x) if it consists of a point x, a sequence $\{x_n\}$ converging to x, and disjoint sequences converging individually to each x_n. Call a subset of a comb a *diagonal* if it is a convergent sequence meeting infinitely many of the sequences converging to the individual x_n and converges to some point in the comb.

S_2 is comb without a diagonal.

Lemma 2.2. *For a space X, σX is homeomorphic to S_2 if and only if X is a comb without a diagonal.*

Proof. Suppose σX is homeomorphic to S_2. Since σX and X have the same convergent sequences and S_2 is a comb without a diagonal, X is a comb without a diagonal. Conversely, suppose X is a comb without a diagonal. Since σX is sequential, σX is homeomorphic to S_2. □

A space X is called a *Fréchet space* [7] (or a *Fréchet–Urysohn space*) if, whenever $x \in \mathrm{cl}_X(A)$, there is a sequence in A converging to x in X. Every Fréchet space is sequential, and the sequential space S_2 is not Fréchet. To characterize the Fréchet property of the sequential coreflection of a space, we introduce the following notations. For a space X and $A \subset X$, define that

$$\mathrm{cl}_\sigma(A) = \mathrm{cl}_{\sigma X}(A),$$
$$\mathrm{cl}_s(A) = \{x \in X\colon \text{there is a sequence in } A \text{ converging to } x\}.$$

The following is well known and easy to show.

Lemma 2.3. *The following are equivalent for a space X:*

(1) *σX is a Fréchet space.*

(2) $\mathrm{cl}_\sigma(A) = \mathrm{cl}_s(A)$ *for each $A \subset X$.*

(3) $\mathrm{cl}_s(A)$ *is sequentially closed in X for each $A \subset X$.*

It is easy to see from this that σX is a Fréchet space if and only if every sequential barrier at any point x in X contains a sequentially open subspace containing x.

Theorem 2.4. *The following are equivalent for a space X:*

(1) *σX is a Fréchet space.*

(2) *Every comb at x of X has a diagonal converging to x for each $x \in X$.*

(3) *Every comb of X has a diagonal.*

(4) *X contains no subspace having S_2 as its sequential coreflection.*

Proof. We only need to prove that (4) ⇒ (1). Suppose σX is not Fréchet. By Lemma 2.3, there is a subset A of X such that $\mathrm{cl}_s(A)$ is not closed in σX. Since σX is sequential, there exists a sequence $\{x_n\}$ in $\mathrm{cl}_s(A)$ converging to $x \in X \setminus \mathrm{cl}_s(A)$. We can assume that the x_n's are all distinct and $x_n \notin A$. Since X is T_2, let $\{V_n\}$ be a sequence of pairwise disjoint open subsets of X with each $x_n \in V_n$. For each $n \in \mathcal{N}$, there is a sequence $\{x_{nm}\}$ in $A \cap V_n$ converging to x_n in X. Put

$$C = \{x\} \cup \{x_n\colon n \in \mathcal{N}\} \cup \{x_{nm}\colon n, m \in \mathcal{N}\}.$$

Then C is a comb at x of X. By (4), σC is not homeomorphic to S_2. By Lemma 2.2, C has a diagonal. Let $\{y_k\}$ be a diagonal of C which converges to y in C. If $y \neq x$, then $y \in V_i$ for some $i \in \mathcal{N}$, and $y_k \in V_i$ for some $j \in \mathcal{N}$ and all $k \geqslant j$, a contradiction. Thus C has a diagonal converging to x, hence $x \in \mathrm{cl}_s(A)$, a contradiction. Therefore σX is Fréchet. □

A point x in a space X is called *regular* G_δ if there is a sequence of neighborhoods of x in X such that the intersections of their closures is $\{x\}$.

Lemma 2.5. *Let X be a space in which each point is regular G_δ. If X contains no closed subspace having S_2 as its sequential coreflection, then σX is a Fréchet space.*

Proof. By Theorem 2.4, we only need to show that if X contains a subspace S such that σS is homeomorphic to S_2, then S contains a closed subspace T of X such that σT is homeomorphic to S_2. Let $S = \{x\} \cup \{x_n\colon n \in \mathcal{N}\} \cup \{x_{nm}\colon n, m \in \mathcal{N}\}$. Take a sequence $\{G_k\}$ of open neighborhoods of x in X such that each $G_{k+1} \subset G_k$ and $\{x\} = \bigcap\{\mathrm{cl}(G_k)\colon k \in \mathcal{N}\}$. Since the sequence $\{x_n\}$ converges to x, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with each $x_{n_k} \in G_k$. Since the sequence $\{x_{n_k m}\}$ converges to x_{n_k} for each $m \in \mathcal{N}$, there is $m_k \in \mathcal{N}$ such that $x_{n_k m} \in G_k$ if $m \geqslant m_k$. Put

$$T = \{x\} \cup \{x_{n_k}\colon k \in \mathcal{N}\} \cup \{x_{n_k m}\colon k \in \mathcal{N},\ m \geqslant m_k\}.$$

If $p \in X \backslash T$, then $p \in X \backslash \mathrm{cl}(G_k)$ for some $k \in \mathcal{N}$. Let

$$F = \{x_{n_i}\colon i < k\} \cap \{x_{n_i m}\colon i < k,\ m \geqslant m_i\}.$$

Then F is compact in X, there is a neighborhood W of p in X such that $W \cap F = \emptyset$, so $W \cap (X \backslash \mathrm{cl}(G_k)) \cap T = \emptyset$, hence T is closed in X, and σT is homeomorphic to S_2. □

Since a closed subspace of a sequential space is sequential, the foregoing proof gives:

Corollary 2.6. *Let X be a space in which each point is regular G_δ. If X contains a copy of S_2, then X contains a closed copy of S_2.*

For a space X, let $\wp$ be a family of subsets of X. $\wp$ is a *network* of x in X if $x \in \bigcap \wp$ and whenever G is open in X with $x \in G$, then $P \subset G$ for some $P \in \wp$.

Definition 2.7. Let $\wp = \bigcup\{\wp_x\colon x \in X\}$ be a family of subsets of X which satisfies that for each $x \in X$,

(1) $\wp_x$ is a network of x in X,

(2) if $U, V \in \wp_x$, then $W \subset U \cap V$ for some $W \in \wp_x$.

$\wp$ is a *sequentially open network* (respectively, a *universal cs-network*) for X if each element of $\wp_x$ is a sequentially open subset (respectively, a sequential barrier of x) in X. A space X is a *sof-countable space* (respectively, a *universally csf-countable space*) if X has a sequentially open network (respectively, universal cs-network) $\wp$ such that each $\wp_x$ is countable.

Obviously, a space is a first-countable space if and only if it is a sof-countable and sequential space. S_2 is not sof-countable. The following two corollaries follow easily from Lemma 2.3, Theorems 2.4 and 2.5.

Corollary 2.8. *The following are equivalent for a space X:*
(1) *σX is a first-countable space.*
(2) *X is a sof-countable space.*
(3) *X is a universally csf-countable space and contains no subspace having S_2 as its sequential coreflection.*

Corollary 2.9. *A (regular) space X has a σ-locally finite sequentially open network if and only if X has a σ-locally finite universal cs-network and contains no (closed) subspace having S_2 as its sequential coreflection.*

Remark 2.10. If a space X has a σ-locally finite sequentially open network, then σX has a σ-locally finite space. But its inverse proposition is not hold. For example, $\sigma(\beta\mathcal{N})$ is a discrete space, and $\beta\mathcal{N}$ is not a σ-space.

Definition 2.11. Let X be a space, and let $\wp$ be a cover of X. $\wp$ is a *k-network* for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \wp' \subset U$ for some finite $\wp' \subset \wp$.

Theorem 2.12. *Suppose X has a point-countable k-network. If σX contains no closed copy of S_2, then σX is a Fréchet space.*

Proof. Suppose $\wp$ is a point-countable k-network for X. If σX is not a Fréchet space, by Theorem 2.4, X contains a subspace C having S_2 as its sequential coreflection. Put

$$C = \{x\} \cup \{x_n\colon n \in \mathcal{N}\} \cup \{x_{nm}\colon n, m \in \mathcal{N}\},$$
$$K = \{x\} \cup \{x_n\colon n \in \mathcal{N}\},$$
$$\mathfrak{R} = \{P \in \wp\colon P \cap \{x_{nm}\colon n, m \in \mathcal{N}\} \neq \emptyset \text{ and } \overline{P} \cap K = \emptyset\}.$$

The $\mathfrak{R}$ is countable. Let $\mathfrak{R} = \{P_k\colon k \in \mathcal{N}\}$. For each $n \in \mathcal{N}$, there is $m_n \in \mathcal{N}$ such that $\{x_{nm}\colon m \geqslant m_n\} \subset X \setminus \bigcup_{k \leqslant n} \overline{P}_k$. Take

$$S = K \cup \{x_{nm}\colon m \geqslant m_n\}.$$

Then σS is homeomorphic to S_2. If σS is not closed in σX, there is a sequence $\{x_{n_i m_i}\}$ in S with $x_{n_i m_i} \to x' \notin S$. We may assume that $n_{i+1} > n_i$. Put

$$K_1 = \{x'\} \cup \{x_{n_i m_i}\colon i \in \mathcal{N}\}.$$

Then $K_1 \cap K = \emptyset$, there is an open set U in X with $K_1 \subset U \subset \overline{U} \subset X \setminus K$, thus $K_1 \subset \bigcup \wp' \subset U$ for some finite $\wp' \subset \wp$, so $P \cap K_1$ is infinite for some $P \in \wp'$, hence $P = P_j$ for some $j \in \mathcal{N}$, and $x_{n_i m_i} \notin P$ for each $n_i > j$, a contradiction. Therefore σS is closed in σX. $\square$

Corollary 2.13. *Suppose X is a k-space with a point-countable k-network, then X is a Fréchet space if and only if X contains no closed copy of S_2.*

Proof. Every k-space with a point-countable k-network is a sequential space [8, Corollary 3.4]. □

Example 2.14. There exist a compact, sequential space X and its subspace M such that

(1) X contains no copy of S_2 or $S(\omega)$.

(2) σM is homeomorphic to S_2.

(3) M has a countable universal cs-network.

Proof. By Example 7.1 in [7], let $\psi(\mathcal{N}) = \mathcal{N} \cup \mathcal{A}$ be the Isbell's space, and let $X = \psi(\mathcal{N}) \cup \{a\}$ be the one-point compactification of $\psi(\mathcal{N})$, then X is a compact, sequential space. X contains no copy of S_2 or $S(\omega)$ by Corollary 3.10 in [16]. Take an infinite subset $\{A_n\colon n \in \mathcal{N}\} \subset \mathcal{A}$, then the $\{A_n\}$ converges to a in X because $\mathcal{A}$ is closed discrete in $\psi(\mathcal{N})$. For each $n \in \mathcal{N}$, put

$$A_n = \{a_{nm}\colon m \in \mathcal{N}\}.$$

Then the $\{a_{nm}\}$ converges to A_n in $\psi(\mathcal{N})$. Let

$$M = \{a\} \cup \{A_n\colon n \in \mathcal{N}\} \cup \{a_{nm}\colon n, m \in \mathcal{N}\}.$$

Since any subsequence of $\{a_{nm}\}$ does not converge to a in X, by Theorem 2.4, σM is homeomorphic to S_2. For each $x \in M$, let

$$\wp_x = \begin{cases} \{\{a\} \cup \{A_n\colon n \geqslant i\}\colon i \in \mathcal{N}\}, & x = a \\ \{\{A_n\} \cup \{a_{nm}\colon m \geqslant i\}\colon i \in \mathcal{N}\}, & x = A_n,\ n \in \mathcal{N} \\ \{\{a_{nm}\}\}, & x = a_{nm},\ n, m \in \mathcal{N}. \end{cases}$$

Then $\bigcup\{\wp_x\colon x \in X\}$ is a countable universal cs-network for M.

M is not sof-countable by Corollary 2.8. $\mathrm{cl}_s(\mathcal{N})$ is not a sequentially closed subset of X because $\mathrm{cl}_s(\mathcal{N}) = \psi(\mathcal{N})$. □

3. On the sequential fan $S(\omega)$

Definition 3.1. Call a subspace of a space a *fan* (at a point x) if it consists of a point x, and a countably infinite family of disjoint sequences converging to x. Call a subset of a fan a *diagonal* if it is a convergent meeting infinitely many of the sequences converging to x and converges to some point in the fan.

A fan at a point x in a space X is called a *countable sheaf* at x in [3,4]. If X is a fan, then each point of X is regular G_δ. $S(\omega)$ is a fan without a diagonal.

Lemma 3.2. *For a space X, σX is homeomorphic to $S(\omega)$ if and only if X is a fan without a diagonal.*

Proof. Suppose σX is homeomorphic to $S(\omega)$. Since $S(\omega)$ is a fan without a diagonal, X is a fan without a diagonal. Conversely, suppose X is a fan without a diagonal. Since σX is sequential, it is homeomorphic to $S(\omega)$. □

Lemma 3.3. *Suppose X contains a fan S at a point x without a diagonal converging to x. If x is regular G_δ in X, then S contains a closed subspace T of X such that σT is homeomorphic to $S(\omega)$.*

Proof. Let $S = \{x\} \cup \{x_{nm}\colon n, m \in \mathcal{N}\}$, where the sequence $\{x_{nm}\}$ converges to x for each $n \in \mathcal{N}$. There is a sequence $\{W_n\}$ of open neighborhoods of x in X with $\{x\} = \bigcap\{\mathrm{cl}(W_n)\colon n \in \mathcal{N}\}$. For each $n \in \mathcal{N}$, there is $m(1,n) \in \mathcal{N}$ with $x_{nm(1,n)} \in W_{n+1}$. Denote $D_1 = \{x_{nm(1,n)}\colon n \in \mathcal{N}\}$, and $V_1 = X\backslash D_1$, then any subsequence of D_1 does not converge to x, thus V_1 is a sequential barrier of x in X. By inductive method, we can construct $D_i = \{x_{nm(1,n)}\colon n \in \mathcal{N}\}$, and $V_i = X\backslash(D_1 \cup D_2 \cup \cdots \cup D_i)$ such that $x_{nm(i+1,n)} \in W_{n+i+1} \cap V_i$ and $m(i,n) < m(i+1,n)$ for each $i \in \mathcal{N}$. Then the sequence $\{x_{nm(i,n)}\colon i \in \mathcal{N}\}$ converges to x for each $n \in \mathcal{N}$, and $x_{nm(i,n)} \in W_k$ if $n+i \geqslant k$. Let

$$T = \{x\} \cup \{x_{nm(i,n)}\colon i, n \in \mathcal{N}\}.$$

Then $T\backslash W_k$ is finite for each $k \in \mathcal{N}$, thus $p \in \mathrm{cl}(W_k)$ when p is an accumulation point of T in X, so $p = x$, i.e., x is a unique accumulation point of T in X. Therefore, T is closed in X, and σT is homeomorphic to $S(\omega)$. □

Corollary 3.4. *Let X be a space in which each point is regular G_δ. If X contains a copy of $S(\omega)$, then X contains a closed copy of $S(\omega)$.*

Definition 3.5.

(1) A space X is an α_1-*space* [3,4] if $T = \{x\} \cup (\bigcup\{T_n\colon n \in \mathcal{N}\})$ is a fan at x of X, where each sequence T_n converges to x, then there exists a sequence S converging to x such that $T_n\backslash S$ is finite for each $n \in \mathcal{N}$.

(2) A space X is an α_4-*space* [3,4] if every fan at x of X has a diagonal converging to x.

(3) A space X is a *countably bisequential space* (or a *strong Fréchet space*) [13] if, whenever $\{A_n\}$ is a decreasing sequence of subsets of X and $x \in \bigcap\{\mathrm{cl}(A_n)\colon n \in \mathcal{N}\}$, there is a sequence $\{x_n\}$ converging to x with $x_n \in A_n$ for each $n \in \mathcal{N}$.

Clearly, each α_1-space is an α_4-space, and a space is countably bisequential if and only if it is a Fréchet and α_4-space. X is an α_4-space if and only if σX is an α_4-space. By Lemma 3.3, we have that

Theorem 3.6. *The following are equivalent for a space X (in which each point is regular G_δ):*

(1) *X is an α_4-space.*

(2) *Every fan of X has a diagonal.*

(3) *X contains no (closed) subspace having $S(\omega)$ as its sequential coreflection.*

Corollary 3.7. *Let X be a space (in which each point is regular G_δ). σX is countably bisequential if and only if X contains no (closed) subspace having S_2 or $S(\omega)$ as its sequential coreflection.*

Theorem 3.8. *Suppose X has a point-countable k-network. If σX contains no closed copy of $S(\omega)$, then X is an α_4-space.*

Proof. Suppose $\wp$ is a point-countable k-network for X. If X is not an α_4-space, by Definition 3.5, there is a fan at x of X without a diagonal converging to x. Put

$$S = \{x\} \cup \{x_{nm}\colon n, m \in \mathcal{N}\},$$
$$\mathfrak{R} = \{P \in \wp\colon P \cap \{x_{nm}\colon n, m \in \mathcal{N}\} \neq \emptyset \text{ and } x \notin \overline{P}\} = \{P_k\colon k \in \mathcal{N}\}.$$

For each $n \in \mathcal{N}$, there is $m_n \in \mathcal{N}$ such that $\{x_{nm}\colon m \geqslant m_n\} \subset X \setminus \bigcup_{k \leqslant n} \overline{P}_k$. Take

$$T = \{x\} \cup \{x_{nm}\colon m \geqslant m_n\}.$$

Then T is a fan at x of X without a diagonal converging to x. If there is a sequence $\{x_{n_i m_i}\}$ in T with $x_{n_i m_i} \to x' \neq x$. We may assume that $n_{i+1} > n_i$. So there exists $P \in \mathfrak{R}$ such that $P \cap \{x_{n_i m_i}\colon i \in \mathcal{N}\}$ is infinite, a contradiction. Hence σT is a closed subspace of σX, and is homeomorphic to $S(\omega)$. □

Corollary 3.9. *Suppose X is a k-space with a point-countable k-network.*
(1) *X is an α_4-space if and only if X contains no closed copy of $S(\omega)$.*
(2) *X is a first-countable space if and only if X contains no closed copy of S_2 and $S(\omega)$.*
(3) *X is a first-countable space if and only if X^ω is a k-space.*

Proof. Since every k-space with a point-countable k-network is sequential [8, Corollary 3.4], (1) holds by Theorem 3.8.

If X contains no closed copy of S_2 and $S(\omega)$, by (1) and Corollary 2.13, X is countably bisequential. For each $p \in X$, declaring every point $x \in X$, $x \neq p$ isolated and p having old neighborhoods we get a regular countably bisequential topology τ on X and X has a point-countable k-network in this topology. By Corollary 3.6 in [8], X is first-countable at p in the topology τ and thus in its original topology, (2) holds.

If X^ω is a k-space, X contains no closed copy of S_2 and $S(\omega)$ by Proposition 4.2 in [19], hence X is first-countable and (3) holds. □

Corollary 3.9(2) answers a question in [12]. By Corollary 3.9(2), Theorem 6.1 in [8] and Theorem 9.8 in [13], we have the following corollary, which improves some theorems in [20].

Corollary 3.10. *Suppose X is a quotient s-image of a metric space. X has a point-countable base if and only if X contains no (closed) copy of S_2 and $S(\omega)$.*

Definition 3.11. Let $\wp = \bigcup\{\wp_x\colon x \in X\}$ be a family of subsets of X which satisfies the conditions (1) and (2) in Definition 2.7. $\wp$ is a *weak base* [2] for X if a necessary and sufficient condition for $G \subset X$ to be open in X is that, for each $x \in G$, $P \subset G$ for some $P \in \wp_x$. $\wp$ is a *cs-network* for X if, given an open neighborhood G of x and a sequence $\{x_n\}$ converging to x, there are $P \in \wp_x$ and $n \in \mathcal{N}$ such that $x_n \in P \subset G$ for all $n \geqslant i$. A space is a *gf-countable space* [2] (respectively, a *csf-countable space*) if X has a weak base (respectively, a cs-network) $\wp$ such that each $\wp_x$ is countable. A space is a *g-metrizable space* [8] (respectively, an *$\aleph$-space* [4]) if it is a regular space having a σ-locally finite weak base (respectively, cs-network).

Every g-metrizable space is gf-countable. Every $\aleph$-space is csf-countable. The following lemma can be checked directly.

Lemma 3.12. *Let $\wp$ be a cover of a space X. If $\wp$ is a weak base for X, then $\wp$ is a universal cs-network for X. If X is a sequential space and $\wp$ is a universal cs-network for X, then $\wp$ is a weak base.*

Theorem 3.13. *The following are equivalent for a space X:*
(1) *σX is a gf-countable space.*
(2) *X is a universally csf-countable space.*
(3) *X is a csf-countable and α_1-space.*
(4) *X is a csf-countable and α_4-space.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (3) Suppose X is a universally csf-countable space. Let $F = \{x\} \cup (\bigcup\{T_n\colon n \in \mathcal{N}\})$ be a fan at x of X, where each sequence T_n converges to x. Let $\{P_n\colon n \in \mathcal{N}\}$ be a decreasing universal cs-network at x in X, and $S_n = T_n \cap P_n$ for each $n \in \mathcal{N}$, $S = \bigcup_{n\in\mathcal{N}} S_n = \{s_n\colon n \in \mathcal{N}\}$, then $\{s_n\}$ is a sequence converging to x, and $T_n \backslash S$ is finite for each $n \in \mathcal{N}$. Hence X is an α_1-space.

(4) $\Rightarrow$ (1) Suppose X is a csf-countable and α_4-space. For each $x \in X$, let $\wp_x$ be a countable cs-network at x in X. Put

$$\mathfrak{R}_x = \Big\{ \bigcup \wp'_x\colon \wp'_x \text{ is a finite subset of } \wp_x \text{ and } \bigcup \wp'_x \text{ is a sequential barrier of } x \text{ in } X \Big\}.$$

If $\mathfrak{R}_x$ is not a network of x in X, then there exists an open subset G in X such that $x \in G$ and $F \not\subset G$ for each $F \in \mathfrak{R}_x$. Denote

$$\{P \in \wp_x\colon P \subset G\} = \{P_i\colon i \in \mathcal{N}\}, \qquad F_n = \bigcup\{P_i\colon i \leqslant n\}, \quad n \in \mathcal{N}.$$

Then F_n is not a sequential barrier of x in X. Since $\wp_x$ is a cs-network at x in X, there are a sequence T_i converging to x and $n_i \in \mathcal{N}$ such that $T_i \subset P_{n_{i+1}} \backslash F_{n_i}$, and $n_{i+1} > n_i$ for each $i \in \mathcal{N}$. Put

$$T = \{x\} \cup \Big(\bigcup\{T_i\colon i \in \mathcal{N}\}\Big).$$

Then T is a fan at x in X. Since X is an α_4-space, T has a diagonal $\{x_k\}$ converging to x, there are i and $m \in \mathcal{N}$ such that $x_k \in P_i$ for all $k \geqslant m$. Take some $k \geqslant m$ and some $j \geqslant i$ with $x_k \in T_j$, then $x_k \in P_i \cap (X \backslash F_{n_j}) = \emptyset$, a contradiction. So $\mathfrak{N}_x$ is a countable universal cs-network at x in X, and $\mathfrak{N}_x$ is a countable universal cs-network at x in σX. Since σX is sequential, σX is gf-countable. □

By Corollary 3.9, Theorem 3.13 and Lemma 7(3) in [10], we have the following corollary which answers a question in [21].

Corollary 3.14. *Suppose X is a sequential space with a point-countable cs-network. X has a point-countable weak base if and only if X contains no (closed) copy of $S(\omega)$.*

Theorem 3.15. *The following are equivalent for a regular space X.*
(1) *X has a σ-locally finite universal cs-network.*
(2) *X is an $\aleph$ and α_1-space.*
(3) *X is an $\aleph$ and α_4-space.*
(4) *X is an $\aleph$-space and contains no (closed) subspace having $S(\omega)$ as its sequential coreflection.*

Proof. (1) implies (2) because of Theorem 3.13. (2) implies (3) by Definition 3.5. (3) is equivalent to (4) by Theorem 3.6. We show that (3) $\Rightarrow$ (1). Suppose X is an $\aleph$ and α_4-space. Let $\wp$ be a σ-locally finite cs-network for X which is closed under finite intersections. By Theorem 3.13, X is universally csf-countable. For each $x \in X$, let $\{Q_n(x)\colon n \in \mathcal{N}\}$ be a universal cs-network at x in X. Let

$$\wp_x = \{P \in \wp\colon Q_n(x) \subset P \text{ for some } n \in \mathcal{N}\}.$$

Then $\wp_x$ is a universal cs-network at x in X by the proof of Lemma 7(3) in [10], thus $\bigcup\{\wp_x\colon x \in X\}$ is a σ-locally finite universal cs-network for X. □

Since k-spaces are equivalent to sequential spaces in which each point is G_δ [13], we have that

Corollary 3.16 [11]. *The following are equivalent for a k-space X:*
(1) *X is a g-metrizable space.*
(2) *X is an $\aleph$ and α_1-space.*
(3) *X is an $\aleph$ and α_4-space.*
(4) *X is an $\aleph$-space and contains no (closed) copy of $S(\omega)$.*

Corollary 3.17. *A space is a metrizable space if and only if it is a countably bisequential $\aleph$-space.*

Theorem 3.18. *The following are equivalent for a space X:*
(1) *X has a countable universal cs-network.*
(2) *X is an α_1-space with a countable cs-network.*

(3) *X is an* α_4*-space with a countable cs-network.*
(4) *X has a countable cs-network and contains no subspace having* $S(\omega)$ *as its sequential coreflection.*

Proof. By Theorem 3.13, Definition 3.5 and Theorem 3.6, we only need to show that (4) implies (1). Let $\wp$ be a countable cs-network for X which is closed under finite unions. For each $x \in X$, put

$$\wp_x = \{P \in \wp\colon\ P \text{ is a sequential barrier at } x \text{ in } X\}.$$

If $\wp_x$ is not a network of x in X, by the proof in Theorem 3.13, we has a fan T at x in X. Using the same notation in the proof in Theorem 3.13, if D is a diagonal of T converging to d, then $\{x, d\} \cup D \subset P \subset G$ for some $P \in \wp$, thus $P = P_i$ for some $i \in \mathcal{N}$. Take some $j \geqslant i$ and $d' \in D \cap T_j$, then $d' \in P_i \cap T_j \subset P_i \cap (X \backslash F_{n_j}) = \emptyset$, a contradiction. This show that T has not a diagonal. By Lemma 3.2, σT is homeomorphic to $S(\omega)$, a contradiction. Hence $\wp_x$ is a network of x in X, and X has a countable universal cs-network. □

Example 3.19. There are a compact, sequential space Y and its subspace T such that
(1) Y contains no copy of S_2 or $S(\omega)$.
(2) σT is homeomorphic to $S(\omega)$.
(3) T has a countable cs-network.

Proof. By the same notation in Example 2.14, let $A = \{A_n\colon\ n \in \mathcal{N}\}$. Take $Y = X/A$ and let $f: X \to Y$ be the natural quotient map, then Y is a compact, sequential space, and Y contains no copy of S_2 or $S(\omega)$ by Corollary 3.10 in [16]. Let $T = f(M)$, then T has a countable cs-network and σT is homeomorphic to $S(\omega)$. □

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