第18卷 第3期 吉首大学学报(自然科学版) Vol. 18 No. 3
1997年9月 Journal of Jishou University (Natural Science Edition) Sept. 1997

A Note on Spacs with a Locally Countable K —network*

Lin Shou
(Department of mathematics, Ningde Teather's College Ningde 352100)

Abstract In this paper it is shown that a regular space has a locally countable k —network if and only if it has a locally countable cs —network. As its application, a perfect preimage theorem on spaces with a locally countable k —network is established.

Key words and phrases k —network, cs —network, locally countable family, perfect map.
AMS Classification 54E20, 54D55

Let us recall some definitions. Let X be a space, and $\mathscr{P}$ be a cover of X. Then $\mathscr{P}$ is called a k —network for X if, whenever $K \subset U$ with K compact and U open in X, $K \subset \bigcup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$. Then $\mathscr{P}$ is called a cs—network for X if, whenever K is a sequence converging to $x \in K$, and U is a neighborhood of X, there is $P \in \mathscr{P}$ such that $x \in P \subset U$, and P contains K eventually.

A collection $\mathscr{P}$ of subsets of a space X is said to be locally countable if for each point of X there is a neighborhood which intersects at most countably many elements of $\mathscr{P}$. $\mathscr{P}$ is said to be star—countable if each element of $\mathscr{P}$ intersects at most countably many elements of $\mathscr{P}$.

For a space X and $P \subset X$. P is sequentially open in X if, whenever $x \in P$ and $\{x_n\}$ is a sequence converging to x in X, then $x_n \in P$ for all but finitely many $n \in N$.

We assume that spaces are regular and T_1, and maps are continuous and onto. Unexplained notions and terminology are the same as [5].

Spaces with certain locally countable covers, for example, spaces with a locally countable k —network, spaces with a locally countable weak base, spaces with a locally countable cs —network, have been widely studied in [4,6,8]. In this paper, we affirmative answer the following question posed by Chuan Liu in [8].

Question. [8, Question 2.1] If a regular space X has a locally countable k —network, then does X have a locally countable (or point —countable) cs —network?

Lemma. [1, Lemma 3.10] Let $\mathscr{P}$ be a star—countable collection of subsets of a set X. Then we can set $\mathscr{P} = \bigcup \{\mathscr{P}_\alpha : \alpha \in \Lambda\}$, where each $\mathscr{P}_\alpha$ is a countable subcollection of $\mathscr{P}$ and for two distinct $\alpha, \beta \in \Lambda$, $(\bigcup \mathscr{P}_\alpha) \cap (\bigcup \mathscr{P}_\beta) = \varnothing$.

We call $\{\mathscr{P}_\alpha : \alpha \in \Lambda\}$ in the Lemma a docomposition of $\mathscr{P}$.

Received June 10, 1997
* This work was Supported by the NNSF of China

Theorem. The following are equivalent for a space X :

(1) X has a locally countable k —network.

(2) X has a locally countable cs —network.

Proof. It only need to show that (1) implies (2). Let $\mathscr{R}$ be a locally countable k —network for X. By the regularity of X, We can assume that each elements of $\mathscr{R}$ is closed in X. For each $x \in X$, there is a neighborhood U_x of x in X such that U_x intersects at most countably many elements of $\mathscr{R}$. Put $\mathscr{P} = \{P \in \mathscr{R} : R \subset U_x$ for some $x \in X\}$. Then $\mathscr{P}$ is a star—countable k —network for X. Let $\{\mathscr{P}_a : \in \Lambda\}$ be a decomposition of $\mathscr{P}$. For each $\alpha \in \Lambda$, let $\mathscr{F}_a = \{\bigcup \mathscr{P}' : a$ finite $\mathscr{P}' \subset \mathscr{P}\}$, $T_a = \bigcup \mathscr{F}_a$. Then $\{T_a : \alpha \in \Lambda\}$ is a locally countable and disjoint cover of X.. Let $\{x_n\}$ be a sequence converging to x in X, there is a unique $\alpha \in \Lambda$ with $x \in T_a$. Let U be a neighborhood of x in X, then $\{x\} \cup \{x_n : n \geq m\} \cup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$ and some $m \in N$. Let $\mathscr{P}'' = \{P \in \mathscr{P}' : x \in P\}$, then $\{x\} \cup \{x_n : n \geq i\} \subset X \setminus \bigcup (\mathscr{P}' \setminus \mathscr{P}'')$ for some $i \geq m$, thus $\{x\} \cup \{x_n : n \geq i\} \subset \bigcup \mathscr{P}'' \subset U$. By $x \in \bigcap \mathscr{P}''$, we have that $\bigcup \mathscr{P}'' \in \bigcap \mathscr{F}_a$ and $\{x\} \cup \{x_n : n \geq i\} \subset \bigcup \mathscr{P}'' \subset U \cap T_a$. Hence $\mathscr{F}_a$ is a countable cs —network for T_a, and T_a is a sequentially open subset of X.

Remarks. (1)The sequential openness of the cover of X in the Theorem is essential. For example, the countable ordinal space ω_1 has a locally countable and disjoint cover $\{\{\alpha\} : \alpha < \omega_1\}$ of singletons, but ω_1 does not have a point—countable k —network by[5].

(2)The sequentially open cover of X can not be improved the open cover of X in the Theorem. It is easy to prove that a space has a locally countable (and disjoint) open cover of countable cs —networks if and only if it is a paracompact space with a locally countable cs —network. A space with a locally countable cs —network may not be a paracompact space by [4, Example].

(3)By the Theorem and its proof, we have that

spaces with locally countable k —networks	$\Leftrightarrow$	spaces with locally countable cs —networks
$\Downarrow$		
spaces with star—countable closed k —networks	$\Rightarrow$	spaces with star—countable closed cs —networks
$\Downarrow$		$\Downarrow$
spaces with star—countable k —networks		spaces with star—countable cs —networks $\Rightarrow$ spaces with point—countable cs —networks.

The Fortissimo space $X(p)$ in [5, Example 2. 5. 19] has a star—countable closed k —network, but it does not have a locally countable k —network. The fan space S_{ω_1} in [8, Example 1. 13] has a star—countable k —network. but it does not have a point —countable cs —network. The Stone—Cech compactification βN has a star—countable closed cs —network, but it does not have a star—countable k —network. The hedgehog space $J(\omega_1)$ in [2, Example 4. 1. 5]has a point—countable cs —network, but it does not have a star—countable cs —network.

Quettion 1. Does a space with a star—countable cs —network have a point—countable closed cs —network?

Corollary. Suppose $f: X \to Y$ is a perfect mapping. If X has a local G_δ —diagonal, and Y has a locally countable k —network, then X has a locally countable k —network.

Proof. By the Theorem, Y has a locally countable and disjoint sequentially open cover $\{Y_\alpha : \alpha \in \Lambda\}$ such that each Y_α has a countable cs —network. For each $\alpha \in \Lambda$, put $X_\alpha = f^{-1}(Y_\alpha)$, $f_\alpha = f_{|X_\alpha} : X_\alpha \to Y_\alpha$, then f_α is a perfect mapping, so X_α is a paracompact space. Thus X_α has a point—finite open cover of G_δ —diagonal subsets. Since G_δ —diagonal property satisfies the point—finite open sum theorem by [3], X_α has a G_δ —diagonal. And since a space with a G_δ —diagonal has a countable cs —network if it is a perfect preimage of a space with a countable cs —network by [7], X_α has a countable cs —network. Therefore $\{X_\alpha : \alpha \in \Lambda\}$ is a locally countable and disjoint sequentially open cover of countable cs —networks, X has a locally countable k —network.

Spaces with a locally countable k —network have a local G_δ —diagonal.

Question 2. Does a space with a locally countable k —network has a G_δ —diagonal?

References

1 Burke D. Covering properties, in Handbook of Set—Theoretic Topology Elsevier Science Publishers, B. V, 1984, 347—422

2 Engelking R. General Topology Polish Scientific Publishers, 1977

3 Gittings R. Open mapping theorems, in Set—Theoretic Topology Academic Press, 1977, 141—191

4 Shou lin. Spaces with a locally countable k— network, Northeastern Math. J. 1990(6):39—44

5 Shou Lin. Generalized Metric Spaces and Mappings China Science Press, 1995

6 Chuan liu and Mumin Dai. Spaces with a locally countable weak base, Math. Japonica 1995, (41):261—267

7 Mancuso V. Inverse images and first countability, General Topology Appl. 1972, (2):29—44

8 Tanaka Y and Shengxiang Xia. Certain s—images of locally separable metric spaces. Questions Answers in General Topology, 1996, (14):217—231

具有局部可数 K 网空间的注记*

林寿

(宁德师范高等专科学校数学系　福建　352100)

[摘要]　证明了一个正则空间有局部可数 K 网当且仅当它有局部可数 CS 网，作为其应用，本文建立了具有局部可数 K 网空间的完备逆映象定理。

关键词　K 网；CS 网；局部可数族；完备映射

中图分类号　O189

* 国家自然科学基金资助课题