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Images on Locally Separable Metric Spaces

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Abstract In this paper internal characterizations on certain quotient images of locally separable metric spaces are discussed. We obtain some descriptions of quotient s-images, pseudo-open s-images, quotient compact images and closed images of locally separable metric spaces, and establish some relations between these and certain quotient images of metric spaces by the local separability of suitable subspaces.

Keywords Locally separable, Metric space, Quotient mapping, Pseudo-open mapping, Closed mapping, cs-network, k-network.
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1 Introduction

To find the internal characterizations of certain images of metric spaces is one of the central questions on general topology. Since A. Arhangel'skii published the famous paper "Mappings and Spaces"[1] in 1966, the behaviour of certain quotient images on separable metric spaces or metric spaces has attracted considerable attention, and some noticeable results have been obtained[2-4]. Then, what are the internal characterizations of certain quotient images of locally separable metric spaces which are important spaces lying between the separable metric spaces and the metric spaces?

We know that every quotient image of a metric space is actually the quotient image of a locally separable metric space[4]. However, every quotient s-image of a metric space need not be the quotient s-image of a locally separable metric space. This arouses our interest in the images of locally separable metric spaces. In Section 2, we describe the quotient s-images, pseudo-open s-images and closed s-images of locally separable metric spaces. In Section 3, we characterize the quotient compact images of locally separable metric spaces. In Section 4, we give the characterizations of closed images of locally separable metric spaces.

In this paper, all spaces are regular and T_1, all mappings are continuous and onto. N denotes the set of all natrual numbers, and $\omega = \{0\} \cup N$. Suppose X is a space. Let $\mathcal{P}$ be a collection of subsets of X. $\mathcal{P}$ is called a k-network for X if whenever $K \subset U$ with K compact

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and U open in X, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is called a cs-network for a convergent sequence $\{x_n\}$ with $x_n \to x$ in X if whenever U is a neighborhood of x in X, there exists $m \in N$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P \subset U$ for some $P \in \mathcal{P}$. $\mathcal{P}$ is called a cs^*-network if whenever $\{x_n\}$ is a convergent sequence with $x_n \to x$ and U is a neighborhood of x in X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$ for some $P \in \mathcal{P}$. $\mathcal{P}$ is called a weak base for X if $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ satisfies that

(1) $x \in \cap\mathcal{P}_x$,

(2) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$,

(3) a subset G of X is open in X if and only if for each $z \in G$, we have $P \subset G$ for some $P \in \mathcal{P}_z$.

Here $\mathcal{P}_x$ is called a weak neighborhood base of x in X. A sequence $\{\mathcal{U}_n\}$ of covers of a space X is called a point-finite weak development of X if each $\mathcal{U}_n$ is point-finite and $\{\mathrm{st}(x, \mathcal{U}_n) : n \in N\}$ is a weak neighborhood base of x in X for each $x \in X$.

One of the applications of the concepts above is to characterize certain quotient images of the metric spaces as follows.

Theorem 1.1[5] *A space X is a quotient s-image of a metric space if and only if it is a sequential space with a point-countable cs^*-network.*

Theorem 1.2[6] *A space X is a quotient compact image of a metric space if and only if it has a point-finite weak development.*

Theorem 1.3[2] *A space X is a closed image of a metric space if and only if it is a Fréchet space with a σ-hereditarily closure-preserving k-network.*

In this paper we shall prove that there are some similarities between certain quotient images of locally separable metric spaces and certain quotient images of metric spaces.

2 On Quotient s-Images

A mapping $f : X \to Y$ is a subsequence-covering mapping if for each convergent sequence S (containing the limit point) in Y, there exists a compact subset L of X such that $f(L)$ is a subsequence of S.

Lemma 2.1 *Suppose $f : X \to Y$ is a subsequence-covering mapping. If Y is a sequential space, then f is a quotient mapping.*

Proof Suppose $A \subset Y$ with $f^{-1}(A)$ being closed in X. If A is not closed in Y, then there is a sequence $\{y_n\} \subset A$ with $y_n \to y \in Y \backslash A$ because Y is a sequential space. Take a compact subset K of X such that $f(K) = \{y\} \cup \{y_{n_k} : k \in N\}$. Since $f^{-1}(A) \cap K$ is compact, $f(f^{-1}(A) \cap K) = A \cap f(K)$ is closed in Y, a contradiction. Hence f is a quotient mapping.

Theorem 2.2 *The following conditions are equivalent for a space X:*

(1) *X is a quotient s-image of a locally separable metric space.*

(2) *X is a sequential space, and there exists a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ of X where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for some subsequence of S.*

Proof (1) $\Longrightarrow$ (2). Suppose $f : M \to X$ is a quotient s-mapping, here M being a locally separable metric space. By 4.4.F in [7], $M = \oplus_{\alpha\in\Lambda} M_\alpha$, where all M_α are separable metric spaces. For each $\alpha \in \Lambda$, let $\mathcal{B}_\alpha$ be a countable base for M_α, and put $X_\alpha = f(M_\alpha)$, $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$. Then $\{X_\alpha\}$ is a point-countable cover of X, and each $\mathcal{P}_\alpha$ is a countable network of X_α. For each convergent sequence S of X, by Lemma 1.6 in [5], there exists a convergent sequence T of M such that $f(T)$ is a subsequence of S. Since $\bigcup_{\alpha\in\Lambda}\mathcal{B}_\alpha$ is a base for M, there is $\alpha\in\Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for $f(T)$.

(2) $\Longrightarrow$ (1). For each $\alpha\in\Lambda$, we can assume that $\{X_\alpha\}\in\mathcal{P}_\alpha$. Put $\mathcal{P}=\bigcup_{\alpha\in\Lambda}\mathcal{P}_\alpha=\{P_\beta:\beta\in A\}$. Then $\mathcal{P}$ is point-countable. The set A is endowed with discrete topology. Define

$$M \quad = \{b=(\beta_i)\in A^\omega: \text{ there is } \alpha\in\Lambda \text{ such that } P_{\beta_0}=X_\alpha, \{P_{\beta_i}\}\subset\mathcal{P}_\alpha$$

$$\text{and } \{P_{\beta_i}\} \text{ is a network of some } x(b) \text{ in } X\},$$

and give M the subspace topology induced from the usual product topology. Then M is a metric space. For each $b\in M, x(b)$ is unique in X because X is a T_1-space. Define $f:M\to X$ by $f(b)=x(b)$. It is easy to check that f is an s-mapping.

(a) M is a locally separable space.

Suppose $b=(\beta_i)\in M$ with $P_{\beta_0}=X_\alpha$ for some $\alpha\in\Lambda$. Denoting $\mathcal{P}_\alpha=\{P_\beta:\beta\in A_\alpha\}$, then A_α is countable. Put $M_\alpha=M\cap\{(\lambda_i)\in A^\omega:\lambda_0=\beta_0\}$. Then M_α is open in M and $b\in M_\alpha\subset A_\alpha^\omega$, hence M_α is a separable neighborhood of b in M. Thus M is a locally separable space.

(b) f is a quotient mapping.

By Lemma 2.1, we need only to prove that f is a subsequence-covering mapping. Suppose $S=\{x_0\}\cup\{x_n:n\in N\}$ is a convergent sequence of X with $x_n\to x_0$. There is $\alpha\in\Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for some subsequence $T=\{x_0\}\cup\{x_{n_k}:k\in N\}$ of S. We can assume that $T\subset X_\alpha$. Let

$$\mathcal{P}'_\alpha=\{P\in\mathcal{P}_\alpha: P\cap T \text{ is a non-empty closed subset in } X\},$$

$$\mathcal{R}=\{\mathcal{P}'\subset\mathcal{P}'_\alpha:\mathcal{P}' \text{ is a finite cover of } T\}.$$

Then $\mathcal{R}$ is countable. Denote

$$\mathcal{R}=\{\mathcal{P}_i:i\in\omega\} \text{ with } \mathcal{P}_0=\{X_\alpha\},$$

$$\mathcal{P}_i=\{P_\beta:\beta\in B_i\},\quad |B_i|<\omega,$$

$$K=\{b=(\beta_i)\in\prod_{i\in\omega}B_i:\{P_{\beta_i}\cap T:i\in\omega\} \text{ has the finite intersection property}\}.$$

Then K is compact in A^ω. To complete the proof of the theorem, we need to show that $K\subset M$ and $f(K)=T$.

In fact, on the one hand, suppose $b=(\beta_i)\in K$. Then $\{P_{\beta_i}\cap T:i\in\omega\}$ is a collection of closed subsets with the finite intersection property. Take $x\in\bigcap_{i\in\omega}(P_{\beta_i}\cap T)$. Let U be a neighborhood of x in X. For each $y\in T$, choose a neighborhood W_y of y in X satisfying that

(i) if $y=x$, then $W_y\subset U$,

(ii) if $y\neq x$, then $x\notin W_y$,

(iii) if $y\neq x_0$, then $W_y\cap T=\{y\}$.

Since $\mathcal{P}_\alpha$ is a cs-network for T, there exists $m\in N$ such that $\{x_0\}\cup\{x_{n_k}:k\geq m\}\subset P\subset W_{x_0}$ for some $P\in\mathcal{P}_\alpha$. Let $D=T\backslash P$. Then D is finite. For each $y\in D$, choose $P_y\in\mathcal{P}_\alpha$ with

$y \in P_y \subset W_y$. Define $\mathcal{P}' = \{P\} \cup \{P_y : y \in D\}$. Then $\mathcal{P}' \in \mathcal{R}$, so $P_{\beta_i} \in \mathcal{P}_i = \mathcal{P}'$ for some $i \in N$. By the construction of $\mathcal{P}', x \in P_{\beta_i} \subset U$. Hence $\{P_{\beta_i} : i \in \omega\}$ is a network of x in X. So $b \in M$ and $f(b) = x$. This shows $K \subset M$ and $f(K) \subset T$. On the other hand, let $x \in T$. For each $i \in \omega$, choose $\beta_i \in B_i$ with $x \in P_{\beta_i}$. Putting $b = (\beta_i) \in A^\omega$, then $x \in \bigcap_{i\in\omega}(P_{\beta_i} \cap T)$, so $b \in K \subset M$, and $f(b) = x$. This shows $T \subset f(K)$. We have shown that there exists a compact subset K of M with $f(K) = T$. Therefore f is a subsequence-covering mapping, and f is a quotient mapping. This completes the proof of the theorem.

A space X is an $\aleph_0$-space if it has a countable cs-network.

Corollary 2.3 *If X is a sequential space with a point-countable cs^*-network consisting of $\aleph_0$-subspaces, then X is a quotient s-image of a locally separable metric space.*

We don't know whether the inverse proposition of Corollary 2.3 holds. In the following, we discuss some conditions which make a quotient s-image of a metric space to be a quotient s-image of a locally separable metric space.

Lemma 2.4 *Suppose $\mathcal{P}$ is a point-countable collection of subsets of X, which is closed under finite intersections. Let*

$$\mathcal{H} = \{P \in \mathcal{P} : P \text{ is a hereditarily separable subspace of } X\}.$$

Then $\mathcal{H}$ is a cs^*-network for X if and only if $\mathcal{P}$ is a cs^*-network for X and every first countable subspace of X is locally separable.

Proof Necessity. Suppose $\mathcal{H}$ is a cs^*-network for X. Let F be a first countable subspace of X. for each $x \in F$, there is a finite $\mathcal{F} \subset \mathcal{H}$ such that $x \in \text{int}_F(\cup\{H \cap F : H \in \mathcal{F}\}) \subset \cup\mathcal{F}$ by Proposition 3.2 in [3]. Since $\cup\mathcal{F}$ is hereditarily separable, F is a locally separable subspace of X.

Sufficiency. Suppose $\mathcal{P}$ is a cs^*-network for X and every first countable subspace of X is locally separable. For a convergent sequence $\{x_n\}$ of X with $x_n \to x$, define $K = \{x\} \cup \{x_n : n \in N\}$. Let U be a neighborhood of K in X. By Lemma in [8], there exists a finite $\mathcal{P}' \subset \mathcal{P}$ with the property $K \subset \cup\mathcal{P}' \subset U$, and $P \cap K$ is closed in X for each $P \in \mathcal{P}'$. The collection of finite subcollections of $\mathcal{P}$ with the property above is countable, and denote it by $\{\mathcal{P}_i : i \in N\}$. For each $n \in N$, put $A_n = \cup(\wedge_{i\leq n}\mathcal{P}_i)$. Then $\{A_n\}$ is a descending network of K in X. If A_n is not hereditarily separable for each $n \in N$, then A_n has an uncountable discrete subspace D_n. Define $A = K \cup \left(\bigcup_{n\in N} D_n\right)$. Then A is a first countable subspace of X, hence A is locally separable. By Proposition 8.8 in [3], A is locally hereditarily separable, a contradiction. Therefore A_m is hereditarily separable for some $m \in N$. Take $P \in \wedge_{i\leq m}\mathcal{P}_i$ such that $x \in P$ and P contains a subsequence $\{x_{n_i}\}$ of $\{x_n\}$; thus P is hereditarily separable and $\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$. Hence $\mathcal{H}$ is a cs^*-network for X.

By Theorem 1.1 and Lemma 2.4, we have

Corollary 2.5 *The following conditions are equivalent for a space X:*

(1) *X is a quotient s-image of a metric space and every first countable subspace of X is locally separable.*

(2) *X is a sequential space with a point-countable cs^*-network consisting of hereditarily separable subspaces.*

By Theorem 2.2 and Corollary 2.5, every first countable subspace of a space X is locally separable if X is a quotient s-image of a locally separable metric space.

Question 2.6 (1) Are the conditions in Corollary 2.5 equivalent to the conditions in Theorem 2.2?

(2) Is a hereditarily separable, sequential space with a point-countable cs^*-network an $\aleph_0$-space?

Theorem 2.7 *The following conditions are equivalent for a space X:*

(1) *X is a closed s-image of a locally separable metric space.*

(2) *X is a closed s-image of a metric space, and every first countable subspace of X is locally separable.*

(3) *X is a pseudo-open s-image of a locally separable metric space.*

(4) *X is a pseudo-open s-image of a metric space, and every first countable subspace of X is locally separable.*

(5) *X is a Fréchet space with a point-countable cs^*-network consisting of separable subspaces.*

(6) *X is a Fréchet space with a point-countable cs^*-network consisting of $\aleph_0$-subspaces.*

(7) *X is a Fréchet space with a point-countable cs^*-network, and every first countable subspace of X is locally separable.*

Proof (1) $\Longrightarrow$ (2) By Corollary 2.5.

(2) $\Longrightarrow$ (4) Obvious.

(4) $\Longrightarrow$ (7) By Theorem 1.1.

(7) $\Longrightarrow$ (5) By Lemma 2.4.

(5) $\Longrightarrow$ (6) By Theorem 5.2 in [3].

(6) $\Longrightarrow$ (3) By Corollary 2.3.

(3) $\Longrightarrow$ (1) By Theorem 2 in [9] or Theorem 2 in [10].

3 On Quotient Compact Images

Theorem 3.1 *The following conditions are equivalent for a space X:*

(1) *X is a quotient compact image of a locally separable metric space.*

(2) *X has a point-finite cover $\{X_\alpha : \alpha \in \Lambda\}$, where each X_α has a sequence $\{\mathcal{P}_{\alpha i}\}$ of countable and point-finite covers such that $\{\bigcup_{\alpha\in\Lambda}\mathcal{P}_{\alpha i}\}$ is a weak development of X.*

Proof (1) $\Longrightarrow$ (2) Suppose $f : M \to X$ is a quotient compact mapping, here M being a locally separable metric space. Then $M = \oplus_{\alpha\in\Lambda}M_\alpha$, where every M_α is a separable metric space. By the metrizability of M, there exists a sequence $\{\mathcal{B}_i\}$ of locally finite open covers of M satisfying that each $\mathcal{B}_i$ is a refinement of $\mathcal{B}_{i+1}$, and whenever $K \subset U$ with K compact and U open in M, then $\mathrm{st}(K, \mathcal{B}_i) \subset U$ for some $i \in N$ by 5.4.E in [7]. Put

$$X_\alpha = f(M_\alpha), \alpha \in \Lambda, \quad \mathcal{P}_{\alpha i} = \{f(B \cap M_\alpha) : B \in \mathcal{B}_i\}, \alpha \in \Lambda, i \in N.$$

Then $\{X_\alpha\}$ is a point-finite cover of X, and $\{\mathcal{P}_{\alpha i}\}$ is a sequence of countable and point-finite covers of X_α. By the construction of $\{\mathcal{B}_i\}$, it is easy to check that $\{\bigcup_{\alpha\in\Lambda}\mathcal{P}_{\alpha i}\}$ is a weak development of X (cf. the proof of Theorem 1 in [6]).

(2) $\Longrightarrow$ (1) Let

$$\mathcal{P}_{\alpha 0} = \{X_\alpha\}, \alpha \in \Lambda, \quad \mathcal{P}_i = \bigcup_{\alpha\in\Lambda} \mathcal{P}_{\alpha i} = \{P_\beta : \beta \in A_i\}, i \in \omega.$$

Then $\{\mathcal{P}_i : i \in \omega\}$ is a sequence of point-finite covers of X. Each A_i is endowed with discrete topology. Let

$$M = \Big\{b = (\beta_i) \in \prod_{i\in\omega} A_i : \text{ there exists } \alpha \in \Lambda \text{ such that } P_{\beta_i} \in \mathcal{P}_{\alpha i} \text{ for each } i \in \omega \text{ and } \{P_{\beta_i} : i \in \omega\} \text{ forms a network at some point } x(b) \text{ in } X\Big\},$$

and give M the subspace topology induced from the usual product topology. Then M is a metric space. Define $f : M \to X$ by $f(b) = x(b)$ for each $b \in M$. From the proof of Theorem 2.2, f is a mapping from a locally separable metric space M onto X. Since $\{\mathcal{P}_i : i \in \omega\}$ is a point-finite weak development of X, by the proof of Theorem 1.2 (cf. [6]), f is a quotient compact mapping.

Remark 3.2 The locally separable metric spaces are preserved by pseudo-open compact mappings by Corollary 4.5 in [11].

4 On Closed Images

Suppose $\mathcal{P}$ is a collection of subsets of a space X. $\mathcal{P}$ is called hereditarily closure-preserving, HCP for short, if whenever $H(P) \subset P \in \mathcal{P}$, then $\cup\{\overline{H(P)} : P \in \mathcal{P}\} = \overline{\cup\{H(P) : P \in \mathcal{P}\}}$.

Lemma 4.1 *If a separable, Fréchet space X has a σ-HCP k-network, then X is a closed image of a separable metric space.*

Proof By Theorem 1.3, there exist a metric space M and a closed mapping $f : M \to X$. By Lemma 5.4 in [12], we can assume that f is an irreducible closed mapping. Let D be a countable dense subset of X, and choose a countable subset C of M with $f(C) = D$. Then $f(\overline{C}) = X$, thus $\overline{C} = M$, hence M is a separable metric space. Therefore X is a closed image of a separable metric space.

Theorem 4.2 *The following conditions are equivalent for a space X :*

(1) X is a closed image of a locally separable metric space.

(2) X is a Fréchet space with a σ-HCP k-network consisting of separable subspaces.

Proof (1) $\Longrightarrow$ (2) Suppose $f : M \to X$ is a closed mapping, where M is a locally separable metric space. Obviously, X is Fréchet[4]. Let $\mathcal{B}$ be a σ-locally finite base of M consisting of separable subspaces. Put $\mathcal{P} = f(\mathcal{B})$. Then $\mathcal{P}$ is a σ-HCP k-network of X consisting of separable subspaces[2].

(2) $\Longrightarrow$ (1) Suppose X is a Fréchet space with a σ-HCP k-network $\mathcal{P}$ consisting of separable subspaces. Denote $\mathcal{P} = \bigcup_{n\in N} \mathcal{P}_n$, where each $\mathcal{P}_n$ is an HCP collection of closed subsets of X[13], and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N$, put

$$P_n = \cup\mathcal{P}_n, \quad F_n = \overline{P_n \backslash P_{n-1}}, \text{ where } P_0 = \emptyset.$$

Then $\{P_n\}$ is a closed cover of X and $P_n \subset P_{n+1}$ for each $n \in N$. Since X is a k-space, X has weak topology with respect to $\{P_n\}$, thus X is dominated by $\{P_n\}$. Since X is a Fréchet space, by Lemma 2.1 in [14], $\{F_n : n \in N\}$ is an HCP closed cover of X. Define

$$\mathcal{F} = \{\overline{P \backslash F_{n-1}} : n \in N, P \in \mathcal{P}_n\} = \{F_\alpha : \alpha \in \Lambda\}, \text{ where } F_0 = \emptyset.$$

Then $\mathcal{F}$ is an HCP closed cover of X. Let

$$Z = \oplus_{\alpha\in\Lambda} F_\alpha, \quad f : Z \to X \text{ be the obvious mapping.}$$

Then f is a closed mapping. For each $\alpha \in \Lambda$, by Lemma 4.1, there exist a separable metric space M_α and a closed mapping $g_\alpha : M_\alpha \to F_\alpha$. Define

$$M = \oplus_{\alpha\in\Lambda} M_\alpha, \quad g = f \circ (\oplus_{\alpha\in\Lambda} g_\alpha) : M \to X.$$

Then g is a closed mapping from a locally separable metric space M onto X. Hence X is a closed image of a locally separable metric space.

In the final part, we shall establish some relations between the closed images of metric spaces and locally separable metric space.

Lemma 4.3 *The following conditions are equivalent for a space X :*

(1) *X is a space with a σ-HCP k-network consisting of $\aleph_0$-subspaces.*

(2) *X is a space with a σ-HCP k-network consisting of hereditarily separable subspaces.*

(3) *X has a σ-HCP k-network and every first countable subspace of X is locally separable.*

Proof (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Suppose X is a space with a σ-HCP k-network consisting of hereditarily separable subspaces. Let A be a first countable subspace of X. By Theorem 4.2, there exist a locally separable metric space M and a closed mapping $f : M \to A$. For each $x \in A$, by Lemma 4.4.16 in [7], $\partial f^{-1}(x)$ is compact in M; we can assume that f is a perfect mapping. Thus A is a locally separable metric subspace of X.

(3) $\Longrightarrow$ (1) Suppose X has a σ-HCP k-network, and every first countable subspace of X is locally separable. We can assume that X has a k-network $\mathcal{P} = \bigcup_{n\in N} \mathcal{P}_n$, where each $\mathcal{P}_n$ is an HCP collection of closed subsets of X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Put

$$\mathcal{R} = \{P \backslash D_n : P \in \mathcal{P}_n, n \in N\} \cup \{\{x\} : x \in D_n, n \in N\},$$

$$\text{where } D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finie at } x\}.$$

Then, by Theorem in [15], we have the following facts:

(a) D_n is a σ-closed discrete subset of X for each $n \in N$,

(b) if K is a compact subset of X, then $K \cap D_n$ is finite for each $n \in N$,

(c) for each finite $\mathcal{F} \subset \mathcal{R}$, there exist $m \in N, P \in \mathcal{P}_m$ and $D \subset D_m$ such that $\cap\mathcal{F} = (P\backslash D_m) \cup D$.

Define

$$\mathcal{H} = \{\overline{R} : R \in \mathcal{R} \text{ and } \overline{R} \text{ is an } \aleph_0\text{-subspace of } X\}.$$

Then, by (a), $\mathcal{H}$ is a σ-HCP collection of closed subsets of X consisting of $\aleph_0$-subspaces. We shall show that $\mathcal{H}$ is a k-network for X. For $K \subset U$ with K compact and U open in X, since $\mathcal{R}$ is a point-countable cover of K, by Miščenko Lemma (cf. 3.12.22 in [7]), there are only countably many minimal finite subcollections of coverings K, denoted by $\{\mathcal{R}_i : i \in N\}$. For each $n \in N$, let

$$A_n = \cup (\wedge_{i\le n} \mathcal{R}_i).$$

Then $\{\overline{A}_n\}$ is a descending sequence of closed subsets of X. If V is open in X with $K \subset V$, then there exists $j \in N$ such that $K \subset \cup\mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}_j$. Thus

$$K \subset (\cup\{P\backslash D_j : P \in \mathcal{P}'\}) \cup (K \cap D_j) \subset (\cup\{\overline{P\backslash D_j} : P \in \mathcal{P}'\}) \cup (K \cap D_j) \subset V.$$

By (b), there is $n \in N$ such that $\mathcal{R}_n \subset \{P \backslash D_j : P \in \mathcal{P}'\} \cup \{\{x\} : x \in K \cap D_j\}$, so $K \subset \overline{A}_n \subset V$. Thus $\{\overline{A}_n\}$ is a network of K in X. By the proof of Lemma 2.4, there is $m \in N$ such that $\overline{A}_m \subset U$ and $\overline{A}_m$ is a hereditarily separable subspace of X. By Lemma 1 in [16], $\overline{A}_m$ is an $\aleph_0$-subspace of X because a hereditarily separable space with a σ-HCP k-network is an $\aleph_0$-space. Since A_m is the finite union of finite intersections of elements of $\mathcal{R}$, by (c), there are a finite $\mathcal{R}' \subset \mathcal{R}, n \in N$ and $D \subset D_n$ with $A_m = (\cup\mathcal{R}') \cup D$. Put $\mathcal{H}' = \{\overline{R} : R \in \mathcal{R}'\} \cup \{\{x\} : x \in K \cap D\}$. Then $\mathcal{H}'$ is a finite subcollection of $\mathcal{H}$ and $K \subset \cup\mathcal{H}' \subset U$. Therefore, $\mathcal{H}$ is a k-network for X, and X is a space with a σ-HCP k-network consisting of $\aleph_0$-subspaces.

Theorem 4.4 *The following conditions are equivalent for a space X:*

(1) *X is a closed image of a locally separable metric space.*

(2) *X is a closed image of a metric space, and every first countable subspace of X is locally separable.*

(3) *X is a Fréchet space with a σ-HCP k-network, and every first countable subspace of X is locally separable.*

Proof (1) $\Longleftrightarrow$ (3). By Theorem 4.2 and Lemma 4.1.

(2) $\Longleftrightarrow$ (3) By Theorem 1.3.

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