Topology Proceedings
Vol. 21, 1996

A Note on Lašnev Decomposition Theorems

Shou Lin*

Abstract

In this paper a Lašnev decomposition theorem for quasi-perfect pre-images of normal semistratifiable spaces is established, which partly answers a question posed by G. Gruenhage in 1992.

N. Lašnev [6] proved that if X is metrizable and $f : X \to Y$ is a closed onto map, then $f^{-1}(y)$ is compact for all $y \in Y$ outside of some σ-closed discrete subset of Y. Some extensions of Lašnev's theorem were reported in [1] and [3]. G. Gruenhage tried to show the following decomposition theorem [3, Theorem 5.14]: Suppose X is a quasi-perfect pre-image of a normal semistratifiable space, and $f : X \to Y$ is a closed onto map, then $f^{-1}(y)$ is countable compact for all $y \in Y$ outside of some σ-closed discrete subset of Y.

But, its proof has a gap, and Gruenhage asks if it is true or not [4, p 264]. In this short note the following decomposition theorem is established.

Theorem A *Suppose X is a quasi-perfect pre-image of a normal semi-stratifiable space, and each point of Y is a G_δ-set. If $f : X \to Y$ is a closed map, then $Y = \bigcup_{n\in\omega} Y_n$, where*

*Project supported by NNSF of China

$f^{-1}(y)$ is countably compact for each $y \in Y_0$, and for each $n \in N$, Y_n is a closed discrete subset of Y.

All spaces in this paper are assumed to be regular and T_2. By map we mean a continuous surjection.

Definition 1 [3, Definition 5.6] A space X is semistratifiable if there is a function G which assigns to each $n \in N$ and closed set $H \subset X$, an open set $G(n,H)$ containing H such that

(a) $H = \bigcap_{n \in N} G(n,H)$;

(b) $H \subset K \Rightarrow G(n,H) \subset G(n,K)$.

Definition 2 [3, Definition 7.7] A space (X,τ) is a β-space if there is a function $g : N \times X \to \tau$ such that

(a) $x \in g(n,x)$;

(b) if $x \in g(n,x_n)$, then the sequence $\{x_n\}$ has a cluster point in X.

g in Definition 2 is called a β-function of X. It is easy to check that a quasi-perfect pre-image of a β-space is a β-space. A key lemma of the proof of Theorem A is that β-spaces are preserved by closed mappings, which appears in [7] written in Chinese.

Lemma 1 *For a space X, the following are equivalent:*

(a) X is a β-space;

(b) For every open set U of X, there is a closed set sequence $\{F_n(U)\}$ of X such that

(1) $F_n(U) \subset U$;

(2) $U \subset V \Rightarrow F_n(U) \subset F_n(V)$;

(3) If $\{U_n\}$ is an increasing open sequence of X with $\bigcup_{n \in N} U_n = X$, then $X = \bigcup_{n \in N} F_n(U_n)$.

(c) *For every closed set F of X, there is an open set sequence $\{U_n(F)\}$ of X such that*

(1) $U_n(F) \supset F$;

(2) $F \subset H \Rightarrow U_n(F) \subset U_n(H)$;

(3) If $\{F_n\}$ is a decreasing closed set sequence of X with $\cap_{n\in N} F_n = \emptyset$, then $\cap_{n\in N} U_n(F_n) = \emptyset$.

Proof: (b) $\leftrightarrow$ (c) is obvious. We will prove only (a) $\leftrightarrow$ (c).

(a) $\rightarrow$ (c). Let g be a β-function of X. For any closed set F of X, $n \in N$, let

$$U_n(F) = \bigcup\{g(n,x) \ : \ x \in F\},$$

then open set sequence $\{U_n(F)\}$ of X satisfies (c). In fact, (1) and (2) are obvious. If there exists a decreasing closed set sequence $\{F_n\}$ of X with $\cap_{n\in N} F_n = \emptyset$ and $\cap_{n\in N} U_n(F_n) \neq \emptyset$, then there is a point $p \in \cap_{n\in N} U_n(F_n)$. For each $n \in N$, there is an $x_n \in F_n$ such that $p \in g(n, x_n)$, thus $\{x_n\}$ has a cluster point. However, $\{F_n\}$ is decreasing and F_n is closed, so we have $\cap_{n\in N} F_n \neq \emptyset$, a contradiction.

(c) $\rightarrow$ (a). Assume that for every closed set F of X and $n \in N$, there is $\{U_n(F)\}$ as in (c). For every $x \in X, n \in N$, let $g(n,x) = U_n(\{x_n\})$, then $g : N \times X \rightarrow \tau(X)$. We now check that g is a β-function of X. Let $p \in X$ and for each $n \in N$, $p \in g(n, x_n)$. If $\{x_n\}$ has no cluster points, let $F_n = \{x_i : i \geq n\}$; then $\{F_n\}$ is a decreasing closed set sequence with $\cap_{n\in N} F_n = \emptyset$. Hence, $\cap_{n\in N} U_n(F_n) = \emptyset$. But, $\cup\{g(n,x) : x \in F_n\} \subset U_n(F_n)$ and $p \in \cap_{n\in N}(\cup\{g(n,x) : x \in F_n\})$, a contradiction.

Lemma 2 (H. Teng, S. Xia and S. Lin) *β-spaces are preserved by closed mappings.*

Proof: Assume that $f : X \rightarrow Y$ is a closed mapping and X is a β-space. For any open set U of $X, \{F_n(U)\}$ corresponding to U satisfies (b) of Lemma 1. For every open set V of Y, let $\{G_n(V)\} = \{f(F_n(f^{-1}(V)))\}$ correspond to V. It is not difficult to check that this correspondence satisfies (b) of Lemma 1. Hence, Y is a β-space.

Proof of Theorem A: Let $h : X \to Z$ be quasi-perfect, where Z is semistratifiable. Let G satisfy the conditions of Definition 1 for Z with $G(n+1, H) \subset G(n, H)$. For a closed set $K \subset X$, let $U_n(K) = h^{-1}(G(n, h(K)))$. Since Z is semistratifiable, Z is a β-space, thus X is a β-space. By Lemma 2, Y is a β-space. Let $g : N \times Y \to \tau(Y)$ be a β-function of Y such that

(1) $g(n+1, y) \subset g(n, y)$;

(2) $\cap_{n\in N} g(n, y) = \{y\}$.

Since f is closed, for each $y \in Y$ we can find an open set $O_n(y)$ containing y such that $O_n(y) \subset g(n, y)$ and $f^{-1}(O_n(y)) \subset U_n(f^{-1}(y))$. For each $n \in N$, let $Y_n = \{y \in Y : y' \neq y \Rightarrow y \notin O_n(y')\}$. It is easy to check that each Y_n is a closed discrete subset of Y.

Let $Y_0 = Y \setminus \cup_{n\in N} Y_n$, and let $y \in Y_0$. We need to show that $f^{-1}(y)$ is countable compact. For each $n \in N$, since $y \notin Y_n$, there exists $y_n \neq y$ such that $y \in O_n(y_n)$. Then $y \in g(n, y_n)$. If $\{y_n : n \in N\}$ is finite, there exists a subsequence $\{y_{ni}\}$ of $\{y_n\}$ such that $y_{ni} = y_{n1}$ for each $i \in N$. Since $y \in g(n_i, y_{ni}) \subset g(i, y_{n1})$, then $y \in \cap_{i\in N} g(i, y_{n1}) = \{y_{n1}\}$, a contradiction. Thus $\{y_n : n \in N\}$ is infinite. We can assume that the y_n's are distinct. By $y \in g(n, y_n)$, any subsequence of the sequence $\{y_n\}$ has a cluster point in Y.

Now suppose $f^{-1}(y)$ is not countable compact. Then there is an infinite discrete set $\{x_n : n \in N\} \subset f^{-1}(y)$. Since h is quasi-perfect, we may assume $h(x_n) \neq h(x_m)$ for $n \neq m$. Then $\{h(x_n) : n \in N\}$ is a closed discrete set of distant points in Z. Since Z is normal, there exists a discrete collection $\{V_n : n \in N\}$ of open subsets of Z such that $h(x_n) \in V_n$. On the other hand, for each $n \in N$,

$$x_n \in f^{-1}(y) \subset \cap_{k\geq n} f^{-1}(O_k(y_k)) \subset \cap_{k\geq n} U_k(f^{-1}(y_k))$$

$$
\begin{aligned}
&= \cap_{k\geq n} h^{-1}(\overline{G(k, h(f^{-1}(y_k)))}) \\
&\subset h^{-1}(\overline{\cap_{k\geq n} G(k, \overline{\cup_{k\geq n} h(f^{-1}(y_k))})}) \\
&= h^{-1}(\overline{\cup_{k\geq n} h(f^{-1}(y_k))}) = h^{-1}\overline{h(\cup_{k\geq n} f^{-1}(y_k))},
\end{aligned}
$$

thus $h(x_n) \in \overline{h(\cup_{k\geq n} f^{-1}(y_k))}$, and $V_n \cap h(\cup_{k\geq n} f^{-1}(y_k)) \neq \emptyset$. So there exist a subsequence $\{y_{ki}\}$ of the sequence $\{y_k\}$ and a subfamily $\{V_{ni}\}$ of the family $\{V_n\}$ with $V_{ni} \cap hf^{-1}(y_{ki}) \neq \emptyset$. For each $i \in N$, take a point $x'_i \in X$ with $x'_i \in f^{-1}(y_{ki})$ and $h(x'_i) \in V_{ni}$. Since $\{y_{ki} : i \in N\}$ has a cluster point in Y, $\{x'_i : i \in N\}$ is not closed discrete in X, and $\{h(x'_i) : i \in N\}$ is not closed discrete, a contradiction.

Remark (1) The condition "X is a quasi-perfect pre-image of a normal semistratifiable space" can not be replaced by "X is a normal β-space". J. Chaber [2, Example 3.1] constructed a paracompact β-space X and a closed mapping of X onto the interval with no Lindelöf fibers.

(2) For a β-function g of Y, let

$$Y_n = \{y \in Y : y' \neq y \Rightarrow y \notin g(n, y')\},$$

then Y_n is a closed discrete subset of Y. If $y \in g(n, y_n)$, then it is possible that $\{y_n : n \in N\}$ is finite. Let Y be the Fortissimo space [8, Example 25] with a particular point p. Let $g : N \times Y \to \tau(Y)$ such that $g(n, y) = Y$ if $y = p$, and $g(n, y) = \{y\}$ if $y \in Y \backslash \{p\}$, then g is a β-function of Y, $Y_n = \{p\}$ for each $n \in N$, and $g(n, y_n) = Y$ if $y_n = p$.

(3) T. Ishii [5] proved that let $f : X \to Y$ be a closed mapping of a ωM-space X onto a space Y, then $Y = \cup_{n\in\omega} Y_n$, where $f^{-1}(y)$ is countable compact for each $y \in Y_0$, and for each $n \in N$, Y_n is a closed discrete subset of Y.

But, T. Ishii's proof [5] has a gap in line 17 on page 19. It is easy to check the following decomposition theorem by a similar method in [5].

Theorem B *Let $f : X \to Y$ be a closed mapping of a ωM-space X onto a space Y. If each point of Y is a G_δ-set, then*

$Y = \cup_{n\in\omega} Y_n$, where $f^{-1}(y)$ is countable compact for each $y \in Y_0$, and for each $n \in N, Y_n$ is a closed discrete subset of Y.

References

[1] D. K. Burke, *Closed mappings*, Surveys in General Topology, G. M. Reed, editor, Academic Press, New York, 1980, 1-32.

[2] J. Chaber, *Generalizations of Lašnev theorem*, Fund. Math., **119** (1983), 85-91.

[3] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan, editors, North-Holland, Amsterdam, 1984, 423-501.

[4] G. Gruenhage, *Generalized metric spaces and metrization*, Recent Progress in General Topology, M. Hušek and J. van Mill, editors, Elsevier Science Publishers B. V. Amsterdam, 1992, 239-274.

[5] T. Ishii, *ωM-spaces and closed maps*, Proc. Japan Acad., **46** (1970), 16-21.

[6] N. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Soviet Math. Dokl., **6** (1965), 1504-1506.

[7] Shou Lin, *Generalized metric spaces and mappings*, China Scientific Publishers, Beijing, 1995, 173.

[8] L.A. Steen, J.A. Seebach, Counterexamples in Topology (Second Edition), Springer-Verlag, New York, 1978.

Ningde Teachers' College
Ningde, Fujian 352100 P. R. CHINA