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On spaces with point-countable cs-networks

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Abstract

In this paper we discuss three questions about the quotient s-images of metric spaces. The main results are:

(1) X is a sequential space with a point-countable cs-network if and only if X is a compact-covering, sequence-covering, quotient and s-image of a metric space.

(2) Let X and Y be sequential spaces with point-countable cs-networks, then $X \times Y$ is a k-space if and only if one of the three properties below holds.

(a) X and Y are first countable spaces.

(b) X or Y is a locally compact space.

(c) X and Y are local k_ω-spaces.

(3) Let $f : X \to Y$ be a pseudo-open s-map. If X is a Fréchet space with a point-countable cs-network, then Y is a Fréchet space with a point-countable cs*-network.

They partly answer three questions posed by Michael and Nagami (1973), Tanaka (1983), and Gruenhage, Michael and Tanaka (1984) respectively.

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0. Introduction

Since generalized metric spaces determined by point-countable covers were discussed by Burke and Michael in [2] and Gruenhage, Michael and Tanaka in [5], point-countable covers have drawn attention in general topology. Partly, that is because point-countable

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covers are closely related to spaces with point-countable bases, and quotient s-images of metric spaces. The problem posed by Arhangel'skii in [1] about the quotient s-images of metric spaces is answered affirmatively by Tanaka in [19].

Lemma 0.1. *A space X is a quotient (pseudo-open) s-image of a metric space if and only if X is a sequential (Fréchet) space with a point-countable cs*-network.*

Though Arhangel'skii's problem is answered, point-countable covers and related questions are noticeable. For example:

Michael–Nagami's question [13]. If a space X is a quotient s-image of a metric space, must X also be a compact-covering quotient s-image of a metric space?

Tanaka's question [18]. For quotient s-images X and Y of metric spaces, what is a necessary and sufficient condition for $X \times Y$ to be a k-space?

Gruenhage–Michael–Tanaka's question [5]. Are pseudo-open s-images of metric spaces preserved by pseudo-open s-maps? By perfect maps?

By Lemma 0.1 the above three questions can recount three equivalent questions by cs*-networks. In this paper we shall establish three similar theorems by means of the concept of cs-networks, which partly answer the three questions mentioned above.

We recall some basic definitions.

Definition 0.2. Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a network if, whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$. A subfamily $\mathcal{P}'$ of $\mathcal{P}$ is a network at $x \in X$ if $x \in \bigcap \mathcal{P}'$ and whenever $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathcal{P}'$.

(2) $\mathcal{P}$ is a cs-network [6] if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then $\{x\} \cup \{x_n\colon n \geqslant m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a cs*-network [3] if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then $\{x\} \cup \{x_{n_i}\colon i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is a k-network [14] if, whenever $K \subset U$ with U open and K compact in X, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

Definition 0.3. Let $f : X \to Y$ be a map.

(1) f is an s-map if every $f^{-1}(y)$ is separable for each $y \in Y$.

(2) f is a compact-covering map [9] if each compact subset of Y is the image of a some compact subset of X.

(3) f is a sequence-covering map [5] if each convergent sequence of Y is the image of some convergent sequence of X.

There is a different definition of a sequence-covering map in [5], namely it requires that each convergent sequence of Y be the image of some compact subset of X. In [5] it is shown that a space X is a quotient s-image of a metric space if and only if X is a sequence-covering quotient s-image of a metric space.

We assume that spaces are regular and T_1, and maps are continuous and onto.

1. On Michael–Nagami's question

Michael–Nagami's question is whether a sequential space with a point-countable cs*-network is a compact-covering quotient s-image of a metric space by Lemma 0.1. The main result of this section is that a sequential space with a point-countable cs-network is a compact-covering quotient s-image of a metric space. First of all, we characterize a space with a point-countable cs-network by maps.

Theorem 1.1. *A space X has a point-countable cs-network if and only if X is a sequence-covering s-image of a metric space.*

Proof. Let X be a space with a point-countable cs-network $\mathcal{P}$. Suppose $\mathcal{P}$ is closed under finite intersections. Denote $\mathcal{P}$ by $\{P_\alpha\colon\ \alpha \in A\}$. Let A_i denote the set A with discrete topology for each $i \in \mathbb{N}$. Put

$$M = \left\{\beta = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i\colon\ \{P_{\alpha_i}\colon\ i \in \mathbb{N}\} \text{ is a network at some point } x(\beta) \text{ in } X\right\},$$

then M is a metric space, and $f : M \in X$ defined by $f(\beta) = x(\beta)$ is a function. It is easy to check that f is an s-map from M onto X. We shall show that f is a sequence-covering map. For a sequence $\{x_n\}$ of X converging to a point x_0 in X, we assume that all x_n's are distinct. Let $K = \{x_m\colon\ m \in \omega\}$, and let $K \subset U$ with U open in X, a subset $\mathcal{F}$ of $\mathcal{P}$ is said to have the property $F(K,U)$ if $\mathcal{F}$ satisfies that

(1) $\mathcal{F}$ is finite;
(2) $\emptyset \neq P \cap K \subset P \subset U$ for each $P \in \mathcal{F}$;
(3) for each $x \in K$ there is a unique $P_x \in \mathcal{F}$ with $x \in P_x$;
(4) if $x_0 \in P \in \mathcal{F}$, then $K \setminus P$ is finite.

Put

$$\{\mathcal{F} \subset \mathcal{P}\colon\ \mathcal{F} \text{ has the property } F(K,X)\} = \{\mathcal{F}_i\colon\ i \in \mathbb{N}\},$$

for each $i \in \mathbb{N}$ and each $m \in \omega$ there is $\alpha_{im} \in A_i$ with $x_m \in P_{\alpha_{im}} \in \mathcal{F}_i$. It can be checked that $\{P_{\alpha_{im}}\colon\ i \in \mathbb{N}\}$ is a network at the point x_m. Let $\beta_m = (\alpha_{im})$ for each $m \in \omega$, then $\beta_m \in M$ and $f(\beta_m) = x_m$. For each $i \in \mathbb{N}$, there is $n(i) \in \mathbb{N}$ such that $\alpha_{in} = \alpha_{i0}$ if $n \geqslant n(i)$, thus the sequence $\{\alpha_{in}\}$ converges to α_{i0} in A_i, and the sequence $\{\beta_n\}$ converges to β_0 in M. This shows that f is a sequence-covering map.

Conversely, suppose that $f: M \to X$ is a sequence-covering s-map, where M is a metric space. Let $\mathcal{B}$ be a σ-locally finite base for M, then $\{f(B)\colon B \in \mathcal{B}\}$ is a point-countable cs-network for the space X. □

Lemma 1.2. *Let $\mathcal{P}$ be a point-countable cs-network for a space X. If $x \in K \cap W$ with W open and K compact, first countable in X, then $x \in \mathrm{int}_K(P \cap K) \subset P \subset W$ for some $P \in \mathcal{P}$.*

Proof. Let $\{V_n\colon n \in \mathbb{N}\}$ be a local base at the point x in K. Put

$$\mathcal{F} = \{P \cap K\colon P \in \mathcal{P}, \text{ and } P \subset W \text{ or } P \subset X \setminus \{x\}\},$$

$$\mathcal{F}' = \{F \in \mathcal{F}\colon V_n \subset F \text{ for some } n \in \mathbb{N}\},$$

then $\mathcal{F}$ is a point-countable cs-network for the subspace K, and $\mathcal{F}'$ is a neighborhood base at x in K by the proof of Lemma 7(3) in [8], thus $x \in \mathrm{int}_K(F) \subset K \cap W$ for some $F \in \mathcal{F}$, i.e., for some $P \in \mathcal{P}$ $x \in \mathrm{int}_K(P \cap K) \subset P \subset W$. □

Let $\mathcal{P}$ be a family of subsets of X, and let $K \subset X$, denote that

$$(\mathcal{P}|K)^0 = \{\mathrm{int}_K(P \cap K)\colon P \in \mathcal{P}\},$$

$$(\mathcal{P}|K)^{0-} = \{\mathrm{cl}_K(\mathrm{int}_K(P \cap K))\colon P \in \mathcal{P}\}.$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be families of subsets of X, denote that

$$\mathcal{P} \wedge \mathcal{Q} = \{P \cap Q\colon P \in \mathcal{P} \text{ and } Q \in \mathcal{Q}\},$$

$\mathcal{P} < \mathcal{Q}$ if for each $P \in \mathcal{P}$ there is $Q \in \mathcal{Q}$ such that $P \in Q$.

Lemma 1.3. *A space X is a compact-covering, sequence-covering and s-image of a metric space if and only if X has a point-countable cs-network and each compact subset of X is metrizable.*

Proof. The "only if" part is clear, so we only need to prove the "if" part. By Theorem 1.1, there are a metric space M and a sequence-covering and s-map $f: M \to X$. We use the same notations as in the proof of Theorem 1.1, and show that f is a compact-covering map. Let K be compact in X, then K is metrizable and $(\mathcal{P}|K)^0$ is a countable base for the subspace K by Lemma 1.2. Put

$$\mathcal{H} = \{P \in \mathcal{P}\colon \mathrm{int}_K(P \cap K) \neq \emptyset\},$$

then $\mathcal{H}$ is countable. Let

$$\Big\{\mathcal{H}' \subset \mathcal{H}\colon \mathcal{H}' \text{ is finite and } \bigcup(\mathcal{H}'|K)^0 = K\Big\} = \{\mathcal{H}_k\colon k \in \mathbb{N}\},$$

then for each $n, m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $\mathcal{H}_k < \mathcal{H}_n \wedge \mathcal{H}_m$. We assert that for each $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $(\mathcal{H}_j|K)^{0-} < (\mathcal{H}_i|K)^0$. In fact, for each $x \in K$, there are $H \in \mathcal{H}_i$, an open set G in K and $Q \in \mathcal{H}$ such that $x \in \mathrm{int}_K(Q \cap K) \subset G \subset \mathrm{cl}_K(G) \subset \mathrm{int}_K(H \cap K)$, thus $\mathrm{cl}_K(\mathrm{int}_K(Q \cap K)) \subset \mathrm{int}_K(H \cap K)$. By the compactness of K, we have a $j \in \mathbb{N}$ with $(\mathcal{H}_j|K)^{0-} < (\mathcal{H}_i|K)^0$. Take a subsequence $\{\mathcal{L}_i\}$ of $\{\mathcal{H}_k\}$ satisfying that $\mathcal{L}_i < \mathcal{H}_i$ and $(\mathcal{L}_{i+1}|K)^{0-} < (\mathcal{L}_i|K)^0$ for each $i \in \mathbb{N}$, then there is a finite $B_i \subset A_i$ with $\mathcal{L}_i = \{P_\alpha\colon \alpha \in B_i\}$. Put

$$L = \left\{\beta = (\alpha_i) \in \prod_{i\in\mathbb{N}} B_i\colon\ \emptyset \neq \mathrm{cl}_K(\mathrm{int}_K(P_{\alpha_{i+1}} \cap K)) \subset \mathrm{int}_K(P_{\alpha_i} \cap K) \text{ for each } i \in \mathbb{N}\right\},$$

then L is closed in $\prod_{i\in\mathbb{N}} B_i$, and L is compact in $\prod_{i\in\mathbb{N}} B_i$.

For each $\beta = (\alpha_i) \in L$, take a point $x \in \bigcap_{i\in\mathbb{N}} \mathrm{int}_K(P_{\alpha_i} \cap K)$. If W is open in X with $x \in W$, then $x \in \mathrm{int}_K(P \cap K) \subset P \subset W$ for some $P \in \mathcal{P}$, and there exists a finite $\mathcal{H}' \subset \mathcal{H}$ such that

$$K \setminus \mathrm{int}_K(P \cap K) \subset \bigcup (\mathcal{H}'|K)^0 \subset \bigcup \mathcal{H}' \subset X \setminus \{x\}$$

because $K \setminus \mathrm{int}_K(P \cap K)$ is compact, thus $\mathcal{H}_i = \mathcal{H}' \cup \{P\}$ for some $i \in \mathbb{N}$, hence $x \in P_{\alpha_i} \subset P \subset W$, i.e., $\{P_{\alpha_i}\colon i \in \mathbb{N}\}$ is a network at the point x, so $\beta \in M$ and $f(\beta) = x$, therefore $L \subset M$ and $f(L) \subset K$.

On the other hand, for each $x \in K$ and each $i \in \mathbb{N}$, put

$$\mathcal{U}_i = \{U \in (\mathcal{L}_i|K)^0\colon x \in U\},$$

then $\mathcal{U}_i$ is finite and nonempty. If $V \in \mathcal{U}_{i+1}$, there exists $U \in \mathcal{U}_i$ with $\mathrm{cl}_K(V) \subset U$. By the König Lemma [7], there exists an $(\alpha_i) \in \prod_{i\in\mathbb{N}} B_i$ with $\mathrm{cl}_K(\mathrm{int}_K(P_{\alpha_{i+1}} \cap K)) \subset \mathrm{int}_K(P_{\alpha_i} \cap K) \in \mathcal{U}_i$ for each $i \in \mathbb{N}$, hence $(\alpha_i) \in L$ and $x \in \bigcap_{i\in\mathbb{N}} \mathrm{int}_K(P_{\alpha_i} \cap K)$, and $\{P_{\alpha_i}\colon i \in \mathbb{N}\}$ is a network at x in X, i.e., $f((\alpha_i)) = x$, so $f(L) \supset K$.

In a word, L is compact in M and $f(L) = K$, hence f is compact-covering. □

Theorem 1.4. *The following are equivalent for a space X.*

(1) *X is a compact-covering, sequence-covering, quotient and s-image of a metric space.*

(2) *X is a sequence-covering, quotient and s-image of a metric space.*

(3) *X is a sequential space with a point-countable cs-network.*

Proof. It suffices to prove that (3) ⇒ (1). Let X be a sequential space with a point-countable cs-network. By Lemma 0.1, X is a quotient s-image of a metric space, thus every compact subset of X is metrizable by [5, Theorem 3.3]. By Lemma 1.3, if there are a metric space M and a compact-covering, sequence-covering and s-map $f: M \to X$, then f is also quotient. □

Remark 1.5. (1) $\beta\mathbb{N}$ is a compact space with a point-countable cs-network, but it is not metrizable.

(2) The subspace $\mathbb{N} \cup \{p\}$ $(p \in \beta\mathbb{N} \setminus \mathbb{N})$ of $\beta\mathbb{N}$ has a point-countable cs-network, and each compact subset is metrizable, but it is not sequential.

(3) There is a compact-covering, quotient and s-image of a metric space such that it is not a space with a point-countable cs-network by Remark 14(2) in [8].

2. On Tanaka's question

Michael, Gruenhage and Tanaka have made an attentive study for products of k-spaces [18], and obtained some beautiful results.

Definition 2.1. Let X be a space.

(1) X is a k_ω-space [11] if X has the weak topology with respect to a covering of countable many compact subsets of X.

(2) X is an $\aleph_0$-space [10] if X has a countable cs-network.

Lemma 2.2 [18]. *Let X and Y be k and $\aleph_0$-spaces, then $X \times Y$ is a k-space if and only if one of the three properties below holds.*

(1) X and Y are first countable spaces.

(2) X or Y is a locally compact space.

(3) X and Y are local k_ω-spaces.

For convenience's sake, a pair (X, Y) of spaces X and Y is said to have Tanaka's condition, if one of the three properties in Lemma 2.2 holds.

Conjecture 2.3 (Tanaka, 1994). For the quotient s-images X, Y of metric spaces, $X \times Y$ is a k-space if and only if the pair (X, Y) has Tanaka's condition.

The main results of this section are to prove that Tanaka's conjecture holds in the spaces with point-countable cs-networks, and to construct an example to show that Tanaka's conjecture does not hold under the set-theoretic hypothesis BF(ω_2).

Theorem 2.4. *Let X and Y be sequential spaces with point-countable cs-networks, then $X \times Y$ is a k-space if and only if the pair (X, Y) has Tanaka's condition.*

Proof. If the pair (X, Y) of spaces X and Y has Tanaka's condition, then $X \times Y$ is a k-space [18]. Conversely, suppose $X \times Y$ is a k-space, by Theorem 4.2 in [16], then the following condition (C_1) or (C_2) holds.

(C_1) For each decreasing sequence $\{A_n\}$ of subsets of X, if a point $x \in \mathrm{cl}(A_n \setminus \{x\})$ for each $n \in \mathbb{N}$, then there exists a nonclosed subset $\{a_n\colon n \in \mathbb{N}\}$ of X with each $a_n \in A_n$.

(C_2) If $\{B_n\colon n \in \mathbb{N}\}$ is a decreasing network at some point in Y, then some $\mathrm{cl}(B_n)$ is countably compact.

By [12, Theorem 9.5], Lemma 0.1 and the condition (C_1), X is a countably bi-quotient s-image of a metric space, thus X has a point-countable base. By the condition (C_2), Y has a point-countable cs-network consisting of separable metrizable subspaces.

By the symmetry of spaces X and Y, to prove that the pair (X, Y) has Tanaka's condition, it suffices to discuss the following two cases.

Case 1: *X has a point-countable base and Y has a point-countable cs-network consisting of separable metrizable subspaces.* Since Y is sequential, Y has the weak topology with respect to the cs-network for Y. If Y contains no closed copy of the sequential fan S_ω and no closed copy of the Arens' space S_2, then Y is metrizable by Theorem 4.6 in [17]. If Y contains a closed copy of S_ω or S_2, then $X \times S_\omega$ is a k-space because S_ω is a perfect image of S_2, thus X is locally compact by [4, Lemma 3 and 4].

Case 2: *X and Y have point-countable cs-networks consisting of separable metrizable subspaces.* First of all, we assert that X and Y are local $\aleph_0$-spaces. Let $\mathcal{P}$ be a point-countable cs-network consisting of separable metric subspaces, and let $D(P)$ be a countable dense subset of P for each $P \in \mathcal{P}$. For each $a \in X$, put

$$\mathcal{P}_1 = \{P \in \mathcal{P}\colon\ a \in P\}, \qquad D_1 = \bigcup\{D(P)\colon\ P \in \mathcal{P}_1\},$$

and for each $n \geqslant 2$ inductively define that

$$\mathcal{P}_n = \{P \in \mathcal{P}\colon\ P \cap D_{n-1} \neq \emptyset\}, \qquad D_n = \bigcup\{D(P)\colon\ P \in \mathcal{P}_n\}.$$

Let $\mathcal{P}' = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, $U = \bigcup \mathcal{P}'$. If $\{x_n\}$ is a sequence in X converging to a point x in U and W is a neighborhood of x in X, then $x \in P$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}_m$, and there is a sequence $\{x'_n\}$ in $D(P)$ with $x'_n \to x$ in P, thus $\{x\} \cup \{x_n, x'_n\colon\ n \geqslant k\} \subset Q \subset W$ for some $k \in \mathbb{N}$ and some $Q \in \mathcal{P}$, so $Q \in \mathcal{P}_{m+1}$ and $\{x\} \cup \{x_n\colon\ n \geqslant k\} \subset Q \subset U \cap W$. This shows that U is a sequentially open subset of X and $\mathcal{P}'$ is a countable cs-network for the subspace U of X. Since X is a sequential space, U is open in X, hence X is a local $\aleph_0$-space. By the same reason, Y is also a local $\aleph_0$-space. Now, by Lemma 2.2, the pair (X, Y) has the Tanaka's condition. □

Let ${}^\omega\omega$ be the set of all functions from ω into ω. For two functions f and g in ${}^\omega\omega$ we define $f \leqslant g$ if and only if the set $\{n \in \omega\colon\ f(n) > g(n)\}$ is finite. BF(ω_2) is the following assertion.

BF(ω_2): If $F \subset {}^\omega\omega$ has cardinality less than ω_2, then there exists $g \in {}^\omega\omega$ such that $f \leqslant g$ for all $f \in F$.

It is known that CH implies that BF(ω_2) is false.

Lemma 2.5 [4]. *The following are equivalent:*

(1) BF(ω_2) *holds.*

(2) $S_\omega \times S_{\omega_1}$ *is a k-space.*

Theorem 2.6. *Under* BF(ω_2), *Tanaka's conjecture does not hold.*

Proof. Let X be the Arens' space S_2, then X is a quotient s-image of a locally compact metric space. Let I_1 be a subset of the unit interval $[0, 1]$ with cardinality ω_1. Let Z be the topological sum of $[0, 1]$ and the collection $\{S(x)\colon\ x \in I_1\}$ of ω_1 convergent sequences $S(x)$, then Z is a locally compact metric space. Let Y be the space obtained from Z by identifying the limit point of $S(x)$ with x for each $x \in I_1$, then Y is a quotient s-image of a locally compact metric space.

Under BF(ω_2), by Lemma 2.5, $S_\omega \times S_{\omega_1}$ is a k-space. Since S_ω is a perfect image of S_2, $X \times S_{\omega_1}$ is a k-space. Let H be the space obtained from Y by identifying $[0, 1]$ to a single point, then H is homeomorphic to S_{ω_1}, and a perfect image of Y, hence $X \times Y$ is a k-space. It is easy to see that the pair (X, Y) does not have Tanaka's condition. □

3. On Gruenhage–Michael–Tanaka's question

This question is whether every Fréchet space with a point-countable cs*-network is preserved by pseudo-open s-maps or perfect maps.

Theorem 3.1. *Let $f: X \in Y$ be a quotient (pseudo-open) s-map. If X is a Fréchet space with a point-countable cs-network, then Y is a sequential (Fréchet) space with a point-countable cs*-network.*

Proof. Since f is a quotient (pseudo-open) map, Y is a sequential (Fréchet) space. Let $\mathcal{P}$ be a point-countable cs-network for the space X. For each $y \in Y$, let D_y be a countable dense subset of $f^{-1}(y)$. Put

$$D = \bigcup \{D_y \colon y \in Y\} \quad \text{and} \quad \mathcal{F} = \{f(P \cap D) \colon P \in \mathcal{P}\},$$

then D is dense in X and $\mathcal{F}$ is a point-countable cover of Y. We shall show that $\mathcal{F}$ is a cs*-network for Y.

Suppose a sequence $\{y_n\}$ in Y converges to a point y, and let U be a neighborhood of y in Y. We assume that all y_n's are distinct. Put $A = \{y_n \colon n \in \mathbb{N}\} \setminus \{y\}$, then A is not closed in Y, and $f^{-1}(A)$ is not closed in X, so there exists $x \in \mathrm{cl}(f^{-1}(A)) \setminus f^{-1}(A) = \mathrm{cl}(f^{-1}(A) \cap D) \setminus f^{-1}(A)$. By the Fréchet property of X, there are a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and $x_i \in f^{-1}(y_{n_i}) \cap D$ with $x_i \to x$ in X, thus $x \in f^{-1}(y)$, and there is a sequence $\{x_i'\}$ in D_y with $x_i' \to x$ in X, hence $\{x\} \cup \{x_i, x_i' \colon i \geqslant m\} \subset P \subset f^{-1}(U)$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$, so $\{y\} \cup \{y_{n_i} \colon i \geqslant m\} \subset f(P \cap D) \subset U$, and $\mathcal{F}$ is a point-countable cs*-network for Y. □

Remark 3.2. (1) A sequential space with a point-countable cs*-network may not be preserved by pseudo-open s-maps [5, Example 9.8].

(2) A space with a point-countable cs-network may not be preserved by perfect maps [20, p. 160].

Lemma 3.3 [8]. *Let $f: X \to Y$ be a closed map, and let each point of X be a G_δ-set.*

(1) *If X has a point-countable k-network, then so has Y.*

(2) *If sequence $\{x_n\}$ in X satisfies that $\{f(x_n)\}$ converges in Y and all $f(x_n)$'s are distinct, then $\{x_n\}$ has a convergent subsequence in X.*

Theorem 3.4. *Let $f: X \to Y$ be a closed map, and let each point of X be a G_δ-set. If X has a point-countable cs-network, then f is compact-covering.*

Proof. Since each point of X is a G_δ-set, each compact subset of X is first countable. By Lemma 1.2, every point-countable cs-network of X is a k-network. By Lemma 3.3,

Y has a point-countable k-network, so each compact subset of Y is metrizable. Let K be compact in Y, then K has a countable dense subset D. For each $y \in D$, take a point $x_y \in f^{-1}(y)$. Put $E = \{x_y\colon\ y \in D\}$, then each sequence in E has a convergent subsequence in X by Lemma 3.3. If a point $x \in \mathrm{cl}(E)$, take a sequence $\{G_n\}$ of open subsets of X such that $\mathrm{cl}(G_{n+1}) \subset G_n$ and $\{x\} = \bigcap_{n\in\mathbb{N}} G_n$, then there is a point $x_n \in E \cap G_n$ for each $n \in \mathbb{N}$. Since each subsequence of $\{x_n\}$ has a cluster point in X and x is the unique cluster point of $\{x_n\}$, sequence $\{x_n\}$ converges to x.

Now, let $\mathcal{P}$ be a point-countable cs-network for X, and put

$$\mathcal{P}' = \{P \cap \mathrm{cl}(E)\colon\ P \in \mathcal{P},\ P \cap E \neq \emptyset\},$$

then $\mathcal{P}'$ is a countable network for the subspace $\mathrm{cl}(E)$. In fact, for each $x \in \mathrm{cl}(E)$, let U be open in X with $x \in U$, then there exists a sequence $\{x_n\}$ in E such that $x_n \to x$ in X, thus $\{x\} \cup \{x_n\colon\ n \geqslant m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$, therefore $P \cap E \neq \emptyset$ and $x \in P \cap \mathrm{cl}(E) \subset U \cap \mathrm{cl}(E)$, and $\mathcal{P}'$ is a countable network for $\mathrm{cl}(E)$. This shows that $\mathrm{cl}(E)$ is paracompact. Since $f(\mathrm{cl}(E)) = K$, $f|_{\mathrm{cl}(E)}\colon \mathrm{cl}(E) \to K$ is a closed map, and it is a compact-covering map, hence there is a compact subset L of $\mathrm{cl}(E)$ with $f(L) = K$, and f is compact-covering. □

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(1) The theorem remains valid if cs-network is replaced by k-network (or cs*-network). The only change in the proof is another definition of the family $\mathcal{P}'$:

$$\mathcal{P}' = \{\mathrm{cl}(P) \cap \mathrm{cl}(E)\colon\ P \in \mathcal{P},\ P \cap E \neq \emptyset\}.$$

By the same method as in the proof of theorem one can show that such $\mathcal{P}'$ is a countable network for $\mathrm{cl}(E)$.

(2) Let ψ^* be the well known Mrowka's space. Then taking $f\colon \psi^* \to S$ where f maps all the nonisolated points of ψ^* into a single point one obtains a closed mapping of a regular first countable space onto a convergent sequence which is not compact-covering. This demonstrates that the point-countable cs-network is essential.

(3) (A well known Frolik's construction.) Let $\mathcal{D} = \{D\colon\ D \subset \omega \text{ is infinite}\}$. For any $D \in \mathcal{D}$ choose $x_D \in \beta\omega \setminus \omega$ such that $xD \in \mathrm{cl}(D)$. Then take $X = \omega \cup \{x_D\colon\ D \in \mathcal{D}\} \subset \beta\omega$. Finally let $f\colon X \to S$ map all the nonisolated points of X into a single point. Then f is a closed noncompact-covering map of a regular space X in which every compact subspace is finite onto the convergent sequence. Thus the condition that every point is G_δ is essential.

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