Cleavability of Non-multiplicative Spaces and Arhangel'skii Problems

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Abstract Arhangel'skii posed the problem: How a class of topological spaces has the orthogonality or cleavability? The partial solution on this problem has been obtained for the class of spaces which are multiplicative by Arhangel'skii, Bella and the others. In this paper we shall prove that some classes of generalized metric spaces which are non-multiplicative also have the orthogonality or cleavability.

Keywords orthogonality; cleavability; co-perfect property; generalized metric space

In this paper all spaces are regular and T_1 topological spaces, maps are continuous and onto. Let Ψ be a class of spaces. Ψ has the orthogonality if, whenever a space X is a perfect preimage of a space in Ψ, and a one-to-one preimage of a space in Ψ, then $X\in\Psi$. Arhangel'skii [1] posed the question: How a class of spaces has the orthogonality? A partial solution was given by Arhangel'skii [2] oneself as follows.

Theorem A A class of spaces which are closed hereditary and finitely multiplicative has the orthogonality.

Let Ψ be a class of spaces. A space X is cleavable (perfectly cleavable) over Ψ if, whenever A is a subspace of X, there are a space Y in Ψ and a map (perfect map) $f: X\to Y$ such that $f^{-1}(f(A))=A$. Ψ has the strong orthogonality if, whenever a space X is a perfect peimage of a space in Ψ, and cleavable over Ψ, then $X\in\Psi$. Ψ is perfectly cleavable if, whenever a space X is perfectly cleavable over Ψ, then $X\in\Psi$. Recently, Arhangel'skii [3] posed the question: How a class of spaces has the strong orthogonality or perfect cleavability? A partial solution was given by Bella, Cammaroto and Kočinac in [5] as follows.

Theorem B Let Ψ be a class of spaces which are closed hereditary and countably multiplicative. If one of the two properties below holds, then Ψ has the strong orthogonality.

(1) $\psi(X)\leqslant\omega$ for each $X\in\Psi$.

(2) $|K|\leqslant c$ for each compact subspace K of X in Ψ.

Theorem A and Theorem B are related to the finitely or countably multiplicative property. In this paper we discuss the strong orthogonality, perfect cleavability or orthogonality of the classes of some non-multiplicative generalized metric spaces as follows:

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Ⅰ: g-metrizable spaces, σMK-spaces, MOBI, quotient s-images of metric spaces, quotient compact images of metric spaces.

Ⅱ: Lašnev spaces, closed s-images of metric spaces, pseudo-open s-images of metric spaces.

First, recall some definitions.

Definition 1 Let Ψ be a class of spaces, and let P be a topological property. P is a co-perfect property on Ψ if, whenever a space X is a perfect preimage of a space in Ψ, then X has P iff $X \in \Psi$.

Definition 2 Let X be a space, and let Θ be a family of subsets of X. Θ is a k-network for X if, whenever $K \subset U$ with Kcompact and U open in X, then $K \subset \bigcup \Theta' \subset U$ for some finite $\Theta' \subset \Theta$.

Lemma 1 Let P be a co-perfect property on a class Ψ. If P has the strong orthogonality, then so has Ψ.

Lemma 2 Spacess with a point-countable k-network have the strong orthogonality.

Proof Obviously, spaces with a point-countable k-network are closed hereditary and countably multiplicative. Let X be a space with a point-countable k-network. If K is a compact subspace of X, K is metrizable by Theorem 3.3 in [7], thus $|K| \leqslant c$. By Theorem B, spaces with a point-countable k-network have the strong orthogonality.

Lemma 3([6]) Let $f: X \to Y$, $g: M \to Y$. Define $Z = \{(x, \alpha) \in X \times M : f(x) = g(\alpha)\}$, $h_1 = p_1|z$, and $h_2 = p_2|z$, then Z is a closed subspace of $X \times M$ and

(1) if f is a perfect map, then so is h_2;

(2) if g is an s-map, then so is h_1;

(3) if g is a compact map, then so is h_1;

(4) if g is an open map, then so is h_1;

(5) if g is a quotient map and X, M are sequential spaces, then h_1 is a quotient map [9].

Lemma 4 The property with a point-countable k-network is a co-perfect property on the class of spaces in Ⅰ.

Proof Let $f: X \to Y$ be a perfect map, where X has a point-countable k-network.

(1) If Y is a g-metrizable space, i.e., Y has a σ-locally finite weak base, thus Y is a strong $\sum$-space, so is X. Since a strong $\sum$-space with a point-countable k-network is a σ-space [7, Corollary 3.8]. X is a σ-space, and X has a G_δ-diagonal, hence X is a g-metrizable space [9, Theorem 2.3].

(2) If Y is a σMK-space, i.e., Y has a countable closed cover $\{Y_i : i \in N\}$ of metrizable subspaces such that if K is compact in X, then $K \subset Y_i$ for some $i \in N$, then Y is a strong $\sum$-space. By (1), X has a G_δ-diagonal. Thus each $f^{-1}(Y_i)$ is a closed paracompact M-subspace with a G_δ-diagonal, and it is a closed metrizable subspace of X. It is easy to check that each compact subset of X is contained in some $f^{-1}(Y_i)$. Hence X is also a σMK-

space.

(3) If $Y \in$ class MOBI, i. e. , there are a metric space M and a finite family $\{g_1, g_2, \cdots, g_n\}$ of open compact maps such that $g_n \circ g_{n-1} \circ \cdots \circ g_1(M) = Y$. Thus Y has a point-countable base. Now we inductively prove that $X \in$ class MOBI for each $n \in N$. For $n = 1$, by Lemma 3, there are a space Z with a point-countable k-network and an open compact map $h_1: Z \to X$, a perfect map $h_2: Z \to M$. By (1), Z has a G_δ-diagonal, hence Z is a metric space, and $X \in$ class MOBI. Suppose the proposition holds for $n-1 \in N$. let $Y_1 = g_{n-1} \circ g_{n-2} \circ \cdots \circ g_1(M)$, then $g_n: Y_1 \to Y$ is an open compact map. By Lemma 3, there are a subspace Z_1 of $X \times Y_1$ and an open compact map $h_1: Z_1 \to X$, a perfect map $h_2: Z_1 \to Y_1$. Since Z_1 has a point-countable k-network, $Z_1 \in$ class MOBI by the inductive hypothesis. And since h_1 is an open compact map, $X \in$ class MOBI.

(4) If Y is a quotient s (quotient compact) image of a metric space, then Y is a k-space, so is X. By Corollary 3. 4 in [7], a k-space with a point-countable k-network is a sequential space, thus X is a sequential space. By Lemma 3, there are a space Z with a point-countable k-network and a quotient s (quotient compact) map $h_1: Z \to X$, a perfect map $h_2: Z \to M$. Thus Z is a metric space, and X is a quotient s(quotient compact) image of a metric space.

On the other hand, g-metrizable spaces, σMK-spaces have σ-locally finite k-networks; spaces in class MOBI have point-countable bases; and quotient s(quotient compact) images of metric spaces have point-countable k-networks [7]. Hence each space in I has a point-countable k-network.

In a word, the property with a point-countable k-network is a co-perfect property on the class of spacess in I.

By Lemma 1, Lemma 2 and Lemma 4, we obtain a cleavability theorem which partly answers Arhangel'skill's question.

Theorem 1 The classes of spaces in I have the strong orthogonality.

Theorem 2 The classes of spaces in II have not the orthogonality.

Proof Let S_ω denote the sequential fan, i. e. , the quotient space obtained from the topological sum of ω convergent sequences by identifying all the limit points. The S_ω is a closed s-image of a metric space. Let I be the unit interval, $f: S_\omega \times I \to S_\omega$ be the projective map, then f is a perfect map. Since $S_\omega \times I$ is a paracompact space with a G_δ-diagonal, it is submetrizable, thus it is a one-to-one preimage of a metric space. Since S_ω is not a strong Fréchet space, $S_\omega \times I$ is not a Fréchet space. However, each space in II is a Fréchet space, hence the classes of spaces in II have not the orthogonality.

Theorem 3 The classes of spaces in II have the perfect cleavability.

Proof We knew that

(1) A space is a Lašnev space iff it is a Fréchet space with a σ-compact-finite k-network [10].

(2) A space is a closed s-image of a metric space iff it is a Fréchet space with a σ-local-

ly finite k-network [8].

(3) A space is a pseudo-open s-image of a metric space iff it is a Fréchet space which is a quotient s-image of a metric space [1].

By Theorem B or Theorem 1, the classes of space with a σ-compact-finitte k-network, spaces with a σ-locally finite k-network, and spaces which are quotient s-images of metric spaces have the strong orthogonality. And Fréchet property has the perfect cleavability [4], hence the classes of spaces in II have the perfect cleavability.

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