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A NOTE ON THE MICHAEL-NAGAMI PROBLEM

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In this paper it is shown that a Hausdorff space Y is a strongly compact-covering quotient *s*-image of a metric space if and only if Y is a sequential space with a point-countable *cs*-network, which is a partial answer to the Michael-Nagami problem.

Keywords Point-countable covers, Compact-covering maps, Sequence-covering maps, s-maps, Sequential spaces, cs-networks.

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In 1973, Michael and Nagami[12] posed a problem: If a space X is a quotient s-image of a metric space, must X also be a compact-covering quotient s-image of a metric space? It interests many research workers in general topology as a classic open problem [1,3,8,11,14,]. Some related results are

Theorem A[12]. The following are equivlent for a T_2 space Y:

(1) Y is an open s-image of a metric space;

(2) Y is a compact-covering open s-image of a metric space;

(3) Y has a point-countable base.

Theorem B[3,14]. The following are equivalent for a T_2 space Y:

(1) Y is a quotient s-image of a metric space;

(2) Y is a sequence-covering quotient s-image of a metric space;

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(3) Y is a sequential space with a point-countable cs^*-network.

These show that a study of certain point-countable covers for a space will be a key to answer the Michael-Nagami problem. In this paper we discuss spaces with a point-countable cs-network, and prove that a sequential space with a point-countable cs-network is a strongly compact-covering quotient s-image of a metric space, which not only deepens Theorem A, but also gives a new way to solve the Michael-Nagami problem.

In this paper all spaces are T_2, maps are continuous and onto, N denotes the set of all natural numbers and $\omega = \{0\} \cup N$. We recall some basic definitions.

Definition 1. Let $f : X \to Y$ be a map.

(1) f is an s-map if each $f^{-1}(y)$ has a countable base in X.

(2) f is a compact-covering map [10] if each compact subset of Y is the image of some compact subset of X.

(3) f is a sequence-covering map [3] if each convergent sequence of Y is the image of some compact subset of X.

(4) f is a strongly sequence-covering map [13] if each convergent sequence of Y is the image of convergent sequence of X.

(5) f is a strongly compact-covering map if f is a strongly sequence-covering map and a compact-covering map.

Obviously,

$$\text{strongly compact-covering map} \Rightarrow \left\{\begin{array}{l}\text{compact-covering map}\\ \text{strongly sequence-}\\ \text{covering map}\end{array}\right\} \Rightarrow \text{sequence-covering map}.$$

Definition 2. Let X be a space, and let $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a network if, whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is a cs-network [4] if, whenever $\{x_n\}$ is a sequence converging to a point x in X and U is a neighborhood of x in X, then $\{x\} \cup \{x_n : n \geq m\} \subset P \subset U$ for some $m \in N$ and some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a cs^*-network [2] if, whenever $\{x_n\}$ is a sequence converging to a point x in X and U is a neighborhood of x in X, then $\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

Theorem 1. A space Y is a strongly sequence-covering s-image of a metric space if and only if Y has a point-countable cs-network.

Proof. Suppose a space Y is a strongly sequence-covering s-image of a metric space, then there exist a metric space X and a strongly sequence-covering s-map $f; X \to Y$. Let $\mathcal{B}$ be a σ-locally finite base for X. Then $\{f(B) : B \in \mathcal{B}\}$ is a point-countable cs-network for the space Y.

Conversely, suppose Y has a point-countable cs-network $\mathcal{P}$. We can assume that $\mathcal{P}$ is closed under finite intersections. Denote $\mathcal{P}$ by $\{P_\alpha : \alpha \in A\}$. Let A_i denote the set A with discrete topology for each $i \in N$. Put

$$X = \left\{\beta = (\alpha_i) \in \prod_{i \in N} A_i : \{p_{\alpha_i} : i \in N\} \text{ is a network at some point } y(\beta) \text{ in } Y\right\}.$$

Then X is a metric space, and $f : X \to Y$ defined by $f(\beta) = y(\beta)$ is a function. It is easy to check that f is an s-map from X onto Y[7]. We shall show that f is a strongly sequence-covering map. For a sequence $\{y_n\}$ converging to a point y_0 in Y, we assume that all $y_m's$ are distinct. Let $K = \{y_m : m \in \omega\}$, and let $K \subset U$ with Uopen in y. A subset $\mathcal{F}$ of $\mathcal{P}$ is said to have the property $F(K, U)$ if $\mathcal{F}$ satisfies that

(1) $\mathcal{F}$ if finite;

(2) $\emptyset \neq P \cap K \subset P \subset U$ for each $P \in \mathcal{F}$;

(3) there is the only $P_z \in \mathcal{F}$ with $z \in P_z$ for each $z \in K$;

(4) if $y_0 \in P \in \mathcal{F}$, then $K \backslash P$ is finite.

Put

$$\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has the property } F(K, X)\} = \{\mathcal{F}_i : i \in N\}.$$

For each $i \in N$ and $m \in \omega$ there is $\alpha_{im} \in A_i$ with $y_m \in P_{\alpha_{im}} \in \mathcal{F}_i$. Let

$$x_m = (\alpha_{im}) \in \prod_{i \in N} A_i.$$

It can be checked that $\{P_{\alpha_{im}} : i \in N\}$ is a network at y_m in Y, thus $x_m \in X$ and $f(x_m) = y_m$ for each $m \in \omega$. For each $i \in N$, there is $n(i) \in N$ so that $\alpha_{in} = \alpha_{io}$ for all $n \geq n(i)$, thus the sequence $\{\alpha_{in}\}$ converges to α_{io} in A_i, and the sequence $\{x_n\}$ converges to x_0 in X. This shows that f is a strongly sequence-covering map, and completes the proof of the Theorem.

Now, we give a characterization of the strongly compact-covering s-image of a metric space.

Lemma. Let $\mathcal{P}$ be a point-countable cs-network for a space Y. If $y \in K \cap W$ with W open and K compact, first countable in Y, then $y \in \text{int}_K(P \cap K) \subset P \subset W$ for some $P \in \mathcal{P}$.

Proof. Let $\{V_n : n \in N\}$ be a local base at the point y in K. Put

$$\mathcal{F} = \{P \cap K : P \in \mathcal{P}, \text{ and } P \subset W \text{ or } P \subset Y \backslash \{y\}\},$$
$$\mathcal{F}' = \{F \in \mathcal{F} : V_n \subset F \text{ for some } n \in N\}.$$

Then $\mathcal{F}$ is a point-countable cs-network for the subspace K, and $\mathcal{F}'$ is a neighborhood base at y in K by the proof of Lemma 7(3) in [9], thus $y \in \text{int}_K(F) \subset K \cap W$ for some $F \in \mathcal{F}$, i.e., $y \in \text{int}_K(P \cap K) \subset P \subset W$ for some $P \in \mathcal{P}$.

Let $\mathcal{P}$ be a family of subsets of a space Y, and let $K \subset Y$. Denote that

$$(\mathcal{P}|K)^0 = \{\text{int}_K(P \cap K) : P \in \mathcal{P}\},$$
$$(\mathcal{P}|K)^{0-} = \{\text{cl}_k(\text{int}_K(P \cap K)) : P \in \mathcal{P}\}.$$

Theorem 2. A space Y is a strongly compact-covering s-image of a metric space if and only if Y has a point-countable cs-network and each compact subset of Y is metrizable.

Proof. The "if" part is clear, so we only need to prove the "only if " part. By Theorem 1, there exist a metric space X and a strongly sequence-covering s-map $f : X \to Y$. We still use the same notations in the proof of Theorem 1, and show that f is a compact-covering map. Let K be compact in Y, then K is metrizable and $(\mathcal{P}|K)^0$ is a countable base for the subspace K by Lemma. Put

$$\mathcal{H} = \{P \in \mathcal{P} : \text{int}_K(P \cap K) \neq \phi\}.$$

Then $\mathcal{H}$ is countable. Let

$$\{\mathcal{H}' \subset \mathcal{H} : \mathcal{H}' \text{ is finite and } \cup(\mathcal{H}'|K)^0 = K\} = \{\mathcal{H}_k : k \in N\}.$$

Then for each $n, m \in N$ there is $k \in N$ such that

$$\mathcal{H}_k < \mathcal{H}_n \wedge \mathcal{H}_m.$$

We assert that for each $i \in N$ there is $j \in N$ such that

$$(\mathcal{H}_j|K)^{0-} < (\mathcal{H}_i|K)^0.$$

In fact, for each $y \in K$, there are $H \in \mathcal{H}_i$, an open subset G in K and $Q \in \mathcal{H}$ such that

$$y \in \text{int}_K(Q \cap K) \subset G \subset \text{cl}_K(G) \subset \text{int}_K(H \cap K),$$

thus

$$\text{cl}_K(\text{int}_K(Q \cap K)) \subset \text{int}_K(H \cap K).$$

By the compactness of K, $(\mathcal{H}_j|K)^{0-} < (\mathcal{H}_i|K)^0$ for some $j \in N$. Take a subsequence $\{\mathcal{L}_i\}$ of $\{\mathcal{H}_k\}$ satisfying that

$$\mathcal{L}_i < \mathcal{H}_i \text{ and } (\mathcal{L}_{i+1}|K)^{0-} < (\mathcal{L}_i|K)^0 \text{ for each } i \in N.$$

Then there is a finite $B_i \subset A_i$ with $\mathcal{L}_i = \{P_\alpha : \alpha \in B_i\}$.

Put

$$L = \Big\{\beta = (\alpha_i) \in \prod_{i \in N} B_i : \phi \neq \text{cl}_K(\text{int}_K(P_{\alpha_{i+1}} \cap K)) \subset \text{int}_K(P_{\alpha_i} \cap K) \text{ for each } i \in N\Big\}.$$

Then L is closed in $\prod_{i \in N} B_i$, and L is compact in $\prod_{i \in N} B_i$.

For each $\beta = (\alpha_i) \in L$, take a point $y \in \bigcap_{i \in N} \text{int}_K(P_{\alpha_i} \cap K)$. If W is open in Y with $y \in W$, then, by Lemma, $y \in \text{int}_K(P \cap K) \subset P \subset W$ for some $P \in \mathcal{P}$, and

$$K \backslash \text{int}_K(P \cap K) \subset \cup(\mathcal{H}'|K)^0 \subset \cup\mathcal{H}' \subset Y \backslash \{y\}$$

for some finite $\mathcal{H}' \subset \mathcal{H}$ because $K \backslash \text{int}_K(P \cap K)$ is compact, thus $\mathcal{H}_i = \mathcal{H}' \cup \{P\}$ for some $i \in N$, hence $y \in P_{\alpha_i} \subset P \subset W$, i.e., $\{P_{\alpha_i} : i \in N\}$ is a network at the point y in Y, so $\beta \in X$ and $f(\beta) = y$, therefore $L \subset X$ and $f(L) \subset K$.

On the other hand, for each $y \in K$ and each $i \in N$, put

$$\mathcal{U}_i = \{U \in (\mathcal{L}_i|K)^0 : y \in U\}.$$

Then $\mathcal{U}_i$ is finite and non-empty. If $V \in \mathcal{U}_{i+1}$, there exists $U \in \mathcal{U}_i$ with $\text{cl}_K(V) \subset U$. By König Lemma[6, Lemma 37.4], there exists $(\alpha_i) \in \prod_{i \in N} B_i$ with

$$\text{cl}_K(\text{int}_K(P_{\alpha_{i+1}} \cap K)) \subset \text{int}_K(P_{\alpha_i} \cap K) \in \mathcal{U}_i$$

for each $i \in N$, hence $(\alpha_i) \in L$ and

$$y \in \bigcap_{i \in N} \text{int}_K(P_{\alpha_i} \cap K),$$

and $\{P_{\alpha_i} : i \in N\}$ is a network at y in Y i.e., $f((\alpha_i)) = y$, so $f(L) \supset K$.

In a word, L is compact in X and $f(L) = K$, hance f is compact-covering.

Theorem 3. The following are equivalent for a space Y:

(1) Y is a strongly sequence-covering quotient s-image of a metric space;

(2) Y is a strongly compact-covering quotient s-image of a metric space;

(3) Y is a sequential space with a point-countable cs-network.

Proof. It suffices to prove (3) $\Longrightarrow$ (2). Let Y be a sequential space with a point-countable cs-network. By Theorem B, Y is a quotient s-image of a metric space, thus every compact subset of Y is metrizable by Theorem 3.3 in [3]. By Theorem 2, there exist a metric space X and a strongly compact-covering s-map $f : X \to Y$, then f is also quotient by Lemma 45.8 in [6].

Remarks.

(1) βN is a compact space with a point-countable cs-network, but it is not metrizable.

(2) The subspace $N \cup \{p\}(p \in \beta N \backslash N)$ of βN has a point-countable cs-network, and each of its compact subsets are metrizable, but it is not sequential.

(3) There is a compact-covering quotient s-image of a metric space such that it is not a space with a point-countable cs-network by Remark 14(2) in [9].

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