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ON SPACES WITH A k-NETWORK CONSISTING OF COMPACT SUBSETS*

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Dedicated to Professor Guoshi Gao on his 75th birthday

1. Introduction

In this paper all spaces are regular and T_1. Suppose X is a topological space and $\mathcal{P}$ is a collection of subsets of X. $\mathcal{P}$ is a k-network for X whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. If $\mathcal{P}$ is a k-network for X, then $\mathcal{P}$ is a closed k-network if P is closed in X for every $P \in \mathcal{P}, \mathcal{P}$ is a compact k-network if P is compact in X for every $P \in \mathcal{P}$. We shall study spaces with a compact k-network because a study of certain CW-complex [9], the closed images of locally compact metric spaces [7], k-space properties of products of generalized metric spaces [8] relates to the concept of compact k-network. For example, Y. Tanaka in [9] discussed some characterizations of certain CW-complexes. The main tool is the following Theorem A.

Theorem A. *Let X be dominated by a cover of compact metric subsets. Suppose that X has a σ-locally finite (resp. σ-HCP) closed k-network, then X has a σ-locally finite (resp. σ-HCP) compact k-network.*

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Theorem A is a relation between closed k-networks and compact k-networks for a topological space in essence. In this paper we discuss these relations.

2. On Point-Countable Covers

Lemma 2.1. *Suppose $\mathcal{P}$ is a point-countable closed k-network for a space X which is closed under finite intersections. Put $\mathcal{F} = \{P \in \mathcal{P} : P$ is countably compact in $X\}$, then $\mathcal{F}$ is a k-network for X if and only if every first countable closed subspace of X is locally compact.*

Proof: Necessity, We can assume that X is a first countable space. For each $x \in X$, let $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\} = \{F_n : n \in N\}$. If $x \in \overline{X \setminus \cup_{i \leq n} F_i}$ for each $n \in N$, there is $x_n \in V_n \bigcap (X \setminus \cup_{i \leq n} F_i)$ where $\{V_n\}$ is a local base for x in X. Thus $x_n \to x$, and $\{x\} \bigcup \{x_n : n \in N\} \subset \bigcup \mathcal{F}'$ for some finite $\mathcal{F}' \subset \mathcal{F}$, and some $F \subset \mathcal{F}'$ contains infinitely many x_n, hence $x \in F$, a contradiction. Hence $x \in (\cup_{i \leq n} F_i)^0$ for some $n \in N$. By Corollary 3.5 in [3], each F_i is compact in X, and X is locally compact.

Sufficiency. If K is compact in X, then K is metrizable by Theorem 3.3 in [3]. By Miščenko's Lemma, a collection of minimal covers of K consisting of finite subcollections of $\mathcal{P}$ is at most countable, say $\{\mathcal{P}_n\}$. For each $n \in N$, let

$$\mathcal{A}_n = \bigwedge_{i \leq n} \mathcal{P}_i, A_n = \bigcup \mathcal{A}_n,$$

so $\mathcal{A}_n \subset \mathcal{P}$ and $K \subset A_{n+1} \subset A_n$. We assert that A_n is countably compact for some $n \in N$. If not, then A_n contains a countable discrete closecd subset D_n. Put

$$H = K \bigcup (\bigcup_{n \in N} D_n).$$

Then H is closed in X because $\{A_n\}$ is a network of K in X, and H is a first countable subspace of X, but H is not locally compact, a contradiction. Hence A_n is countable compact for some $n \in N$. Let $K \subset U$ with U open in X. There exists

$m \geq n$ such that $K \subset A_m \subset U$, i.e., a finite $\mathcal{A}_m \subset \mathcal{F}$ and $K \subset \bigcup \mathcal{A}_m \subset U$, thus $\mathcal{F}$ is a k-network for X. $\square$

Theorem 2.2. *The following conditions are equivalent for a space X:*

(1) *X has a σ-discrete compact k-network.*
(2) *X has a σ-locally finite compact k-network.*
(3) *X has a σ-discrete closed k-network, and every first countable closed subspace of X is locally compact.*
(4) *X has a σ-locally finite closed k-network, and every first countable closed subspace of X is locally compact.*

Proof: Since every countable comapct closed subspace of a space with a σ-locally finite closed k-network is compact, (1) is equivalent to (3), and (2) is equivalent to (4) by Lemma 2.1. By Theorem 4 in [1], we have that (3) is equivalent to (4). $\square$

If a space X is dominated by a cover of compact metric subspaces, then first countable closed subspace of X is locally compact by Lemma 14 in [9], thus Theorem 2.2 is a generalization of Theorem A.

Question 2.3. Suppose a space X has a point-countable closed k-network. Is X a space with a point-countable compact k-network if every first countable closed subspace of X is locally compact?

Using Theorem 2.4 in [6] and Lemma 2.1 we have a product theorem on k-spaces by the same proof of Theorem 3.1 in [8] as follows.

Theorem 2.4. *Suppose X and Y are k-spaces with a σ-locally countable k-network, then $X \times Y$ is a k-space if and only if one of the following three properties holds:*

(1) *X and Y are first countable space.*
(2) *X or Y is locally compact.*
(3) *X and Y are spaces with a σ-locally finite compact k-network.*

3. On σ-HCP Covers

Suppose $\mathcal{P}$ is a collection of subsets of a space X. $\mathcal{P}$ is hereditarily closure-preserving if $H(P) \subset P \in \mathcal{P}$ implies that $\bigcup\{\overline{H(P)} : P \in \mathcal{P}\} = \overline{\bigcup\{H(P) : P \in \mathcal{P}\}}$. A σ-hereditarily closure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use "HCP (resp. σ-HCP)" instead of "hereditarily closure-preserving (resp. σ-hereditarily closure-preserving)".

Theorem 3.1. *A space X has a σ-*HCP *compact k-network if and only if X has a σ-*HCP *closed k-network, and every first countable closed subspace of X is locally compact.*

Proof: Necessity. We can assume that X is a first countable space. Suppose $\bigcup_{n\in N}\mathcal{P}_n$ is a σ-HCP compact k-network for X, where each $\mathcal{P}_n$ is HCP in X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in N$. Put $P_n = \bigcup\mathcal{P}_n$, so that $P_n \subset P_{n+1}$. If there exists $x \in X\setminus\bigcup_{n\in N}P_n^0$, then there is $x_n \in V_n\bigcap(X\setminus P_n)$ where $\{V_n\}$ is a local base for x in X. Because $x_n \to x$, $\{x\}\bigcup\{x_n : n \in N\} \subset P_m$ for some $m \in N$, a contradiction. Thus $X = \bigcup_{n\in N}P_n^0$. To complete the proof of the necessity we need only prove that each P_n is locally compact. Let $q_n : \oplus\mathcal{P}_n \to P_n$ be the obvious mapping. Note that $\oplus\mathcal{P}_n$ is a locally compact metric space. Since $\mathcal{P}_n$ is HCP, q_n is a closed mapping. We have that $\partial q_n^{-1}(x)$ is compact for each $x \in P_n$ because P_n is first countable. We can assume that q_n is a perfect mapping. Thus P_n is locally compact, and X is locally compact.

Sufficiency. Suppose $\mathcal{P} = \bigcup_{n\in N}\mathcal{P}_n$ is a σ-HCP closed k-network for X, where each $\mathcal{P}_n$ is HCP in X, and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N$, put

$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\}$,

$\mathcal{R} = \{P\setminus D_n : P \in \mathcal{P}_n, n \in N\}\bigcup\{\{x\} : x \in D_n, n \in N\}$.

From the proof of Theorem in [5], we have the following facts:

(1) D_n is σ-discrete in X.

(2) $K \cap D_n$ is finite if K is compact in X.

(3) For a finite $\mathcal{F} \subset \mathcal{R}$, there are $m \in N, P \in \mathcal{P}_m$ and $D \subset D_m$ such that $\bigcap \mathcal{F} = (P \backslash D_m) \bigcup D$

Define

$$\mathcal{H} = \{R \in \mathcal{R} : \overline{R} \text{ is compact in } X\},$$
$$\mathcal{K} = \{\overline{H} : H \in \mathcal{H}\}.$$

By (1), $\mathcal{K}$ is a σ-HCP collection of compact subsets of X. We shall prove that $\mathcal{K}$ is a k-network for X.

For $K \subset U$ with K compact and U open in X, since $\mathcal{R}$ is a point-countable cover of K, by Miščenko's lemma there are only countable many minimal finite subfamilies of $\mathcal{R}$ covering K, say $\{\mathcal{R}_i\}$. For each $n \in N$, let $A_n = \bigcup(\bigwedge_{i \leq n} \mathcal{R}_i)$. Thus $\{\overline{A}_n\}$ is a descending sequence of closed subsets of X. If V is open in X with $K \subset V$, then $K \subset \bigcup \mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}_i$. Thus

$$K \subset (\bigcup\{\underline{P \backslash D_i} : P \in \mathcal{P}'\}) \bigcup (K \bigcap D_i)$$
$$\subset (\bigcup\{\overline{P \backslash D_i} : P \in \mathcal{P}'\}) \bigcup (K \bigcap D_i) \subset V.$$

By (2), there is $n \in N$ such that

$$\mathcal{R}_n \subset \{P \backslash D_i : P \in \mathcal{P}'\} \bigcup \{\{x\} : x \in K \bigcap D_i\},$$

so $K \subset \overline{A}_n \subset V$, and $\{\overline{A}_n\}$ is a network of K in X. We have that $\overline{A}_n$ is countably compact for some $n \in N$ by the proof of Lemma 2.1. Hence there exists $m \in N$ such that $\overline{A}_m \subset U$ and $\overline{A}_m$ is countably compact. Since X is subparacompact, $\overline{A}_m$ is compact. Since $\overline{A}_m$ is a finite union of finite intersections of elements of $\mathcal{R}$, by (3), there are a finite $\mathcal{R}' \subset \mathcal{R}$ and some $D \subset D_j$ such that $A_m = (\bigcup \mathcal{R}') \bigcup D$. Put

$$\mathcal{H}' = \mathcal{R}' \bigcup \{\{x\} : x \in K \bigcap D\}.$$

Now $\mathcal{H}'$ is a finite subfamily of $\mathcal{H}$ and $K \subset \bigcup\{\overline{H} : H \in \mathcal{H}'\} \subset U$. Therefore, $\mathcal{K}$ is a k-network for X, and X has a σ-HCP compact k-network.

Theorem 3.1 is a generalization of Theorem A.

Corollary 3.2. *The following conditions are equivalent for a space X;*

(1) *X is a closed image of a locally compact metric space.*

(2) *X is a closed image of a metric space, and every first countable closed subspace of X is locally compact.*

(3) *X is a Fréchet space with a σ-HCP compact k-network.*
(4) *X is a Fréchet space with a σ-HCP closed k-network, and every first countable closed subspace of X is locally compact.*

Proof: By Corollary 1.4 in [7]. (1) is equivalent to (2). By Theorem 3.1, (3) is equivalent to (4). by Theorem in [2], (2) is equivalent to (4). □

As for the relation between spaces with a σ-HCP compact k-network and spaces with a σ-locally finite compact k-network, we have that a space X has a σ-locally finite compact k-network if and only if X has a σ-HCP compact k-network, and X contains no closed copy of S_{ω_1} by Theorem 2.6 in [4].

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