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CARDINAL FUNCTIONS ON $C(X)$ WITH THE EPI-TOPOLOGY*

By Shou LIN

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Abstract

In this paper we establish some cardinal functions on $C(X)$ with the epi-topology. The main results are that

(1) $nw(C_e(X)) = knw(X)$;

(2) $\psi(C_e(x)) = ww(C_e(X)) = \pi w(X)$;

(3) $d(C_e(X)) = ew(X)$.

It improves some basic theorems on properties of $C(X)$ with the epi-topology.

KEY WORDS: Function space, cardinal function, epi-topology.
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Convergence in the set $L(X)$ of lower semicontinuous extended real-value functions on space X with the epi-topology is a basic tool in minimization problems [1]. The subspace $C(X)$ of $L(X)$ with the epi-topology, where $C(X)$ is the set of all continuous real-valued functions on X, has many important properties. McCoy and Ntantu [3] studied the separation axioms, countability properties, completeness properties and metrizability of $C(X)$ with the epi-topology. In this paper we establish cardinal functions on $C(X)$ with the epi-topology, which show some basic relations between topological properties on X and $C(X)$ and improve some basic theorems on properties of $C(X)$ with epi-topology.

Throughout the paper all spaces are T_1 and X denotes a Tychonoff space. Also R and Q denote the spaces of real numbers and rational numbers, respectively, all having the usual topology. For each $A \subset X$ and $r \in R$, define

$$[A,t]^- = \{f \in C(X) : f(x) < t \text{ for some } x \in A\}$$

and

$$[A,t]^+ = \{f \in C(X) : f(x) > t \text{ for all } x \in A\}.$$

The epi-topology on $C(X)$ is the topology generated by the

$$\{[U,t]^- : U \text{ is open in } X \text{ and } t \in R\} \cup$$
$$\{[K,t]^+ : K \text{ is compact in } X \text{ and } t \in R\}$$

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as subbase. We use $C_e(X)$ to denote this space with the epi-topology, $C_k(X)$ to denote this space with the compact-open topology. We knew that $C_e(X)$ is always T_1 and the epi-topology is weaker than the compact-open topology on $C(X)$[3].

For a space X, $w(X)$, $nw(X)$, $knw(X)$, $\pi w(X)$, $\psi(X)$ and $d(X)$ denote the weight, netweight, k-netweight, π-weight, pseudo-character and density of X, respectively. The weak weight of X is defined to be $ww(X) = \omega + \min\{w(Y)$: there is a continuous bijection from X onto $Y\}$. Also define the epi-weight of X by $ew(X) = \omega + \min\{w(Y)$: there is a continuous epi-mapping from X onto $Y\}$ where a continuous function $f : X \to Y$ is called an epi-mapping provided that for each open set U and each compact set K in X such that $f(U) \subset f(X)$ then $U \subset K$. We refer the reader to [2] for concepts and terminologies undefined.

THEOREM 1. $nw(C_e(X)) = knw(X)$.

PROOF. By Theorem 4.1.2 in [2] we have that $nw(C_k(X)) = knw(X)$. Since the identity mapping $i : C_k(X) \to C_e(X)$ is continuous, and since the property of networks of a space is preserved by continuous mappings, then $nw(C_e(X)) \leq knw(X)$. On the other hand, if $\mathcal{P}$ is a network for $C_e(X)$ with $|\mathcal{P}| = nw(C_e(X))$, for each $P \in \mathcal{P}$, define $P^* = \{x \in X : P \subset [\{x\}, 0]^+\}$. It suffices to show that $\{P^* : P \in \mathcal{P}\}$ is a k-network for X. To this end, let $K \subset U$ with K compact and U open in X. Then there exists a function $f \in C_e(X)$ such that $f(K) = \{1\}$ and $f(X \backslash U) = \{0\}$. Let $W = [K, 0]^+$, then $f \in W$ and $f \in P \subset W$ for some $P \in \mathcal{P}$. If $x \in K$, and $g \in P$, then $g(x) > 0$, thus $P \subset [\{x\}, 0]^+$, and $x \in P^*$. Therefore $K \subset P^*$. If $x \in P^*$, then $f(x) > 0$, thus $x \in U$, and $P^* \subset U$. So $K \subset P^* \subset U$. This is that $\{P^* : P \in \mathcal{P}\}$ is a k-network for X, and $knw(X) \leq nw(C_e(X))$. Hence we prove that $nw(C_e(X)) = knw(X)$.

THEOREM 2. $ww(C_e(X)) = \psi(C_e(X)) = \pi w(X)$.

PROOF. Obviously, $\psi(C_e(X)) \leq ww(C_e(X))$. It suffices to show that $ww(C_e(X)) \leq \pi w(X) \leq \psi(C_e(X))$.

First, we prove that $ww(C_e(X)) \leq \pi w(X)$. Suppose $\mathcal{B}$ is a π-base for X with $|\mathcal{B}| = \pi w(X)$. For each $B \in \mathcal{B}$, take $d_B \in B$. Put $D = \{d_B : B \in \mathcal{B}\}$, then D is a dense subset of X. Let $C_\tau(X)$ denote $C(X)$ endowed with the topology generated by the family

$$\{[B, s]^- : B \in \mathcal{B} \text{ and } s \in Q\} \cup \{[\{x\}, t]^+ : x \in D \text{ and } t \in Q\}$$

as subbase. It is easy to check that $C_\tau(X)$ is a T_1-space, and that the identity mapping $i : C_e(X) \to C_\tau(X)$ is continuous. Hence we have that $ww(C_e(X)) \leq w(C_\tau(X)) \leq \pi w(X)$.

Secondly, we prove that $\pi w(X) \leq \psi(C_e(X))$. Suppose that f_0 is the zero-function on X. If $\{f_0\} = \cap\{B_\alpha : \alpha \in \Lambda\}$, where $|\Lambda| = \psi(C_e(X))$ and each B_α is open in $C_e(X)$. We can assume that for each $\alpha \in \Lambda$, $B_\alpha = [U_\alpha, s_\alpha]^- \cap [K_\alpha, t_\alpha]^+$, where U_α is open in X, K_α is compact in X and s_α, $t_\alpha \in R$. To show that $\{U_\alpha : \alpha \in \Lambda\}$ is a π-base for X, let U be any nonempty open subset of X such that each $U_\alpha \not\subset U$. Take $x_0 \in U$ and a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(X \backslash U) = \{0\}$. For each $\alpha \in \Lambda$, we have that $s_\alpha > 0$ and $t_\alpha < 0$ because $f_0 \in B_\alpha$, then $f \in B_\alpha$, and $f \in \cap\{B_\alpha : \alpha \in \Lambda\} = \{f_0\}$, which is a contradiction. Hence $\{U_\alpha : \alpha \in \Lambda\}$ is a π-base for X, and we have that $\pi w(X) \leq \psi(C_e(X))$.

THEOREM 3. $d(C_e(X)) = ew(X)$

PROOF. Suppose that $f : X \to Y$ is an epi-mapping, where $w(Y) = ew(X)$. By Theorem 4.2.1 in [2], $d(C_k(Y)) = ww(Y) \leq w(Y)$. Define the induced function $f^* : C(Y) \to C(X)$ by $f^*(g) = g \circ f$ for every $g \in C(Y)$. By the continuity of f, $f^* : C_k(Y) \to C_k(X)$ is continuous (Theorem 2.2.7 in [2]), and $f^* : C_k(Y) \to C_e(X)$ is continuous. Since f is an epi-mapping, $f^*(C(Y))$ is dense in $C_e(X)$ (Theorem 4.2 in [3]), and $d(C_e(X)) \leq d(C_k(Y)) \leq w(Y) = ew(X)$. On the other hand, suppose that D is dense in $C_e(X)$ with $|D| = d(C_e(X))$, define $F : X \to R^D$ by $\pi_g \circ F(x) = g(x)$ for all $x \in X$ and $g \in D$, where $\pi g : B^D \to R$ is the projection onto the g-th factor. Obviously, F is continuous, and $w(F(X)) \leq w(R^D) = |D| = d(C_e(X))$. Now to show that F is an epi-mapping, let K be compact in X and let U be open in X with $x_0 \in U \backslash K$. Put $B = [U, 0]^- \cap [K, 0]^+$, then there is a continuous function $f : X \to [-1, 1]$ such that $f(x_0) = -1$ and $f(K) = \{1\}$, thus $f \in B$, and B is an nonempty open subset of $C_e(X)$. So we have some $g \in B \cap D$. Then since $g \in [U, 0]^-$, take $x_1 \in U$ such that $\pi_g \circ F(x_1) = g(x_1) < 0$. But also since $g \in [K, 0]^+$, for each $x \in K$, $\pi_g \circ F(x) = g(x) > 0$, and thus $F(x_1) \neq F(x)$. Hence $F(x_1) \in F(U) \backslash F(K)$. It follows that F is an epi-mapping, and $ew(X) \leq w(F(X)) \leq d(C_e(X))$.

PROBLEM 4. Is it true that $knw(C_e(X)) = knw(X)$?

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Department of Mathematics
Ningde Teachers' College
Ningde, Fujian 352100
P. R. China