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Point-countable k-networks, closed maps, and related results

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Abstract

We give some conditions for closed images of spaces with a point-countable k-network to have a point-countable k-network, and their applications.

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1. Introduction

Let X be a space, and let $\mathscr{P}$ be a cover (not necessarily open or closed) of X. We recall that $\mathscr{P}$ is a *k-network* [16], if whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$. If we replace "compact" by "single point" then such a cover is called a "network". A *closed* (respectively *compact*) k-network is a k-network consisting of closed subsets (respectively compact subsets). k-networks have played a role in $\aleph_0$-spaces [13] (i.e., spaces with a countable k-network), $\aleph$-spaces [16] (i.e., spaces with a σ-locally finite k-network).

As a modification of k-networks, we recall that a cover $\mathscr{P}$ of X is a *cs-network* (i.e., convergent sequence network) [9], if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a nbd of x, then for some $P \in \mathscr{P}$ and some $n \in N$, $\{x\} \cup \{x_m: m \geqslant n\} \subset P \subset U$. Also, we recall that a cover $\mathscr{P}$ is a *cs*-network* [7], if we replace "$\{x_m: m \geqslant n\}$" by "some subsequence of $\{x_n\}$)". We shall call a cover $\mathscr{P}$ a *wcs*-network*, if we replace "$\{x\} \cup \{x_m: m \geqslant n\}$" by "some subsequence of $\{x_n\}$".

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In [20], cs*-networks, or wcs*-networks were called covers satisfying (C_1), or (C_2) respectively. We recall that a cover is *point-countable* if every point is in at most countably many elements of it. A point-countable cover is a wcs*-network if and only if it is a wcs-network [15] (= Fcs-network [6]). Spaces with a point-countable k-network play an important role in the theory of generalized metric spaces, certain quotient spaces, and their metrizability; see [8,11,20,21], for example.

Now, spaces with a point-countable cs-network, cs*-network, or closed k-network are not necessarily preserved by closed maps (even if the domains are locally compact metric). But, spaces with a point-countable k-network are preserved by perfect maps [8], and closed images of $\aleph$-spaces have a point-countable k-network [20]. Then the following question arises:

Question. Does every closed image of a space with a point-countable k-network have a point-countable k-network?

In this paper, we show that this question is affirmative when the domain is a k-space, a paracompact space, or a space in which every point is a G_δ-set. Also, we give some applications of these partial answers.

We assume that spaces are regular T_1, and maps are continuous and onto.

2. Results

First let us give some lemmas. The following lemma is due to [20].

Lemma 1. *Let $\mathscr{P}$ be a point-countable cover of X. Then $\mathscr{P}$ is a k-network if and only if it is a wcs*-network, and every compact subset of X is sequentially compact.*

Let X be a space, and let $\mathscr{C}$ be a cover of X. We recall that *X is determined by $\mathscr{C}$* [8] (or X has the weak topology with respect to $\mathscr{C}$) if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathscr{C}$.

We recall that a space X is a *sequential space* (respectively *k-space*) if X is determined by the cover of all compact metric (respectively compact) subsets. A space is *Fréchet* if whenever $x \in \overline{A}$, then there exists a sequence in A converging to the point x. Any Fréchet space is sequential, and any sequential space is a k-space. It is well known that every sequential space (respectively k-space) is precisely the quotient image of a metric space (respectively locally compact space); see, [3, 2.4.G], etc.

Lemma 2. *Let $f: X \to Y$ be a closed map. Let K be a countably compact subset of Y, and let $S = \{x_n: n \in N\}$ be a sequence in $f^{-1}(K)$ such that $f(x_m) \neq f(x_n)$ if $m \neq n$. If* (a) *or* (b) *below holds, then there exists a convergent subsequence of S.*

(a) *X is a sequential space,*

(b) *each point of X is a G_δ-set.*

Proof. Any subsequence of $\{f(x_n)\colon n \in N\}$ has an accumulation point in K. Thus, since f is closed, it follows that any subsequence of S has also an accumulation point in $f^{-1}(K)$. If (a) holds, since S is assumed to be not closed in X, there exists a convergent subsequence of S. If (b) holds, let x be an accumulation point of S, and let $\{V_n\colon n \in N\}$ be a sequence of open nbds of x such that $V_n \supset \overline{V}_{n+1}$. Let $T = \{t_k\colon k \in N\}$ be a subsequence of S such that $t_k \in V_k$. Let P be any subsequence of T. Then P has an accumulation point p. Then $p \in \bigcap \overline{V}_n$, hence $p = x$. This implies that the point x is a limit point of T. Hence T is a convergent subsequence of S. □

Lemma 3. *Let $f: X \to Y$ be a closed map. Let $\{y_n\colon n \in N\}$ be a sequence converging to $y \in Y$ with $y_n \neq y$. Let $\{x_n\colon n \in N\}$ be a sequence with $x_n \in f^{-1}(y_n)$. If $Bf^{-1}(y)$ (boundary of $f^{-1}(y)$) is compact, then so is $C = \{x_n\colon n \in N\} \cup Bf^{-1}(y)$.*

Proof. Let $\mathscr{G}$ be an open covering of C. Since $Bf^{-1}(y)$ is compact, there exists a finite $\mathscr{G}' \subset \mathscr{G}$ such that $Bf^{-1}(y) \subset \bigcup \mathscr{G}'$. Hence $f^{-1}(y) \subset V = \bigcup \mathscr{G}' \cup \operatorname{int} f^{-1}(y)$. Since f is closed, there exists an open nbd W of y such that $f^{-1}(W) \subset V$. But, any $x_n \notin \operatorname{int} f^{-1}(y)$, and there exists $n \in N$ such that $f^{-1}(W)$ contains x_m for $m \geqslant n$. Hence, $\{x_m\colon m \geqslant n\} \cup Bf^{-1}(y) \subset \bigcup \mathscr{G}'$. This implies that C is a compact subset of X. □

We recall that a space is *isocompact* if every closed countably compact subset is compact. Also, we recall that a map $f: X \to Y$ is *compact covering* if each compact subset of Y is the image of some compact subset of X.

Lemma 4. *Let $f: X \to Y$ be a closed map. If* (a) *or* (b) *below holds, then f is compact covering.*

(a) *X is normal, isocompact,*
(b) *each $Bf^{-1}(y)$ is Lindelöf.*

Proof. If (a) holds, by the same way as in [12, Corollary 1.2], we see that f is compact covering (indeed, every closed countably compact subset of Y is the closed image of a compact subset of X). If (b) holds, since f is closed, we can assume that every $f^{-1}(y)$ is Lindelöf. Let K be a compact subset of Y. Then it is easy to show that $f^{-1}(K)$ is Lindelöf, hence is normal and isocompact. Since $f \mid f^{-1}(K)$ is closed, it is compact covering by (a), hence so is f. □

Now, we give partial answers to the question in the Introduction.

Theorem 5. *Let $f: X \to Y$ be a closed map such that X has a point-countable k-network. If one of the following properties holds, then Y has a point-countable k-network.*

(a) *X is a k-space,*
(b) *each point of X is a G_δ-set,*

(c) *X is a normal, isocompact space,*
(d) *each* $Bf^{-1}(y)$ *is Lindelöf.*

Proof. Let (a) or (b) hold. We note that every compact subset of X is metric in view of [2, Theorem 3.1]. Thus if (a) holds, then X is sequential. Thus (a) or (b) implies that each compact subset of Y is sequential compact by Lemma 2. Let (c) or (d) hold. Then each compact subset of Y is also sequentially compact by Lemma 4. Hence, in view of Lemma 1, it suffices to show that Y has a point-countable wcs*-network. To show this, let $\mathscr{P}$ be a point-countable k-network for X. For each $y \in Y$, choose $x_y \in f^{-1}(y)$, and let $A = \bigcup\{x_y\colon y \in Y\}$. Let $\mathscr{P}^* = \{f(A \cap P)\colon P \in \mathscr{P}\}$. Then $\mathscr{P}^*$ is a point-countable cover of Y. To show $\mathscr{P}^*$ is a wcs*-network, let $S = \{y_n\colon n \in N\}$ be a sequence converging to a point $y \in Y$, and U be a nbd of y. Choose $x_n \in f^{-1}(y_n) \cap A$ for each $n \in N$. If (a) or (b) holds, by Lemma 2 there exists a convergent subsequence T of $\{x_n\colon n \in N\}$ in $f^{-1}(U)$. Thus there exists $P \in \mathscr{P}$ such that P contains a subsequence of T and $P \subset f^{-1}(U)$. Hence, $f(A \cap P) \subset U$ contains a subsequence of S. This shows that $\mathscr{P}^*$ is a wcs*-network. For (c) and (d), we note that $g = f \mid f^{-1}(S \cup \{y\})$ is a closed map, and $S \cup \{y\}$ is compact. Hence, if (c) or (d) holds, then $Bg^{-1}(y)$ is compact as in the proof of [12, Theorem 1.1]. Thus $C = \{x_n\colon n \in N\} \cup Bg^{-1}(y)$ is a compact subset of X by Lemma 3. Then C is metric, so there exists a convergent subsequence of $\{x_n\colon n \in N\}$ in X. Then, as is seen above, $\mathscr{P}^*$ is a wcs*-network. Consequently, $\mathscr{P}^*$ is a point-countable k-network. □

Remark 6. (1) Every closed, and Lindelöf image (i.e., every point-inverse is Lindelöf) of a metric space (more generally, ℵ-space) is an ℵ-space by [7], hence it has a point-countable cs-, cs*-, and closed k-network by [5,6]. But,

(2) Every closed image of a locally compact metric space doesn't have any point-countable cs-, cs*-, nor closed k-network.

Indeed, let S_{ω_1} be the quotient space obtained from the topological sum of ω_1 convergent sequences by identifying all the limit points to a single point. Then S_{ω_1} is the closed image of a locally compact metric space. But, S_{ω_1} doesn't have any point-countable cs-, cs*-, nor closed k-network by [20, Lemma 2.4].

Next, let us consider some applications of Theorem 5. First, we give some definitions.

Let X be a space. For each $x \in X$, let T_x be a collection of subsets of X such that any element of T_x contains x. Following Arhangel'skii [1], the collection $T_C = \bigcup\{T_x\colon x \in X\}$ is a *weak base* for X if (a) and (b) below are satisfied. We call each element of T_x a *weak nbd* of x.

(a) For each $A, B \in T_x$, there exists $C \in T_x$ such that $C \subset A \cap B$,
(b) a subset U of X is open in X if and only if for each $x \in U$, there exists $A \in T_x$ such that $A \subset U$.

A space X is *g-first countable* [17] ($=X$ is weakly first countable; or X satisfies the weak first axiom of countability, or briefly, the gf-axiom of countability [1]), if X has a weak base T_C such that each T_x is countable.

Any first countable space or any symmetric space is g-first countable. Every g-first countable space X is sequential, and if X is moreover Fréchet, then X is first countable; see [1].

Lemma 7. (1) *Every weak base for X is a cs-network.*

(2) *Every point-countable weak base for X is a k-network.*

(3) *Let X be g-first countable. Then X has a point-countable cs-network if and only if it has a point-countable weak base.*

Proof. To prove (1), it suffices to show that for $x \in X$ and any sequence $\{x_n: n \in N\}$ converging to x with $x_n \neq x$, any weak nbd B of x contains all but finitely many x_n. Indeed, suppose not. Then there exists a subsequence S of $\{x_n: n \in N\}$ such that $S \cap B = \emptyset$. Thus, since $S \cup \{x\}$ is closed in X, S is closed in X, a contradiction. For (2), since X is g-first countable, for each $x \in X$, let $\{Q_n(x): n \in N\}$ be a weak nbd of x in X. To show that every compact subset C of X is sequentially compact, let S be an infinite sequence in C wich is not closed in X. Then there exists $x \in S$ with $Q_n(x) \cap S \neq \emptyset$. This suggests that S has a subsequence converging to the point x. Hence every compact subset of X is sequentially compact. Thus (2) holds by (1) and Lemma 1. For (3), the "if" part holds by (1), so we prove the "only if" part. Let $\mathscr{P}$ be a point-countable cs-network for X which is closed under finite intersections, and for each $x \in X$, let $\{Q_n(x): n \in N\}$ be a weak nbd of x in X. Let $\mathscr{P}_x = \{P \in \mathscr{P}: Q_n(x) \subset P$ for some $n \in N\}$. Then $\mathscr{P}_x$ is a weak nbd of x in X. To show this, let G be an open subset of X. Then there exists $P \in \mathscr{P}_x$ with $P \subset G$. Otherwise, let $\{P \in \mathscr{P}: x \in P \subset G\} = \{P_m(x): m \in N\}$. Then $Q_n(x) \not\subset P_m(x)$ for each $n, m \in N$, so choose $x_{nm} \in Q_n(x) - P_m(x)$. For $n \geqslant m$, let $x_{nm} = y_k$, where $k = m + n(n-1)/2$. Then the sequence $\{y_k: k \in N\}$ converges to the point x. Thus, there exists $m, i \in N$ such that $\{y_k: k \geqslant i\} \cup \{x\} \subset P_m(x) \subset G$. Take $j \geqslant i$ with $y_j = x_{nm}$ for some $n \geqslant m$. Then $x_{nm} \in P_m(x)$. This is a contradiction. If $G \subset X$ satisfies that for each $x \in G$ there exists $P \in \mathscr{P}_x$ with $P \subset G$, then G is open in X. Hence $T_C = \bigcup \mathscr{P}_x$ is a point-countable weak base for X. □

From Theorem 5 and Lemma 7(2), the following holds.

Corollary 8. *Let $f: X \to Y$ be a closed map such that X has a point-countable weak base. Then Y has a point-countable k-network.*

As a characterization of countably bi-quotient images of paracompact M-spaces (respectively M-spaces), Michael [14] introduced the notion of countably bi-k-spaces (respectively countably bi-quasi-k-spaces). For these definitions, see Section 4 in

[14]. Any first countable space or any locally compact space is a countably bi-k-space. The following lemma is due to [8].

Lemma 9. *Every countably bi-k-space with a point-countable k-network has a point-countable base.*

From Corollary 8 and Lemma 9, the following holds.

Corollary 10. *Let $f: X \to Y$ be an open and closed map. If X has a point-countable base, then so does Y.*

We note that every countably compact space with a point-countable k-network (and cs-network) is not metrizable; see [8, Example 9.1]. But Proposition 11 below holds. We recall that a space is an M-space if and only if it is the inverse image of a metric space by a quasi-perfect map. A space is *weakly sequential* [19] if it is determined by the cover of all sequential compact subsets. Every sequential space is weakly sequential.

Proposition 11. *Let X be an M-space with a point-countable, wcs*-network (respectively k-network) $\mathscr{P}$. Then X is metrizable if and only if X is a weakly sequential space (respectively k-space).*

Proof. The "only if" part is clear, so we prove the "if" part. Since X is an M-space, to show that X is metrizable, by [8, Corollary 4.2], it suffices to prove that for a countably compact subset K of X, if $K \subset U = X - \{x\}$, then there exists a finite $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \bigcup \mathscr{P}' \subset U$.

We show that the countably compact set K of X is sequentially compact. Let S be an infinite sequence in K. We can assume that S is not closed in X. Since X is weakly sequential, $S \cap C$ is not closed in C for some sequentially compact subset C of X. Thus there exists a subsequence T of S in C converging to a point $p \in C$. But T has an accumulation point $q \in K$, so $p = q$, hence $p \in K$. This shows that K is sequentially compact. Now, let $\mathscr{P}_U = \{P \in \mathscr{P}\colon P \subset U\}$. Since $\mathscr{P}$ is a point-countable wcs*-network for X, and K is sequentially compact, K is contained in a finite union of elements of $\mathscr{P}_U$ in view of the proof of [20, Proposition 1.2(1)].

For the parenthetic part, since X is a k-space, the open subset U of X is a k-space. But, since $\mathscr{P}$ is a k-network for X, every compact subset of U is contained in a finite union of elements of $\mathscr{P}_U$. Thus U is determined by the collection of all finite unions of $\mathscr{P}_U$. Hence the countably compact set K is also contained in a finite union of elements of $\mathscr{P}_U$ by [8, Proposition 2.1]. □

Theorem 12. *Let $f: X \to Y$ be a closed map such that X has a point-countable k-network, and let Y be an M-space (respectively countably bi-quasi-k-space). If property* (a), (b) *or* (c) *in Theorem* 5 *holds, then Y is metrizable (respectively Y has a point-countable base).*

Proof. In view of the proof of Theorem 5, we see that every closed countably compact subset of Y is sequentially compact. But, since Y is an M-space, Y is determined by a cover of closed countably compact subsets in view of [18, Lemma 1.3]. Then X is weaky sequential. Since Y has a point-countable k-network (hence wcs*-network) by Theorem 5, Y is metrizable by Proposition 11.

For the parenthetic part, to see that any countably compact subset of Y is compact, let K be a countably compact subset of Y. Let (a) hold. Then X is sequential, because every compact subset of X is metrizable by Proposition 11 (or [2, Theorem 3.1]). Thus Y is sequential by [3, 2.4.G]. Then the countably compact set K is closed in Y. Next, let (c) hold. Since Y is normal, $\overline{K}$ is countably compact by [18, Lemma 1.2]. Thus, if (a) or (c) holds, then (a) or (c) holds with respect to a closed subset $f^{-1}(\overline{K})$ of X (for (a), $\overline{K}=K$), and $f \mid f^{-1}(\overline{K})$ is a closed map with $\overline{K}$ countably compact. Thus $\overline{K}$ is metrizable by the first paragraph, hence K is compact. If (b) holds, since (b) holds with respect to a subset $f^{-1}(K)$ of X, K is also metrizable, hence K is compact. Thus property (a), (b), or (c) implies that every countably compact subset of X is compact. Then the countably bi-quasi-k-space Y is bi-k in view of [14, Definition 1.2]. While, Y has a point-countable k-network by Theorem 5. Thus Y has a point-countable base by Lemma 9. □

Corollary 13. *Let $f: X \to Y$ be a closed map such that X has a point-countable weak base. If Y is an M-space, then Y is metrizable.*

Finally, let us consider the quotient s-images of certain metric spaces, and the preservation of spaces with point-countable cs-, cs*-, closed k-networks, or weak bases under quotient finite-to-one maps or perfect maps.

Remark 14. (1) Every quotient s-image of a metric (respectively locally compact metric) space has a point-countable k-network (respectively compact k-network), and every Fréchet space which is the quotient s-image of a locally separable metric space has a point-countable cs-, cs*-, and closed k-network; see [8]. But,

(2) Every quotient finite-to-one image of a locally compact metric space doen't have a point-countable cs-network, nor a point-countable weak base.

Indeed, let X be the topological sum of a collection $\{I, S_\alpha: \alpha \in I\}$, where I is the closed unit interval, and each S_α is a convergent sequence. Let Y be the space obtained from X by identifying the limit point of S_α with $\alpha \in I$ for each $\alpha \in I$. Let $f: X \to Y$ be the obvious map. Then Y is the quotient finite-to-one image of a locally compact metric space X under f, and Y is a paracompact space with a point-countable compact k-network. To show that Y has no point-countable cs-networks, suppose that Y has a point-countable cs-network. Since Y is g-first countable, Y has a point-countable weak base T_C by Lemma 7(3). We note that the subspace I of Y has a countable base $\mathscr{B}$, and for any $y \in I$ and $T \in T_y \subset T_C$, $y \in B \subset T$ for some $B \in \mathscr{B}$. Then it follows that Y has a σ-discrete weak base, which is a cs-network by Lemma 7(1). Thus Y is an $\aleph$-space by [10, Theorem 5]. Let

Z be the space obtained from Y by identifying all points of I to a single point. Then Z is the perfect image of Y, hence Z is an $\aleph$-space by Remark 6(1). But, Z contains a closed subspace which is homeomorphic to S_{ω_1}. This is a contradiction to Remark 6(2).

Remark 15. (1) Spaces with a point-countable k-network are preserved under perfect maps by [8]. But,

(2) Spaces with a point-countable, closed k-network (compact k-network, cs*-network, or weak base) are not necessarily preserved under perfect maps by the proof in Remark 14, and the fact that S_{ω_1} has no point-countable weak bases.

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