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On *ss*-Mappings *)

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Abstract In this paper we prove the following two results: (1) A T_2 topological space has a locally countable network if and only if it is an *ss*-image of a metric space; (2) A T_3 topological space has a locally countable weak base if and only if it is a quotient, compact (or quotient, π) *ss*-image of a metric space.

Key Words and Phrases *ss*-mapping; π-mapping; Compact Mapping; Weak Base; Network

§ 1. Introduction

A central question of Alexandroff's idea is that by means of various mappings the relationships between various topological spaces and metric spaces are established. We know that a topological space has a locally countable k-network if and only if it is a compact-covering *ss*-image of a metric space (see [1]). It initially shows an efficiency by *ss*-mappings to realize Alexandroff's idea. The purpose of this paper is to further discuss the images of metric spaces by *ss*-mappings. In section 2, we establish a relation between spaces with a locally countable network and metric spaces, which improves a classical result of Michael in [2] on continuous images of separable metric spaces. In section 3, we establish a relation between spaces with a locally countable weak base and metric spaces by *ss*-mappings, an important corollary of which is that a space with a countable weak base if and only if it is a quotient, compact (or quotient, π) image of a separable metric space. It affirmatively answers an open problem posed by Arhangel'skii in [3].

In this paper all spaces are T_2, and all mappings are continuous and onto.

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§ 2. Spaces with a Locally Countable Network

A mapping $f: X \to Y$ is an *ss*-mapping (see [1]), if for each $y \in Y$, there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a separable subspace of X.

Theorem 2.1 A space X has a locally countable network if and only if X is an image of a metric space under an *ss*-mapping.

Proof Suppose X is an image of a metric space under an *ss*-mapping. Then there exist a metric space M and an *ss*-mapping f from M onto X. Since M is a metric space, by the Nagata-Smirnov metrization theorem, M has a σ-locally finite base. Let $\mathscr{B}$ be a σ-locally finite base for M, put $\mathscr{P} = \{f(B): B \in \mathscr{B}\}$; then $\mathscr{P}$ is a network for X because a mapping preserves a network. Since f is an *ss*-mapping, it is easy to check that $\mathscr{P}$ is a locally countable family of X. Hence X has a locally countable network.

Conversely, suppose X has a locally countable network $\mathscr{P} = \{P_\alpha: \alpha \in A\}$. For each $n \in N$, let A_n be a copy of A, and it is endowed with discrete topology. Put

$$M = \{\alpha = (\alpha_n) \in \prod_{n \in N} A_n: \{P_{\alpha_n}: n \in N\} \text{ is a network of some point } x_\alpha \text{ in } X\},$$

and give M the subspace topology induced from the usual product topology of the product space $\prod_{n \in N} A_n$. x_α is unique in M because X is T_2. We define $f: M \to X$ by $f(\alpha) = x_\alpha$. Obviously, M is a metric space. We will show that f is an *ss*-mapping.

For each $x \in X$, $\mathscr{P}_x = \{P \in \mathscr{P}: x \in P\}$ is a countable network of x in X because $\mathscr{P}$ is a point-countable network for X. Suppose $\mathscr{P}_x = \{P_{\alpha_n}: n \in N\}$. Let $\alpha = (\alpha_n)$; then $\alpha \in M$ and $f(\alpha) = x$. Thus f is onto. For each $\alpha = (\alpha_n) \in M$, one has $f(\alpha) = x_\alpha$. If U is an open neighborhood of x_α in X, then there exists $n \in N$ with $x_\alpha \in P_{\alpha_n} \subset U$, for $\{P_{\alpha_n}: n \in N\}$ is a network of x_α in X. Put

$$W = \{\beta \in M: \text{the } n\text{-th coordinate of } \beta \text{ is } \alpha_n\};$$

then W is an open neighborhood of α in M, and $f(W) \subset P_{\alpha_n} \subset U$. Hence f is continuous. For each $x \in X$, by the locally countable property of $\mathscr{P}$, there exists an open neighborhood V of x in X such that $\{\alpha \in A: V \cap P_\alpha \neq \phi\}$ is countable. Put

$$L = \left(\prod_{n \in N} \{\alpha \in A_n: V \cap P_\alpha \neq \phi\}\right) \cap M;$$

then L is a hereditarily separable subspace of M and $f^{-1}(V) \subset L$. Thus $f^{-1}(V)$ is a separable subspace of M. Hence f is an *ss*-mapping.

By Theorem 2.1, we can obtain the following classical result due to Michael in [2] (cf. the proof of Corollary 1 in [1]).

Corollary 2.1 A space X has a countable network if and only if X is an image of a separable metric space under a continuous mapping.

§ 3. Spaces with a Locally Countable Weak Base

In this section, all spaces are regular ones and T_1. A family $\mathscr{P}$ of subsets of a space X is a weak base for X (see [3]), if for each $x\in X$, there exists $\mathscr{P}_x\subset\mathscr{P}$ satisfying:

(1) $\mathscr{P}=\bigcup\{\mathscr{P}_x: x\in X\}$;

(2) $x\in\bigcap\mathscr{P}_x$;

(3) if U, $V\in\mathscr{P}_x$, then $W\subset U\cap V$ for some $W\in\mathscr{P}_x$;

(4) a subset G of X is open in X if and only if for each $x\in G$, there exists $P\in\mathscr{P}_x$ with $P\subset G$.

A space X is g-first countable (gf-countable, for short) if X has a weak base $\mathscr{P}=\bigcup\{\mathscr{P}_x: x\in X\}$ such that each $\mathscr{P}_x$ is countable. A mapping f from a metric space (X, d) onto a space Y is a π-mapping (see [3]), if $d(f^{-1}(y), X\setminus f^{-1}(U))>0$ for each $y\in Y$ and each neighborhood U of y in Y.

Lemma 3. 1 The following statements are equivalent for a space X:

(1) X is a gf-countable space with a locally countable k-network;

(2) X is a topological sum of spaces with a countable weak base;

(3) X has a locally countable weak base.

Proof (1)⇒(2). Since a gf-countable space is a k-space (see [4]), by Theorem 1 in [5], X is a topological sum of $\aleph_0$-spaces. Since an $\aleph_0$-space satisfying a gf-countable axiom has a countable weak base (see [4]), X is a topological sum of spaces with a countable weak base.

(2)⇒(3). It is obvious.

(3)⇒(1). Suppose X is a space with a locally countable weak base. It is obvious that X is a gf-countable space, and each point of x is a G_δ-set in X. By Lemma 2. 1 in [6], a locally countable weak base of X is a locally countable k-network of X.

Lemma 3. 2 Suppose f is a quotient mapping from a sequence space X onto a space Y and $\mathscr{P}$ is a k-network for X. Put $\mathscr{B}=\{f(P): P\in\mathscr{P}\}$. If $\mathscr{B}$ is point-countable in Y, then $\mathscr{B}$ is a k-network of Y.

Proof By Propositions 1. 5 and 1. 6 in [7].

Lemma 3. 3 Suppose Y is a quotient ss-image of a sequence space X with a locally countable k-network, then Y has a locally countable k-network.

Proof Suppose $f: X\to Y$ is a quotient ss-mapping. Let $\mathscr{P}$ be a locally countable k-network for X. Put

$$\mathscr{B}=\{f(P): P\in\mathscr{P}\}.$$

By Lemma 3. 2, $\mathscr{B}$ is a locally countable k-network for Y (Since f is an ss-mapping, $\mathscr{B}$ is locally countable).

Theorem 3. 1 The following statements are equivalent for a space X:

(1) X is a quotient, compact, *ss*-image of a metric space;

(2) X is a quotient, π, *ss*-image of a metric space;

(3) X has a locally countable weak base.

Proof (1)⇒(2). It is obvious.

(2)⇒(3). Since a quotient, π-image of a metric space is gf-countable (see [4]), by Lemmas 3. 1 and 3. 3, X has a locally countable weak base.

(3)⇒(1). By Theorem 2 in [8], we know that a space with a countable weak base is a quotient, compact image of a separable metric space. Suppose a space X has a locally countable weak base. By Lemma 3. 1, $X=\bigoplus_{\alpha\in A} X_\alpha$, where each X_α has a countable weak base. For each $\alpha\in A$, there exist a separable metric space M_α and a quotient, compact mapping f_α: $M_\alpha\to X_\alpha$. Define f: $\bigoplus_{\alpha\in A} M_\alpha\to\bigoplus_{\alpha\in A} X_\alpha$ such that each $f|_{M_\alpha}=f_\alpha$; then f is a quotient, compact, *ss*-mapping from the metric space $\bigoplus_{\alpha\in A} M_\alpha$ onto X. Hence X is a quotient compact, *ss*-image of a metric space.

Corollary 3. 1 The following statements are equivalent for a space X:

(1) X is a quotient, compact image of a separable metric space;

(2) X is a quotient, π-image of a separable metric space;

(3) X has a countable weak base.

Remark Corollary 3. 1 affirmatively answers the following open problem posed by Arhangel'skii in [3]: Find an internal characterization of a quotient, compact image of a separable metric space.

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