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A NEW CHARACTERIZATION OF DEVELOPABLE SPACES*

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Abstract In this paper we give a new characterization of developable spaces. As an application, we show that a first countable closed image of a developable space is developable.

Key Words developable space, closed mapping.

Class. : 54E30, 54C10

The class of developable spaces, as a generalizaion of metric spces, has many important properties. Its mapping theorems have been provoking many mathematicians' interest. For example, Burke[1] proved that developability is preserved by perfect mappings. In this paper, we give a new characterization of developable spaces. As its an application, we show that a first countable closed image or an open and closed image of a developable space is developable.

In this paper, all spaces are T_1 and all mappings are continuous and onto. A space X is developable, if there exists a sequence $\{\mathscr{U}_n\}$ of open covers of X such that for each $x \in X$, $\{st(x, \mathscr{U}_n): n \in N\}$ is a base of neighborhoods of x in X. In order to state the characterization of developable spaces it will be necessary to define the idea of a pair-network and develop some companion notation.

A collection $\mathscr{P} = \{(Q_\alpha, R_\alpha): \alpha \in \Lambda\}$ of pairs of subsets of a space X is called a pair-network for X if whenever $x \in W$, with W open in X, there is some $P = (Q_\alpha, R_\alpha) \in \mathscr{P}$ such that $x \in Q_\alpha \subset R_\alpha \subset W$. If $\mathscr{P}$ is a pair-network for X and $P \in \mathscr{P}$ we let P' denote the first element in the pair P and P'' denote the second element. If $\mathscr{R} \subset \mathscr{P}$, let $\mathscr{R}' = \{P' : P \in \mathscr{R}\}$ and $\mathscr{R}'' = \{P'' : P \in \mathscr{R}\}$. If $x \in X$ and $\mathscr{R} \subset \mathscr{P}$ let

$$st(x, \mathscr{R}) = \bigcup \{P'' : P \in \mathscr{R}, x \in P'\},$$

and if $A \subset X$, then $st(A, \mathscr{R}) = \bigcup \{P'' : P \in \mathscr{R}, A \cap P' \neq \varnothing\}$.

Burke[1] obtained the following theorem.

Theorem 1 The following properties of a space Y are equivalent:

(a) Y is developable.

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(b) Y has a pair-network $\mathscr{P} = \bigcup_{n\in N} \mathscr{P}_n$ satisfying:

(1) Each $\mathscr{P}'_n$ is a locally finite collection of closed sets and $\mathscr{P}''_n$ is a collection of open sets.

(2) Whenever $C \subset U \subset Y$ where C is compact and U is open, there is some $n \in N$ such that $C \subset st(C, \mathscr{P}_n) \subset U$.

(c) Y has a pari-network $\mathscr{R} = \bigcup_{n\in N} \mathscr{R}_n$ satisfying:

(1) Each $\mathscr{R}'_n$ is a locally finite collection of closed sets.

(2) Whenever $y \in U \subset Y$ with U open, there is some $n \in N$ such that $y \in st^{\circ}(y, \mathscr{R}_n) \subset U$.

Let $\mathscr{H}$ be a collection of subsets of a space X, $\mathscr{H}$ is hereditarily closure-preserving, if whenever $P(H) \subset H \in \mathscr{H}$, then $\overline{\bigcup \{P(H) : H \in \mathscr{H}\}} = \bigcup \{\overline{P(H)} : H \in \mathscr{H}\}$.

Lemma 2 Suppose Y is a first countable space. If $\mathscr{H}$ is a hereditarily closure-preserving collection of closed subsets of Y, then

$$\{\overline{H \setminus D(\mathscr{H})} : H \in \mathscr{H}\} \cup \{\{y\} : y \in D(\mathscr{H})\}$$

is locally finite in Y, where

$$D(\mathscr{H}) = \{y \in Y : \mathscr{H} \text{ is not point-finite at } y\}.$$

Proof We first prove that $D(\mathscr{H})$ is discrete in Y. Otherwise, there exists a subset A of $D(\mathscr{H})$ such that A is not closed in Y. Take a $y \in cl(A) \setminus A$. Since Y is first countable, let $\{U_n : n \in N\}$ be a countable base of neighborhoods of y in Y, choose $y_n \in U_n \cap A$ such that all y_n' s are distinct, then sequence $\{y_n\}$ converges point y. Put $B = \{y_n : n \in N\}$. Then B is not closed in Y. On the other hand, for each $n \in N, y_n \in D(\mathscr{H})$, there exists a subcollection $\{H_n : n \in N\}$ of $\mathscr{H}$ with $y_n \in H_n$, then B is closed in Y becausse $\mathscr{H}$ is hereditarily closure-preserving, a contradiction. Hence $D(\mathscr{H})$ is discrete in Y.

Secondly, we prove that $\{\overline{H \setminus D(\mathscr{H})} : H \in \mathscr{H}\}$ is locally finite in Y. Otherwise, $\{\overline{H \setminus D(\mathscr{H})} : H \in \mathscr{H}\}$ is not point-finite because it is closure-preserving. There are a point $y \in Y$ and a countable subcollection $\{H_n : n \in N\}$ of $\mathscr{H}$ such that $y \in \bigcap_{n\in N} \overline{H_n \setminus D(\mathscr{H})}$, thus $y \in \bigcap_{n\in N} H_n$, so $y \in D(\mathscr{H})$ and $y \in \bigcap_{n\in N} \overline{H_n \setminus \{y\}}$. Suppose $\{V_n : n \in N\}$ is a countable base of neighborhoods of y in Y, then $V_n \cap (H_n \setminus \{y\})$ is infinite for each $n \in N$. Choose $y_n \in V_n \cap (H_n \setminus \{y\})$ such that all y_n' s are distinct. Then sequence $\{y_n\}$ converges point y, and $\{y_n : n \in N\}$ is discrete in Y, a contratiction. This completes the proof of Lemma.

Theorem 3 A space Y is developable if and only if Y has a pair-network $\mathscr{P} = \bigcup_{n\in N} \mathscr{P}_n$ satisfying:

(1) Each $\mathscr{P}'_n$ is hereditarily closure-preserving collection of closed sets of Y.

(2) Whenever $y \in U \subset Y$ with U open, there is some $n \in N$ such that $y \in st^{0}(y, \mathscr{P}_n) \subset U$.

Proof It is sufficient to prove sufficiency. Suppose Y has a pair-network $\mathscr{P} = \bigcup_{n\in N} \mathscr{P}_n$ satisfying the above condition (1) and (2). By (2), Y is first countable. For each $n \in N$, let

$$\mathscr{R}_n = \{(\overline{P' \setminus D_n}, P'') : P \in \mathscr{P}_n\} \cup \{(\{y\}, st(y, \mathscr{P}_n)) : y \in D_n\},$$

where $D_n = \{y \in Y : \mathscr{P}'_n$ is not point-finite at $y\}$. By Lemma 2, $\mathscr{R}'_n$ is a locally finite collection of closed subsets of Y, and for each $y \in Y, st(y, \mathscr{R}_n) = st(y, \mathscr{P}_n)$. Hence $\bigcup_{n\in N} \mathscr{R}_n$ is a pair-network of Y, and it satisfies the condition (c) in Theorem 1. Therefore Y is a developable space.

Theorem 4 A first countable closed image of a developable space is developable.

Proof Suppose $f: X \to Y$ is a closed mapping, where X is a developable space, and Y is a first countable space. Then there exists a σ-closed discrete subspace Z of Y such that $f^{-1}(y)$ is compact in X for each $y \in Y \setminus Z$ by a Worrell's theorem [3](cf. Corollary 1.1 in[2]). Denote Z by $\bigcup_{n \in N} Z_n$, where each Z_n is discrete in Y. For each $y \in Z$, let $\{U(y,n): n \in N\}$ be a countable base of neighborhoods of y in Y. Suppose $\bigcup_{n \in N} \mathscr{P}_n$ is a pair-network of developable space X, which satisfies the condition (b) in Theorem 1. For each $n, j \in N$, put

$$\mathscr{R}_n = \{(f(P'), f(P'')): P \in \mathscr{P}_n\},$$

$$\mathscr{H}_{n,j} = \{(\{y\}, U(y,j)): y \in Z_n\},$$

then $(\bigcup_{n \in N} \mathscr{R}_n) \cup (\bigcup_{n,j \in N} \mathscr{H}_{n,j})$ is a pair-network of Y satisfying the condition (1) in Theorem 3. Whenever $y \in U$ with U open in Y, if $y \in Z$, then there are $n, j \in N$ such that $y \in Z_n$ and $U(y,j) \subset U$, thus $y \in st^0(y, \mathscr{H}_{n,j}) \subset st(y, \mathscr{H}_{n,j}) = U(y,j) \subset U$. If $y \in Y \setminus Z$, then $f^{-1}(y)$ is compact in X, hence $f^{-1}(y) \subset st(f^{-1}(y), \mathscr{P}_n) \subset f^{-1}(U)$ for some $n \in N$, so $y \in st^0(y, \mathscr{R}_n) \subset st(y, \mathscr{R}_n) \subset U$. By Theorem 3, Y is a developable space.

Corollary 5 An open and closed image of a developable space is developable.

References

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可展空间的一个新刻划

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摘要　本文给出可展空间的一个新刻划。作为它的应用，我们证明可展空间的第一可数闭映象是可展空间。

关键词　可展空间，闭映射。

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环与半群的自内射性

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摘要 本文首先刻划了环与其直和分解项之间自内射性的联系。然后研究了半群环的自内射性。给出了半群环自内射性的一个必要条件；并且对某些逆半群和主因子均 c−非奇异的半群，刻划了其半群环的自内射性。最后，对半群环的 Dedekind 有限性进行了一些讨论。

关键词 环，半群环，自内射性，逆半群，Dedekind 有限性

§1 环的自内射性

本文中没有解释的概念和符号，可参见[1]和[2]。

环 R 称为**左自内射**的，如果 R 作为左 R−模是内射模。本文中左自内射环一律**简称自内射环**。若环 R 既是左自射的 α 是右自内射的，则称为**双边自内射环**。

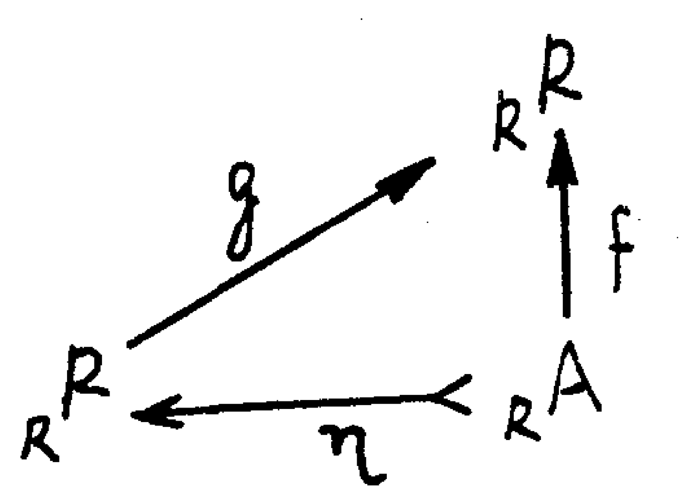
下述定理 1.1(Letami)及定理 1.3 在对半群环自内射性的研究中将是重要的。

定理 1.1[3] 对任何环 R 和正整数 n，R 是自内射环当且仅当全矩阵环 $M_n(R)$ 是自内射环。

引理 1.2 对于环 R 的含幺理想 I(I 的幺元未必是 R 的幺元)，若 I 的左理想均为 R 的左理想，那么当 R 是自内射环时，I 亦是自内射环。

证 任取 I 的左理想 A，则 A 是 R 的左理想。令 f 是任意 I−模同态：${}_IA\xrightarrow{f}{}_II$。当然，$f$ 亦是 A 到 R 的一个 I−模同态，即 $f\in \mathrm{Hom}_I(A,R)$。令 l_I 是 I 的幺元。对任何 $r\in R, a\in A, ra=r(l_I\cdot a)=(r\cdot l_I)\cdot a$，其中 $rl_I\in I$。因此 $f(ra)=f((r\cdot l_I)\cdot a)=(r\cdot l_I)f(a)=r\cdot(l_I\cdot f(a))=rf(l_I\cdot a)=rf(a)$，从而 $f\in \mathrm{Hom}_R(A,R)$。

由于 R 是自内射环，故对 f，存在 $g\in \mathrm{Hom}_R(R,R)$ 使下图交换：



其中 η 是嵌入，左下标 R 表示此图是作为左 R−模同态的交换图。由此，易见下图交换：