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The Sequence-covering s-images of Metric Spaces*)

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Abstract It is shown that a Hausdorff space is the sequence-covering s-image of a metric space if and only if it has a point-countable cs^*-network.

Key Words and Phrases Sequence-covering Mapping; s-mapping; cs^*-network

In this paper, all spaces are assumed to be Hausdorff topological spaces and all mappings are continuous and surjective.

At Prague symposium on topology in 1961, Alexandroff conceived that by means of various mappings the relationships between various classes of topological spaces can be established. Specially, one may ask: Can a particular topological property be expressed as an image of a metric space under some mapping? As is well known the space with a point-countable base can be described as an image of a metric space under an open s-mapping. The quotient s-mappings are a generalization of open s-mappings. In 1966, Arhangel'skii[1] posed the following question: what internal characterization do the quotient s-images of metric spaces have?

Hoshina[2] and Gruenhage, Michael, and Tanaka[3] have given some internal characteriza tions of the quotient s-images of metric spaces respectively. However, their characterizations are complicated in statement, and it is difficult to use their characterizations to further study the property of spaces which can be expressed as quotient s-images of metric spaces. Recently, Tanaka[4] has given a simple internal characterization for the quotient s-images of metric spaces in terms of the concept of cs^*-networks, i. e., a topological space is a quotient s-image of a metric space if and only if it is a sequential space with a point-countable cs^*-network. Thus the spaces with a point-countable cs^*-network play an important role in studying the quotient s-images of metric spaces. According to Alexandroff's hypothesis, the following question can be raised: What mappings can be used to represent the relationship between spaces with a point-countable cs^*-network and metric spaces? We know that the quotient s-images of metric spaces coincide with tht sequence-covering, quotient s-images of metric spaces[3]. One can image that an internal characterization of sequence-covering s-images of metric spaces will play positive role for seeking an in-

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ternal characterization of quotient s-images of metric spaces. It is the purpose of this paper that by means of sequence-covering s-mappings the relationship between spaces with a point-countable cs^*-network and metric spaces is established.

Let X be a topological space. A collection $\mathscr{P}$ of subsets of X is called a cs^*-network[5] for X if for each $x\in X$, for its open neighborhood U in X, and for a sequence $\{x_n\}$ in X which converges to x, there is a subsequence $\{x_{n_i}\}$ such that $\{x\}\cup\{x_{n_i}: i\in N\}\subset P\subset U$ for some $P\in\mathscr{P}$. For convenience' sake, a sequence $\{x_n\}$ which converges to x is sometimes denoted by $\{x\}\cup\{x_n\}$. A mapping $f: X\to Y$ is a sequence-covering if every convergent sequence (including its limit) of Y is the image of some compact subset of X under f; f is an s-mapping if $f^{-1}(y)$ is separable for each $y\in Y$.

By the proof of Lemma 1.17 of [5], we have the following lemma.

Lemma Suppose $\mathscr{P}$ is a point-countable cs^*-network for X. For a convergent sequence $\{x\}\cup\{x_n\}$ of X, let $K=\{x\}\cup\{x_n: n\in N\}$. If U is an open neighborhood of K, then there is a subcollection $\mathscr{F}$ of $\mathscr{P}$ having the following property, which is denoted by $F(K,U)$:

(1) $\mathscr{F}$ is finite;

(2) $K\subset\bigcup\mathscr{F}\subset U$;

(3) For each $P\in\mathscr{F}$, $P\cap K\neq\phi$. And if P contains a subsequence of $\{x_n\}$, then $x\in P$.

Theorem A space X has a point-countable cs^*-network if and only if X is the sequence-covering s-image of a metric space.

Proof Sufficiency. Suppose f is a sequence-covering s-mapping from a metric space M onto X. Since M is a metric space, there is a σ-locally finite base for M, which is denoted by $\mathscr{B}$. Put

$$\mathscr{P}=\{f(B): B\in\mathscr{B}\}.$$

Then $\mathscr{P}$ is point-countable in X because f is an s-mapping. For a sequence $\{x_n\}$ of X which converges to x, put

$$K=\{x\}\cup\{x_n: n\in N\}.$$

Let U be an open neighborhood of K. Then $f(L)=K$ for some compact subset L of M because f is a sequence-covering mapping. Take $z_n\in L\cap f^{-1}(x_n)$ for each $n\in N$. Since L is a compact metrizable subspace of M, the sequence $\{z_n\}$ has a convergent subsequence $\{z\}\cup\{z_{n_i}\}$ in M. Thus $f(z)=x$, and $\{z\}\cup\{z_{n_i}: i\in N\}\subset f^{-1}(U)$. There is an $i_0\in N$ such that

$$\{z\}\cup\{z_{n_i}: i\geqslant i_0\}\subset B\subset f^{-1}(U)$$

for some $B\in\mathscr{B}$. Let $P=f(B)$. Then $P\in\mathscr{P}$ and

$$\{x\}\cup\{x_{n_i}: i\geqslant i_0\}\subset P\subset U.$$

Therefore $\mathscr{P}$ is a point-countable cs^*-network for X.

Necessity. Suppose $\mathscr{P}$ is a point-countable cs^*-network for X. Denote $\mathscr{P}$ by $\{P_\alpha: \alpha\in A\}$. We assume that $\mathscr{P}$ is closed under finite intersections. Let A_i denote the set A with discrete topology for each $i\in N$. Put

$$M=\{\beta=(\alpha_i)\in\Pi\{A_i:\ i\in N\}:\ \{P_{\alpha_i}:\ i\in N\}\subset\mathscr{P},$$
$$\text{which forms a descending network at some point}\quad x(\beta)\in X\},$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{A_i:\ i\in N\}$ of discrete spaces. Since X is Hausdorff, $x(\beta)$ is unique in X for each $\beta\in M$. We define $f:M\to X$ by $f(\beta)=x(\beta)$ for each $\beta\in M$. Thus $\{x(\beta)\}=\bigcap\{P_{\alpha_i}:\ i\in N\}$. It is well known and easy to check that f is an s-mapping from M onto X (cf. [6]). We will show that f is a sequence-covering mapping. For a convergent sequence $\{x\}\cup\{x_n\}$ of X, we assume that all x_n's are distinct, and that $x_n\neq x$ for each $n\in N$. Put

$$K=\{x\}\cup\{x_n:\ n\in N\},$$
$$\mathscr{P}'=\{\mathscr{F}\subset\mathscr{P}:\ \mathscr{F}\ \text{has the property}\quad F(K,X)$$
$$\text{which is described in the Lemma}\}.$$

Then, $\mathscr{P}'$ is countable because $\mathscr{P}$ is point-countable in X and K is a countable subset of X. Denote $\mathscr{P}'$ by $\{\mathscr{F}_i:\ i\in N\}$. For each $n\in N$, let

$$\mathscr{P}_n=\{P\in\wedge\{\mathscr{F}_i:\ i\leqslant n\}:\ P\cap K\neq\phi\}.$$

Then $\mathscr{P}_n$ has the property $F(K,X)$. Since $\mathscr{P}$ is closed under finite intersections, there is a finite subset B_n of A_n such that

$$\mathscr{P}_n=\{P_\alpha:\ \alpha\in B_n\}.$$

Put

$$L=\{\beta=(\alpha_i)\in\Pi\{B_i:\ i\in N\}:\ P_{\alpha_{i+1}}\subset P_{\alpha_i}\quad\text{for each}\quad i\in N\}.$$

Then L is a compact subset of $\Pi\{B_i:\ i\in N\}$ for L is closed in $\Pi\{A_i:\ i\in N\}$. To end the proof of the theorem, we will check that $L\subset M$ and $f(L)=K$.

Suppose $\beta=(\alpha_i)\in L$, and let

$$K(\beta)=K\cap(\bigcap\{P_{\alpha_i}:\ i\in N\}).$$

Then $\{K\cap P_{\alpha_i}:\ i\in N\}$ is a sequence of non-empty descending closed subsets of the compact subset K of X by the property $F(K,X)$ and the definition of L, so $K(\beta)\neq\phi$. Take $y\in K(\beta)$. Then $\{P_{\alpha_i}:\ i\in N\}$ forms a network at y in X. In fact, let V be an open neighborhood of y in X. If $y=x$, then

$$\{y\}\cup\{x_n:\ n\geqslant m\}\subset V$$

for some $m\in N$. Put

$$K_1=\{y\}\cup\{x_n:\ n\geqslant m\},$$
$$K_2=K\setminus K_1.$$

By the Lemma, there is a subcollection $\mathscr{F}'$ of $\mathscr{P}$ with the property $F(K_1,V)$. Since K_2 is finite and $K_2\subset X\setminus\{y\}$, there is a finite subcollection $\mathscr{F}''$ of $\mathscr{P}$ with $K_2\subset\bigcup\mathscr{F}''\subset X\setminus\{y\}$ and $P\cap K_2\neq\phi$ for each $P\in\mathscr{F}''$. Put

$$\mathscr{F}=\mathscr{F}'\cup\mathscr{F}''.$$

Then $\mathscr{F}$ has the property $F(K,X)$, so $\mathscr{F}=\mathscr{F}_i$ for some $i\in N$. Since $y\in P_{\alpha_i}\in\mathscr{P}_i$ and $y\notin\bigcup\mathscr{F}''$, we have $y\in P_{\alpha_i}\subset\bigcup\mathscr{F}'\subset V$, and consequently $\{P_{\alpha_i}\}$ forms a network at y in X. If $y\neq$

x, then $y \in P \subset V \setminus (K \setminus \{y\})$ for some $P \in \mathscr{P}$. By the Lemma, there exists a subcollection $\mathscr{F}'$ of $\mathscr{P}$ with the property $F(K \setminus \{y\}, X \setminus \{y\})$. Put

$$\mathscr{F} = \mathscr{F}' \cup \{P\}.$$

Then $\mathscr{F}$ has the property $F(K, X)$, and $\mathscr{F} = \mathscr{F}_i$ for some $i \in N$. Hence $y \in P_{\alpha_i} \subset P \subset V$ because $y \in P_{\alpha_i} \in \mathscr{P}_i$ and $y \notin \bigcup \mathscr{F}'$. So $\{P_{\alpha_i}: i \in N\}$ also forms a network of y in X. Therefore, $\beta = (\alpha_i) \in M$, $f(\beta) = y \in K$. Thus $L \subset M$, and $f(L) \subset K$.

On the other hand, suppose $y \in K$. Then there is $F_i \in \mathscr{F}_i$ with $y \in F_i$ because F_i has the property $F(K, X)$ for each $i \in N$. Thus, for each $n \in N$, there is $\alpha_n \in \beta_n$ such that $P_{\alpha_n} = \bigcap \{F_i: i \leqslant n\}$. Let $\beta = (\alpha_i)$. Then $\beta \in L$. Since $y \in K \cap (\bigcap \{P_{\alpha_i}: i \in N\})$, $\{P_{\alpha_i}: i \in N\}$ forms a descending network at y in X, and $y = f(\beta)$ by the same method used in the proof of the above paragraph. Hence $K \subset f(L)$.

In a word, $f(L) = K$. Therefore, f is a sequence-covering mapping from the metric space M onto X. The proof is complete.

Corollary[3,4] The following statements are equivalent for a space X:

(1) X is a sequence-covering quotient s-image of a metric space;

(2) X is a quotient s-image of a metric space;

(3) X is a sequential space with a point-countable cs^*-metwork.

Proof (1)⇒(2) is obvious.

(2)⇒(3). Suppose f is a quotient s-mapping from a metric space M onto X. Since the property of sequential spaces is preserved under quotient mappings (cf. [7]), X is a sequential space. Let $\mathscr{B}$ be a σ-locally finite base of the metric space M, and put

$$\mathscr{P} = \{f(B): B \in \mathscr{B}\}.$$

Then $\mathscr{P}$ is point-countable in X. For a sequence $\{x_n\}$ of X which converges to x, let U be an open neighborhood of x in X. We assume that all x_n's are distinct, and $x_n \in U \setminus \{x\}$ for each $n \in N$. Let $A = \{x_n: n \in N\}$. Then A is not a closed subset of X. Since M is a k-space, there is a compact subset Z of M such that $f(Z) \cap A$ is not a closed subset of X (cf. [8]), and so $f(Z) \cap A$ is infinite. Thus, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\{x_{n_i}\} \subset f(Z) \cap A$. Take $z_i \in Z$ with $f(z_i) = x_{n_i}$ for each $i \in N$. Since Z is a compact metrizable space, we can assume that the sequence $\{z_i\}$ converges to a point z fo Z. Thus $f(z) = x$. Since $\{z\} \cup \{z_i: i \in N\} \subset f^{-1}(U)$, there is an $i_0 \in N$ and a $B \in \mathscr{B}$ such that $\{z\} \cup \{z_i: i \geqslant i_0\} \subset B \subset f^{-1}(U)$, and hence

$$\{x\} \cup \{x_{n_i}: i \geqslant i_0\} \subset f(B) \subset U.$$

Therefore, $\mathscr{P}$ is a point-countable cs^*-network for X.

(3)⇒(1). Suppose X is a sequential space with a point-countable cs^*-network. There is a metric space M and a sequence-covering s-mapping f from M onto X by the Theorem. We will prove that f is a quotient mapping. Let $A \subset X$ such that $f^{-1}(A)$ is closed in M. For a convergent sequence K of X, there is a compact subset L of M with $f(L) = K$. Let $L_1 = f^{-1}(A) \cap L$. Then L_1 is compact in M, and so $f(L_1)$ is compact in X, i. e., $A \cap K$ is compact in X, and thus $A \cap K$ is closed in X. Since X is a sequential space, A is closed in X. Therefore, f is a quotient mapping.

Remark The condition "X is a sequential space" in the above Corollary cannot be weakened to "X is a k-space". Since βN is a non-metrizable compact space, βN cannot be the quotient s-image of a metric space by Corollary 3. 7 in [2]. However, βN is a k-space having a point-countable cs^*-network because there is no non-trivial convergent sequence in βN (cf. [9]).

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References

[1] Arhangel'skii, A., Mappings and spaces, *Russian Math. Surveys*, **21**(4)(1966), 115—162.

[2] Hoshina, T., On the quotient s-images of metric spaces, *Sci. Rep. Tokyo Kyoiku Daigaku Sect.*, **A10**(1970), 265—268.

[3] Gruenhage, G., Michael, E., Tanaka, Y., Spaces determined by point-countable covers, *Pacific J. Math.*, **113**(1984), 303—332.

[4] Tanaka, Y., Point-countable covers and k-networks, *Top. Proc.*, **12**(2)(1987), 327—349.

[5] Nagata, J., Generalized metric spaces I, in Topics in General Topology, Morita, K. and Nagata, J. ed., North-Holland, Amsterdam, 1989.

[6] Lin Shou, On a generalization of Michael's theorem, *Northeastern Math. J.*, **4**(2)(1988), 162—168.

[7] Franklin, S. P., Spaces in which sequences suffice, *Fund. Math.*, **57**(1965), 107—115.

[8] Michael, E., $\aleph_0$-spaces, *J. Math. Meth.*, **15**(1966), 983—1002.

[9] Engelking, R., General Topology, PWN, Warszawa, 1977, p. 229.

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On Strongly Semisimple Radicals of Rings*)

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Abstract In this paper we discuss the strongly semisimple radicals of associative rings, give a sufficient and necessary condition for a radical to be strongly semisimple, and characterise the hereditary strongly semisimple radicals. We prove that the union radical of strongly semisimple radicals is strongly semisimple.

Key Words and Phrases Ring; Radical; Semisimple Class

Let R be a radical property. R is said to be hereditary if for every ring A and every ideal I of A we have $R(I)=I\cap R(A)$. R is said to be strongly semisimple if every homomorphic image of every R-semisimple ring is again R-semisimple. We have

Theorem 1 For every radical property R the following conditions are equivalent:

(1) R is strongly semisimple;

(2) For every ring A and every ideal I of A, $A/(R(A)+I)$ is R-semesimple;

(3) For every ring A and every ideal I of A, $R(A/I)=(R(A)+I)/I$;

(4) For every ring A and arbitrary ideals I_1 and I_2 of A, $(I_1+I_2)/(I_1+R(I_2))$ is R-semisimple.

Proof (1)⇒(2). We have

$$A/(R(A)+I)\cong(A/R(A))/((R(A)+I)/R(A)).$$

$(A/R(A))/((R(A)+I)/R(A))$ is a homomorphic image of R-semisimple ring $A/R(A)$. Thus $A/(R(A)+I)$ is R-semisimple.

(2)⇒(3). We have

$$(A/I)/((R(A)+I)/I)\cong A/(R(A)+I).$$

$A/(R(A)+I)$ is R-semisimple. Thus $R(A/I)\subseteq(R(A)+I)/I$. Since

$$(R(A)+I)/I\cong R(A)/(I\cap R(A))$$

and $R(A)/(I\cap R(A))$ is a radical ring, we have also $(R(A)+I)/I\subseteq R(A/I)$. Hence $R(A/I)=(R(A)+I)/I$.

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