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MAPPING THEOREMS ON ℵ-SPACES

Shou LIN
Department of Mathematics, Suzhou University, Suzhou, People's Republic of China

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We prove two mapping theorems on ℵ-spaces: (1) ℵ-spaces are preserved under closed, Lindelöf mappings; (2) a perfect inverse image of an ℵ-space is an ℵ-space if and only if it has a G_δ-diagonal.

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k-network	G_δ-diagonal
ℵ-space	closed Lindelöf mapping
Fréchet space	compact-covering mapping
Lašnev space	perfect mapping

1. Introduction

The concept of ℵ-spaces was first introduced by Meara in [7] as a generalization of metric spaces and $\aleph_0$-spaces (Michael [6]). The main results of this paper are two mapping theorems on ℵ-spaces:

(1) ℵ-spaces are preserved under closed Lindelöf mappings. This affirmatively answers a question posed by Tanaka in [8].
(2) A perfect inverse image of an ℵ-space is an ℵ-space if and only if it has a G_δ-diagonal.

Throughout this paper, all spaces are assumed to be at least T_1 and regular. All mappings are continuous and surjective. A mapping f from X onto Y is to be denoted by $f: X \to Y$. N denotes the set of positive integers.

Let X be a topological space. A family $\mathscr{F}$ of closed subsets of X is a k-network for X if for every compact set $K \subset X$ and neighborhood U of K, there is a finite $\mathscr{F}' \subset \mathscr{F}$ so that $K \subset \bigcup \mathscr{F}' \subset U$. $\mathscr{F}$ is a cs-network for X if for every convergent sequence Z in X and neighborhood U of Z, there is a $F \in \mathscr{F}$ so that Z is eventually in F and $F \subset U$. A regular space with σ-locally-finite k-network is called an ℵ-space [7].

2. Closed images

Mapping $f:X\to Y$ is called Lindelöf if for each $y\in Y$ fiber $f^{-1}(y)$ is a Lindelöf subspace of X; f is called compact-covering [6] if every compact subset of Y is the image of a compact subset of X.

Lemma 2.1. *If $f:X\to Y$ is closed Lindelöf, then f is a compact-covering.*

Proof. Let K be a compact subset of Y; then $f^{-1}(K)$ is a Lindelöf subset of X. But if $g=f|_{f^{-1}(K)}$, then g is a closed mapping from the paracompact space $f^{-1}(K)$ onto K. By Proposition 7.2 in [6], g is compact-covering. Since K is compact, there exists a compact subset K of $f^{-1}(K)$ such that $g(L)=K$. Also, L is a compact subset of X, and $f(L)=K$. □

Theorem 2.2. *$\aleph$-spaces are preserved under closed Lindelöf mappings.*

Proof. Suppose X is an $\aleph$-space, and $f:X\to Y$ is closed Lindelöf. X has a σ-locally-finite closed k-network $\mathcal{P}$. Put $\mathcal{F}=\{f(P)\mid P\in\mathcal{P}\}$. Since f is closed Lindelöf, $\mathcal{F}$ is a σ-closure-preserving and locally-countable collection of closed subsets of Y. It is clear that the compact-covering image of a k-network is a k-network.

Hence, by Lemma 2.1, $\mathcal{F}$ is a σ-closure-preserving and σ-locally-countable closed k-network. Foged [1, Theorem 4, (a)→(d)] proved that a space with σ-locally-finite closed k-network has a σ-discrete cs-network. It is not difficult to check that, in his proof, the condition "σ-locally-finite closed k-network" can be replaced by "σ-locally-countable and σ-closure-preserving closed k-network". Therefore a space with σ-locally-countable and σ-closure-preserving closed k-network is an $\aleph$-space. Therefore Y is an $\aleph$-space. □

Remark 1. The following question is posed by Tanaka in [8]: Are the spaces which are closed Lindelöf images of metric spaces $\aleph$-spaces? Theorem 2.2 answers the question affirmatively.

Remark 2. For each $\alpha<\omega_1$, let $I_\alpha=[0,1]$ with usual topology, and let X be quotient space of $\oplus_{\alpha<\omega_1} I_\alpha$ obtained by identifying $\{0\}$. Then X is a Lašnev space and is not an $\aleph$-space (by [5, Proposition 6.4]). Hence $\aleph$-spaces are not preserved under closed mappings.

Theorem 2.3. *The following properties of a space are equivalent:*
(a) *X is a Fréchet and $\aleph$-space.*
(b) *X is a closed Lindelöf image of a metric space.*

Proof. (b)→(a). It is known that closed mappings preserve the Fréchet property. By Theorem 2.2, X is an $\aleph$-space.

(a)$\to$(b). Suppose X is a Fréchet and ℵ-space. Foged [2, Theorem 1] has shown that X is a Fréchet space with σ-hereditarily closure-preserving k-network if and only if X is a Lašnev space (a space which is a closed image of a metric space). Let M be a metric space, $f: M\to X$ a closed mapping. Since M is a paracompact ℵ-space, and X a k-space with point-countable closed k-network, according to [5, Proposition 6.4] for each $y\in Y$, $\partial f^{-1}(y)$ (boundary of $f^{-1}(y)$) is Lindelöf. Thus there exists a closed subset M' of M such that $g=f|_{M'}: M'\to X$ is closed Lindelöf with $g(M')=X$. Hence X is a closed Lindelöf image of a metric space. □

3. Perfect inverse images

For a topological space X, let $\mathcal{K}(X)=\{K\subset X \mid K$ is a nonempty compact subset of $X\}$. If $\mathcal{U}$ and $\mathcal{V}$ are collections of subsets of X, let $\mathcal{U}\wedge\mathcal{V}=\{U\cap V \mid U\in\mathcal{U}$ and $V\in\mathcal{V}\}$. For any $A\subset X$, let $(\mathcal{U})_A=\{U\in\mathcal{U}\mid U\cap A\neq\emptyset\}$ and $\mathrm{st}(A,\mathcal{U})=\bigcup(\mathcal{U})_A$.

We consider the following properties of space X.

(A) For any open cover of X there exists a σ-discrete refinement $\mathcal{F}$ such that every compact subset of X is covered by a finite subcollection of $\mathcal{F}$.

(B) For any open cover of X there exists a sequence $(\mathcal{G}_n)$ of open refinements which satisfies the condition that for each $K\in\mathcal{K}(X)$, there exist $K_i\in\mathcal{K}(X)_{(i\leq m)}$ such that $K=\bigcup_{i\leq m}K_i$ and $|(\mathcal{G}_{n_i})_{K_i}|=1_{(i\leq m)}$.

(C) There exists a sequence $(\mathcal{G}_n)$ of open covers such that for each $K\in\mathcal{K}(X)$, $K=\bigcap_n \overline{\mathrm{st}(K,\mathcal{G}_n)}$.

Lemma 3.1. *If Y is an ℵ-space and $f: X\to Y$ is a perfect mapping, then X has property* (A).

Proof. Since Y is an ℵ-space, Y has a σ-discrete k-network (by Foged [1, Theorem 4]). Suppose $\mathcal{P}=\bigcup_n\mathcal{P}_n$ is a k-network for Y, each $\mathcal{P}_n$ is a discrete collection of subsets of Y.

Suppose $\mathcal{U}$ is any open cover of X. For each $y\in Y$ we can find a finite subcollection $\mathcal{U}(y)\subset\mathcal{U}$ such that $f^{-1}(y)\subset\bigcup\mathcal{U}(y)$. Let $G(y)=Y-f(X-\bigcup\mathcal{U}(y))$, then $\mathcal{G}=\{G(y)\mid y\in Y\}$ is an open cover of Y. By the definition of k-network and the regularity of Y, without loss of generality, we may assume $\mathcal{P}$ is a refinement of $\mathcal{G}$. Consequently for each $P\in\mathcal{P}$ there exist $U(i,P)\in\mathcal{U}$ such that $f^{-1}(P)\subset\bigcup_{i\leq m_P}U(i,P)$. Let $\mathcal{F}(n,i)=\{f^{-1}(P)\cap U(i,P)\mid P\in\mathcal{P}_n\}$. Then $\mathcal{F}=\bigcup_{n,i}\mathcal{F}(n,i)$ satisfies (A). □

Lemma 3.2. (A)$\to$(B).

Proof. Let $\mathcal{U}$ be an open cover of a space X and take a σ-discrete refinement $\mathcal{F}=\bigcup_n\mathcal{F}_n$ of $\mathcal{U}$ with the property (A). Let $\mathcal{F}_n=\{F(n,\alpha)\mid\alpha\in A_n\}$. By regularity, we may assume each element of $\mathcal{F}$ is a closed subset of X. For each $n\in\mathbb{N}$, $\alpha\in A_n$, pick $U(n,\alpha)\in\mathcal{U}$ such that $F(n,\alpha)\subset U(n,\alpha)$, and put $W(n,\alpha)=U(n,\alpha)-\bigcup\{F(n,\beta)\mid\beta\in A_n-\{\alpha\}\}$. We define

$$\mathcal{W}_n=\{W(n,\alpha)\mid\alpha\in A_n\}\cup\{U-\bigcup\mathcal{F}_n\mid U\in\mathcal{U}\}.$$

It follows that $(\mathcal{W}_n)$ satisfies (B).

It is clear that $(\mathcal{W}_n)$ is the sequence of open refinement of $\mathcal{U}$. To see that $(\mathcal{W}_n)$ satisfies (B), let $K\in\mathcal{K}(X)$, by the property (A), there exists a finite subcollection $\mathcal{F}'=\{F_i\mid i\leq m\}$ of $(\mathcal{F})_K$ which covers K. For each $i\in\{1,2,\ldots,m\}$, there exists a $n_i\in\mathbb{N}$ such that $F_i\in\mathcal{F}_{n_i}$. Then $K\cap F_i\in\mathcal{K}(X)_{(i\leq m)}$, $K=\bigcup_{i\leq m}K\cap F_i$ and $|(\mathcal{W}_{n_i})_{K\cap F_i}|=1$. □

Lemma 3.3. (B) + G_δ-diagonal → (C).

Proof. Suppose a space X with property (B) has a G_δ-diagonal. Clearly X is a submetacompact (i.e., θ-refinable) space with a G_δ-diagonal, so X has a G_δ^*-diagonal [4, Theorem 2.11]. Let $(\mathcal{G}_n)$ be a G_δ^*-diagonal sequence, i.e., $\{x\}=\bigcap_n\overline{\mathrm{st}(x,\mathcal{G}_n)}$ for each $x\in X$. We may assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Now we prove for each $K\in\mathcal{K}(X)$, $K=\bigcap_n\mathrm{st}(K,\mathcal{G}_n)$. Suppose $x\in X-K$; then $\{X-\overline{\mathrm{st}(x,\mathcal{G}_n)}\mid n\in\mathbb{N}\}$ is an open cover of the compact subset K, so there exists a $n\in\mathbb{N}$ such that $K\subset X-\overline{\mathrm{st}(x,\mathcal{G}_n)}$. Therefore $K\cap\overline{\mathrm{st}(x,\mathcal{G}_n)}=\emptyset$, i.e., $x\notin\mathrm{st}(K,\mathcal{G}_n)$. Hence $K=\bigcap_n\mathrm{st}(K,\mathcal{G}_n)$.

Now, we use the regularity of X and property (B) to inductively define, for each $m\in\mathbb{N}$, a sequence $(\mathcal{V}_{m,n})_n$ of open covers for X such that

(a) for each $n\in\mathbb{N}$, $\{\bar{V}\mid V\in\mathcal{V}_{m,n}\}$ is a refinement of $(\bigwedge_{i,j<m}\mathcal{V}_{i,j})\wedge(\bigwedge_{k\leq m}\mathcal{G}_k)$;

(b) $(\mathcal{V}_{m,n})_n$ is a sequence satisfying the condition of property (B).

We prove for each $K\in\mathcal{K}(X)$, $\bigcap_{m,k}\overline{\mathrm{st}(K,\mathcal{V}_{m,k})}=K$. For each $n\in\mathbb{N}$, take $s>n$. Since the sequence $(\mathcal{V}_{s,k})_k$ satisfies (b), there exists $K_i\in\mathcal{K}(X)_{(i\leq h)}$ such that $K=\bigcup_{i\leq h}K_i$ with $|(\mathcal{V}_{s,k_i})_{K_i}|=1$. Then

$$\overline{\mathrm{st}(K_i,\mathcal{V}_{s,k_i})}=\bigcup\{\bar{V}\mid V\in(\mathcal{V}_{s,k_i})_{K_i}\}\subset\mathrm{st}(K_i,\mathcal{V}_{n,n})\subset\mathrm{st}(K_i,\mathcal{G}_n).$$

Pick $r>\max\{s,k_1,k_2,\ldots,k_n\}$; consequently,

$$\begin{aligned}\bigcap_{m,k}\overline{\mathrm{st}(K,\mathcal{V}_{m,k})}&\subset\overline{\mathrm{st}(K,\mathcal{V}_{r,1})}\\&=\bigcup_{i\leq h}\overline{\mathrm{st}(K_i,\mathcal{V}_{r,1})}\subset\bigcup_{i\leq h}\overline{\mathrm{st}(K_i,\mathcal{V}_{s,k_i})}\subset\mathrm{st}(K,\mathcal{G}_n).\end{aligned}$$

Hence

$$\bigcap_{m,k}\overline{\mathrm{st}(K,\mathcal{V}_{m,k})}\subset\bigcap_n\mathrm{st}(K,\mathcal{G}_n)=K.$$

So $K=\bigcap_{m,k}\overline{\mathrm{st}(K,\mathcal{V}_{m,k})}$.

Theorem 3.4. *Suppose there exists a perfect mapping f from a topological space X onto an ℵ-space Y. Then X is an ℵ-space if and only if it satisfies any of the following:*

(a) *X has a G_δ-diagonal.*
(b) *X has a point-countable k-network.*

Proof. Necessity is obvious.

Sufficiency: Since a σ-space has a G_δ-diagonal, by Corollary 3.8 in [5], it is sufficient to show that if X has a G_δ-diagonal, then X is an $\aleph$-space.

Suppose X has a G_δ-diagonal. By Lemmas 3.1, 3.2, and 3.3, there exists a sequence $(\mathscr{G}_n)$ of open covers for X such that for each $K \in \mathscr{K}(X)$, $K = \bigcap_n \overline{\mathrm{st}(K, \mathscr{G}_n)}$. We can assume $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$. For each $n \in \mathbb{N}$, by Lemma 3.1, $\mathscr{G}_n$ has a σ-locally-finite closed refinement $\mathscr{F}(n)$ such that every compact subset of X is covered by a finite subcollection of $\mathscr{F}(n)$. Denote by $\mathscr{F}(n) = \bigcup_m \mathscr{F}(n, m)$ where each $\mathscr{F}(n, m)$ is a locally-finite collection of subsets of X. We can assume $\mathscr{F}(n, m) \subset \mathscr{F}(n, m+1)$ for each $m \in \mathbb{N}$.

Since Y is an $\aleph$-space, let $\bigcup_k \mathscr{X}(k)$ be a k-network for Y where each $\mathscr{X}(k)$ is locally-finite and $\mathscr{X}(k) \subset \mathscr{X}(k+1)$ for each $k \in \mathbb{N}$. Let $\mathscr{D}(k) = \{f^{-1}(Q) \mid Q \in \mathscr{X}(k)\}$; then $\mathscr{D}(k)$ is a locally-finite collection of closed subsets of X. Put

$$\mathscr{P}(n, m, k) = \mathscr{F}(n, m) \wedge \mathscr{D}(k).$$

Clearly $\mathscr{P}(n, m, k)$ is locally-finite for each $n, m, k \in \mathbb{N}$.

We complete the proof by showing that $\mathscr{P} = \bigcup_{n,m,k} \mathscr{P}(n, m, k)$ is a k-network for X. For an open subset W and a compact subset $K \subset W \subset X$, since $K = \bigcap_n \overline{\mathrm{st}(K, \mathscr{G}_n)}$, $\{W\} \cup \{X - \overline{\mathrm{st}(K, \mathscr{G}_n)} \mid n \in \mathbb{N}\}$ is an open cover of compact subset $f^{-1}f(K)$ of X. Thus there exists a $n \in \mathbb{N}$ such that $f^{-1}f(K) \subset W \cup (X - \overline{\mathrm{st}(K, \mathscr{G}_n)})$, so $\overline{\mathrm{st}(K, \mathscr{G}_n)} \cap f^{-1}f(K) \subset W$. For each $x \in f^{-1}f(K) - W$, since $x \notin \overline{\mathrm{st}(K, \mathscr{G}_n)}$, there exists an open set $V(x)$ containing x with $V(x) \cap \overline{\mathrm{st}(K, \mathscr{G}_n)} = \emptyset$. Let $G = W \cup (\bigcup\{V(x) \mid x \in f^{-1}f(K) - W\})$, then $f(K) \subset Y - f(X - G)$. So there exists a finite $\mathscr{X}'(k) \subset \mathscr{X}(k)$ such that $f(K) \subset \bigcup \mathscr{X}'(k) \subset Y - f(X - G)$ for some $k \in \mathbb{N}$. Take $\mathscr{D}'(k) = \{f^{-1}(Q) \mid Q \in \mathscr{X}'(k)\}$; then $f^{-1}f(K) \subset \bigcup \mathscr{D}'(k) \subset G$. On the other hand, by the property of $\mathscr{F}(n)$, there exists a finite $\mathscr{F}'(n, m) \subset (\mathscr{F}(n, m))_K$ such that $K \subset \bigcup \mathscr{F}'(n, m) \subset \mathrm{st}(K, \mathscr{G}_n)$ for some $m \in \mathbb{N}$. Put $\mathscr{P}'(n, m, k) = \mathscr{F}'(n, m) \wedge \mathscr{D}'(k)$. It is easy to check that $K \subset \bigcup \mathscr{P}'(n, m, k) \subset W$. □

Corollary 3.5. *Suppose Y is an $\aleph$-space and $f: X \to Y$ is an open, closed, and finite-to-one mapping. Then X is an $\aleph$-space.*

Proof. Since $\aleph$-space is a σ-space, X is a σ-space [3]. Then X has a G_δ-diagonal. By Theorem 3.4, X is an $\aleph$-space. □

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