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On the Quotient Compact Images of Metric Spaces*

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Abstract

This paper gives an internal characterization of the quotient compact images of metric spaces, which answers a question posed by Arhangel'skii; and an example is constructed to show that the quotient compact images of metric spaces are not preserved by perfect maps.

The open compact images of metric spaces can be characterized by the T_1-spaces with uniform bases. A. Arhangel'skii [1] posed a question to find an internal characterization of the quotient compact images of metric spaces. The purpose of the present paper is to give a solution to this question in terms of the weak bases introduced by A. Arhahgel'skii [1].

In this paper, all spaces are assumed to be T_1-topological spaces. Let all maps be continuous and surjective, and N denote the set of positive integers. Suppose $\mathscr{P}$ is a collection of subsets of a space X. $\mathscr{P}$ is said to be a weak base for X provided that $\mathscr{P}=\bigcup\{\mathscr{P}_x : x\in X\}$ such that (1) $x\in\bigcap\mathscr{P}_x$; (2) if $U,V\in\mathscr{P}_x$, then exists $W\in\mathscr{P}_x$ with $W\subset U\cap V$; (3) a subset G of X is open if and only if given $x\in G$, there exists $P\in\mathscr{P}_x$ with $P\subset G$. The collection $\mathscr{P}_x$ is called to be a local weak base at point x of X. If each $\mathscr{P}_x$ is countable, then X is weakly first countable.

Theorem A topological space X is the quotient compact image of a metric space if and only if X has a sequence $\{\mathscr{P}_i\}$ of point finite covers such that $\{\mathrm{st}(x,\mathscr{P}_i): i\in N\}$ forms a local weak base at x for each $x\in X$.

Proof Necessity. Suppose a space X is an image of a metric space M under a quotient compact map f. Since M is metrizable, there is a sequence $\{\mathscr{B}_i\}$ of open covers for M such that if $K\subset U$ with K compact and U open in M, then $\mathrm{st}(K,\mathscr{B}_i)\subset U$ for some $i\in N$ [2]. We can assume that $\mathscr{B}_{i+1}$ is a refinement of $\mathscr{B}_i$, and that $\mathscr{B}_i$ is an open locally finite cover of M for each $i\in N$ because M is a paracompact space. For each $i\in N$, put

$$\mathscr{P}_i=\{f(B): B\in\mathscr{B}_i\},$$

then $\mathscr{P}_i$ is a point finite cover of X. If $x\in V$ with V open in X, then $f^{-1}(x)\subset f^{-1}(V)$, thus $\mathrm{st}(f^{-1}(x),\mathscr{B}_i)\subset f^{-1}(V)$ for some $i\in N$, so $\mathrm{st}(x,\mathscr{P}_i)\subset V$. On the other

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hand, if a subset G of X satisfies that for each $x\in G$, there is $i\in N$ with $st(x,\mathscr{P}_i)\subset G$, then for each $z\in f^{-1}(G)$, there is $i\in N$ with $st(f(z),\mathscr{P}_i)\subset G$, thus $st(z,\mathscr{B}_i)\subset f^{-1}(G)$, hence $f^{-1}(G)$ is a neighborhood of z, and accordingly $f^{-1}(G)$ is open in M. Therefore, G is open in X. This shows that $\{st(x,\mathscr{P}_i):i\in N\}$ forms a local weak base at x for each $x\in X$.

Sufficiency. Suppose X is a space having a sequence $\{\mathscr{P}_i\}$ of point finite covers such that $\{st(x,\mathscr{P}_i):i\in N\}$ forms a local weak base at x for each $x\in X$. First, notice that if $x\in P_i\in\mathscr{P}_i$ for each $i\in N$, then $\{P_i:i\in N\}$ forms a net at x. In fact, for an open neighborhood V of x in X, there is $i\in N$ such that $st(x,\mathscr{P}_i)\subset V$, thus

$$x\in P_i\subset st(x,\mathscr{P}_i)\subset V.$$

For each $i\in N$, put

$$\mathscr{P}_i=\{P_a:a\in A_i\},$$

and we can assume that all A_i's are disjoint. Each A_i is endowed with discrete topology. Let

$$M=\{b=(a_i)\in\Pi\{A_i:i\in N\}:\{P_{a_i}\}\text{ forms a net at some point }x(b)\in X\},$$

and give M the subspace topology induced from the usual product topology of the collections $\{A_i:i\in N\}$ of discrete spaces. Then M is a metric space. For each $b\in M$, $x(b)$ is unique in X because X is a T_1-space. Define $f:M\to X$ by $f(b)=x(b)$. Thus $\{x(b)\}=\cap\{P_{a_i}:i\in N\}$. We will show that f is a continuous, quotient and compact map from M onto X.

1. f is surjective. For each $x\in X$, $i\in N$, there is $a_i\in A_i$ with $x\in P_{a_i}$. Thus $\{P_{a_i}\}$ forms a net at x. Let $b=(a_i)$, then $b\in M$ and $f(b)=x$.

2. f is continuous. For each $b=(a_i)\in M$, and a neighborhood V of $f(b)$ in X, there exists $n\in N$ with $P_{a_n}\subset V$. Put

$$W=\{c\in M:\text{the }n\text{-th coordinate of }c\text{ is }a_n\},$$

then W is open in M and $f(W)\subset P_{a_n}\subset V$.

3. f is compact. For each $x\in X$, $i\in N$, put

$$B_i=\{a\in A_i:x\in P_a\},$$

then $\Pi\{B_i:i\in N\}$ is compact in $\Pi\{A_i:i\in N\}$. To prove that f is compact, it is sufficient to show that $f^{-1}(x)=\Pi\{B_i:i\in N\}$. If $b=(a_i)\in\Pi\{B_i:i\in N\}$, then $x\in P_{a_i}$ and $a_i\in B_i$ for each $i\in N$, thus $\{P_{a_i}\}$ forms a net at x, so $b\in M$ and $f(b)=x$. On the other hand, if $b=(a_i)\in f^{-1}(x)$, then for each $i\in N$, $x\in P_{a_i}$, thus $a_i\in B_i$, hence $b\in\Pi\{B_i:i\in N\}$.

4. f is quotient. Suppose U is a subset of X such that $f^{-1}(U)$ is open in M. We will show that U is open in X. For each $x\in U$, $i\in N$, put

$$B_i=\{a\in A_i:x\in P_a\},$$

then $f^{-1}(x)=\Pi\{B_i:i\in N\}\subset f^{-1}(U)$. Since $f^{-1}(x)$ is compact in M, for each $i\in N$, there exists an open subset V_i of A_i such that

$$\Pi\{B_i:i\in N\}\subset\Pi\{V_i:i\in N\}\cap M\subset f^{-1}(U),$$

where $V_i \neq A_i$ for only finitely many $i \in N$. Thus

$$(\Pi\{B_i : i \leqslant n\}) \times (\Pi\{A_i : i > n\}) \cap M \subset f^{-1}(U)$$

for some $n \in N$. For each $z \in \bigcap\{\mathrm{st}(x, \mathscr{P}_i) : i \leqslant n\}$, if $i \leqslant n$, picking $a_i \in B_i$ makes $z \in P_{a_i}$; if $i > n$, picking $a_i \in A_i$ makes $z \in P_{a_i}$, then $\{P_{a_i}\}$ forms a net at z. Let $b = (a_i)$, then $z = f(b)$ and

$$b \in (\Pi\{B_i : i \leqslant n\}) \times (\Pi\{A_i : i > n\}) \cap M \subset f^{-1}(U),$$

thus $z \in U$. This is that

$$\bigcap\{\mathrm{st}(x, \mathscr{P}_i) : i \leqslant n\} \subset U.$$

Since $\{\mathrm{st}(x, \mathscr{P}_i) : i \in N\}$ forms a local weak base at x, U is open in X. Therefore, f is quotient. This completes the proof of the Theorem.

It is known that the open compact images of metric spaces are preserved by perfect maps. It is easy to check that the quotient compact images of metric spaces are preserved by finite-to-one quotient maps. But, the following example shows that the quotient compact images of metric spaces are not preserved by perfect maps.

Example The quotient compact images of metric spaces are not preserved by perfect maps.

Construction Take $X = \{0\} \cup N \cup (N \times N)$. For each $n \in N$, put

$$F_n = \{1, 2, \cdots, n\},$$
$$\mathscr{F} = \{F_n : n \in N\}.$$

(N, N) denotes the set of all correspondences from N into N. For each $n, m, k \in N$, and $f \in (N, N)$, put

$$V(n, m) = \{n\} \cup \{(n, k) : k \geqslant m\},$$
$$H(n, f) = \{0\} \cup (\bigcup\{V(k, f(k)) : k \in N \setminus F_n\}).$$

For each $x \in X$, take

$$\mathscr{N}_x = \begin{cases} \{H(n, f) : n \in N, f \in (N, N)\}, & x = 0 \\ \{V(x, m) : m \in N\}, & x \in N \\ \{\{x\}\}, & x \in N \times N, \end{cases}$$

and

$$\mathscr{B}_x = \begin{cases} \{\{0\} \cup (N \setminus F_n) : n \in N\}, & x = 0 \\ \mathscr{N}_x, & x \in N \cup (N \times N). \end{cases}$$

X is endowed with the topology generated by the neighborhood system $\{\mathscr{N}_x : x \in X\}$. Then $\mathscr{B}_x$ forms a local weak base at x [3] for each $x \in X$. For each $i \in N$, put

$$\mathscr{P}_i = \{\{0\} \cup (N \setminus F_i)\} \cup \{V(x, i) : x \in N\} \cup \{\{x\} : x \in N \times N\},$$

then $\mathscr{P}_i$ is a sequence of point finite covers of X, and for each $x \in X$,

$$\mathrm{st}(x, \mathscr{P}_i) = \begin{cases} \{0\} \cup (N \setminus F_i), & x = 0 \\ V(x, i) \cup \{0\} \cup (N \setminus F_i), & x \in N, \ x > i \\ V(x, i), & x \leqslant i \\ V(x, i), & x = (x_1, x_2) \in N \times N, \ x_2 \geqslant i \\ \{x\}, & x_2 < i, \end{cases}$$

hence $\{\mathrm{st}(x, \mathscr{P}_i) : i \in N\}$ forms a local weak base at x. Consequently, X is the quotient compact image of a matric space by the Theorem.

Let $K=\{0\}\cup N$, and $Y=X/K$. Then the natural quotient map $q: X\rightarrow Y$ is a perfect map. It is shown that Y is not weakly first countable in [3], hence Y need not be the quotient compact image of a metric space, and accordingly perfect maps cannot preserve the quotient compact images of metric spaces.

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