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A Note on Metrization Theorem*

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Abstract. In this short note it is shown that there exists a regular and T_1 space with a σ-linearly hereditarily closure-preserving base which is not a metrizable space.

Throughout this paper, all spaces are assumed to be at least regular and T_1. Metrizable problem occupies an important position in the study of the theory of topological spaces. In [1] it is shown that a space is metrizable if and only if it has a countable pseudo-character and has a σ-linearly hereditarily closure-preserving base. The purpose of this short note is to generalize this result by showing that a space is metrizable if and only if it has a countable pseudo-character and has a σ-countably hereditarily closure-preserving base. We also give the following counterexamples:(1) there exists a space with a σ-linearly hereditarily closure-preserving base which is not of countable pseudo-character; (2) there exists a space with a σ-countably hereditarily closure-preserving base which has not any σ-linearly hereditarily closure-preserving base.

A collection $\mathcal{P}$ of subsets of a space X is called hereditarily closure-preserving if, for each $Q(P) \subset P \in \mathcal{P}$, $\{Q(P) : P \in \mathcal{P}\}$ is closure-preserving; it is linearly hereditarily closure-preserving if $\mathcal{P}$ is endowed with a linear order $<$ such that every subcollection of $\mathcal{P}$ having an upper bound with respect to $<$ is hereditarily closure-preserving; it is countably herediatrily closure-preserving if each of its countable subcollection is hereditarily closure-preserving.

A hereditarily closure-preserving(respectively,linearly hereditarily closure-preserving, countably hereditarily closure-preserving) collection of subsets of a space is abbreviated to an HCP(respectively, LHCP,CHCP) collection. The following relations hold by [1, Lemma 3.5].

$$\sigma - \text{HCP collection} \Longrightarrow \sigma - \text{LHCP collection} \Longrightarrow \sigma - \text{CHCP collection}.$$

First, we show a positive result.

Proposition 1 *A space is a metrizable space if and only if it has a countable pseudo-character and has a σ-CHCP base.*

Proof The necessity follows directly from the classical Nagata-Smirnov metrization theorem.

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On the contrary, suppose that X has a countable pseudo-character and has a σ-CHCP base. By [2, Corollary 2.4], a space is a metrizable space if and only if it is a first countable space with a σ-CHCP base, to end the proof, it is sufficient to show that X is first countable. For each $p \in X$, if p is an isolated point of X, then p has a countable local base in X, if p is a limit point of X, then p has also a countable local base in X, since any CHCP collection of open subsets of X is point-finite at p by the proof of [3,Lemma 4]. Hence X is first countable, and accordingly it is a metrizable space. This completes the proof. □

For an uncountable set X, let p be a particular point of X. $X(p)$ is called a Fortissimo space[4] if X is endowed with a Fortissimo topology as follows: for $F \subset X$, F is closed in X if and only if $p \in F$, or $|F| \leq \aleph_0$.

Proposition 2. *Fortissimo space $X(p)$ has the following properties:*

(1) $X(p)$ is a regular and T^1 space.

(2) $X(p)$ has not any countable pseudo-character.

(3) Any collection of subsets of $X(p)$ is CHCP.

(4) For any uncountable subset A of $X(p)$, $\{\{x\} : x \in A\}$ is not HCP for $X(p)$.

Proof. (1) Since $\{\{x\} : x \in X(p) - \{p\}\} \cup \{U \subset X(p) : p \in U \text{and} \mid X - U \mid \leq \aleph_0\}$ forms an open and closed base of $X(p)$, $X(p)$ is a regular and T_1 space.

(2) For any sequence $\{G_n\}$ of open neighborhoods of the point p, we have $|X - G_n| \leq \aleph_0$ since $p \in G_n$. Then $|X - \cap\{G_n : n < \omega\}| \leq \aleph_0$, and accordingly $\cap\{G_n : n < \omega\} \neq \{p\}$. Hence the point p is not of countable pseudo-character. Therefore $X(p)$ has not any countable pseudo-character.

(3) Suppose that $\mathcal{P}$ is a collection of subsets of $X(p)$. Let $\{P_n : n < \omega\}$ be a countable subcollection of $\mathcal{P}$. For each $n < \omega$, take $Q_n \subset P_n$. Put $F = \cup\{\overline{Q_n} : n < \omega\}$. If $p \in \overline{Q_n}$ for some $n < \omega$, then $p \in F$, and F is closed in X, if $p \notin \overline{Q_n}$ for each $n < \omega$, then $|\overline{Q_n}| \leq \aleph_0$, so $|F| \leq \aleph_0$, and F is also closed in X. Hence $\mathcal{P}$ is CHCP.

(4) Since $|A| > \aleph_0, p \in \overline{A - \{p\}}$, so $A - \{p\}$ is not closed in X, and $\{\{x\} : x \in A\}$ is not HCP for $X(p)$. □

In the second part of this note, we use Fortissimo spaces to construct two counterexamples on metriztion problem.

Example 1 $\sigma-$LHCP base $\not\Rightarrow$ countable pseudo-character.

Take $X = \omega_1 + 1$. Let $<$ denote the linear order on $\omega_1 + 1$. The set X is endowed with the Fortissimo topology to form a Fortissimo space $X(\omega_1)$. $X(\omega_1)$ has not any countable pseudo-character by Proposition 2. Put

$$\mathcal{B}_1 = \{\{\alpha\} : \alpha < \omega_1\},$$

$$\mathcal{B}_2 = \{P_\alpha : \alpha < \omega_1\}, \text{ where } P_\alpha = \{\beta : \alpha \leq \beta < \omega_1 + 1\}.$$

Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a base of $X(\omega_1)$.

Define an order $<_1$ on $\mathcal{B}_1$ as follows:

$$P_\alpha <_1 P_\beta \text{ iff } \alpha < \beta.$$

Define an order $<_2$ on $\mathcal{B}_2$ as follows:

$$P_\alpha <_2 P_\beta \text{ iff } \alpha < \beta.$$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are linear orders on $<_1$ and $<_2$ respectively, and they are LHCP collections by Proposition 2. Thus $X(\omega_1)$ has a $\sigma-$LHCP base.

Example 2 $\sigma-$CHCP base $\not\Rightarrow$ $\sigma-$LHCP base.

Take $Y = \omega_2 + 1$. Let $<$ denote the linear order on $\omega_2 + 1$. The set Y is endowed with the Fortissimo topology to form a Fortissimo space $Y(\omega_2)$. $Y(\omega_2)$ has a $\sigma-$CHCP base by Proposition 2. If $Y(\omega_2)$ has a $\sigma-$LHCP base, let $\cup\{\mathcal{B}_n : n < \omega\}$ be a $\sigma-$LCHP base of $Y(\omega_2)$, where each $(\mathcal{B}_n, <_n)$ is LCHP. Since $\{\alpha\}$ is open in $Y(\omega_2)$ for each $\alpha < \omega_2$, there exists an $m < \omega$ and a subset A of $Y(\omega_2) - \{\omega_2\}$such that $|A| = \aleph_2$ and $\mathcal{A} = \{\{x\} : x \in A\} \subset \mathcal{B}_m$ Thus there is an uncountable subcollection $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{A}'$ has an upper bound with respect to $<_m$, and $\mathcal{A}'$ is HCP for $Y(\omega_2)$, a contradiction to Proposition 2.

References

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度量化定理的一个注记

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