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(mod$K$)-bases and Paracompact p-spaces

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Abstract In this paper, it is shown that a T_1 and regular space is a paracompact p-space if and only if it is a k-space with a σ-countably hereditarily closure-preserving (mod$K$)-base.

Key Words (mod$K$)-base; Hereditarily Closure-preserving Collection; Paracompact p-space; k-space

§ 1. Introduction

Throughout this paper, all spaces are assumed to be at least T_1 and regular and N denotes the set of positive integers.

Since metric spaces play an important role in a multitude of fields of mathematics, one of the main questions in studying general topology is to search for a general metrizable condition for topological spaces. It has been fruitful in the research of this field since nineteen fifties. Some important results are that a topological space is a metrizable space if and only if it satisfies one of the following conditions:

(1) X has a σ-locally finite base (Nagata-Smirnov metrization theorem, see [1],[2]);

(2) X has a σ-hereditarily closure-preserving base (Burke-Engelking-Lutzer metrization theorem, see [3]);

(3) X has countable pseudo-character and has a σ-linearly hereditarily closure-preserving base (Jiang metrization theorem, see [4]).

The perfect preimage of a metric space can be portrayed by paracompact p-spaces. Can the class of paracompact p-spaces be portrayed by open subset collections which have some particular properties? Michael[5] portrayed the class of paracompact p-spaces by the concept of (mod$K$)-bases and the concept of locally finite collections. In this paper, we obtain a weaker characterization of paracompact p-spaces in form by the concept of (mod$K$)-bases and the concept of countably hereditarily closure-preserving collections.

Definition 1.1 A collection $\mathscr{P}=\{P_\alpha: \alpha\in A\}$ of subsets of a topological space X is called

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locally finite if every $x\in X$ has a neighborhood which intersects only finitely many $P_a\in\mathscr{P}$; it is closure-preserving if, for every subset $A'\subset A$, $\overline{\bigcup\{P_a:\ \alpha\in A'\}}=\bigcup\{\overline{P}_a:\ \alpha\in A'\}$; it is hereditarily closure-preserving if, for every subset $Q_a\subset P_a$, $\{Q_a:\ \alpha\in A\}$ is closure-preserving; it is linearly hereditarily closure-preserving (see [4]) if $\mathscr{P}$ endows with a linear order $<$ such that every subcollection of $\mathscr{P}$ having an upper bound with respect to $<$ is hereditarily closure-preserving; it is countably hereditarily closure-preserving (see [6]) if every its countable subcollection is hereditarily closure-preserving.

A locally finite (closure-preserving, hereditarily closure-preserving, linearly hereditarily closure-preserving, countably hereditarily closure-preserving) collection of subsets of a space is abbreviated to an LF (CP, HCP, LHCP, CHCP) collection. By Lemma 3. 5 of [4], an LHCP collection is either a σ-HCP collection or a CHCP collection, hence

$$\sigma\text{-LF}\Rightarrow\sigma\text{-HCP}\Rightarrow\sigma\text{-LHCP}\Rightarrow\sigma\text{-CHCP}.$$

Definition 1. 2[5] A collection $\mathscr{P}$ of open subsets of a space X is called a (mod$K$)-base for X, if there is a covering $\mathscr{K}$ of X, which is composed of compact subsets and such that, if $K\subset U$ with $K\in\mathscr{K}$ and U open in X, then $K\subset P\subset U$ for some $P\in\mathscr{P}$.

§ 2. Auxiliary Propositions

Lemma 2. 1 Every space with a σ-CP (mod$K$)-base is a paracompact space.

Proof Suppose X is a space with a σ-CP (mod$K$)-base. Let $\mathscr{P}$ be a σ-CP (mod$K$)-base for X with respect to $\mathscr{K}$ which is a covering of X and is composed of compact subsets. We know that a space is paracompact if and only if every directed open cover of the space has a σ-closure-preserving refinement by closed subsets whose interiors cover the space by Theorem 3. 4 of [7]. Suppose $\mathscr{U}$ is a directed open cover of X. Then there exist $U(K)\in\mathscr{U}$ and $P(K)\in\mathscr{P}$ such that $K\subset P(K)\subset\overline{P(K)}\subset U(K)$ for each $K\in\mathscr{K}$ by the regularity of X. Obviously, $\{\overline{P(K)}:\ K\in\mathscr{K}\}$ is a σ-closure-preserving refinement of $\mathscr{U}$ by closed subsets whose interiors cover X, and so X is a paracompact space.

Lemma 2. 2[8] If $\{P_a:\ \alpha\in A\}$ is an HCP collection for X, then $\{\overline{P}_a:\ \alpha\in A\}$ is also an HCP collection for X.

Lemma 2. 3 If X is a k-space, then a CHCP collection for X is an HCP collection.

Proof Let $\mathscr{P}$ be a CHCP collection for X. We can assume that $\mathscr{P}$ is a collection of closed subsets for X by Lemma 2. 2. Put

$$Y=\{y\in X:\ \mathscr{P}\text{ is not point finite at }y\}.$$

Then $K\cap Y$ is finite for each compact subset K of X. In fact, if there exists $\{y_n:\ n\in N\}\subset K\cap Y$, by induction principle we can take $P_1\in\mathscr{P}$, $P_{n+1}\in\mathscr{P}-\{P_i:\ i\leqslant n\}$ such that $y_n\in P_n$ for each $n\in N$ because $\mathscr{P}$ is not point finite at each y_n. So $\{y_n:\ n\in N\}$ is an infinite discrete subset of K, a contradiction.

Denote $\mathscr{P}$ by $\{P_a:\ \alpha\in A\}$. If $\mathscr{P}$ is not an HCP collection, then there are $A'\subset A$ and a non-empty closed subset Q_α of P_α for each $\alpha\in A'$ such that $\bigcup\{Q_\alpha: \alpha\in A'\}$ is not closed in X, and so $K\cap(\bigcup\{Q_\alpha: \alpha\in A'\})$ is not closed in K for some compact subset K of X because X is a k-space. For each $B\subset A'$, put

$$R(B) = K \cap (\bigcup \{Q_\alpha: \alpha \in A' - B\}).$$

Then

$$K \cap (\bigcup \{Q_\alpha: \alpha \in A'\})$$
$$= (R(B) - Y) \cup (R(B) \cap Y) \cup (K \cap (\bigcup \{Q_\alpha: \alpha \in B\})).$$

If B is a finite subset of A', then $R(B)-Y$ is not closed in K, and so it is infinite. Take $z_1\in R(\phi)-Y$ and put

$$A_1 = \{\alpha \in A': z_1 \in Q_\alpha\}.$$

Then A_1 is a non-empty finite subset of A'. Take $z_2\in R(A_1)-Y$ and put

$$A_2 = \{\alpha \in A' - A_1: z_2 \in Q_\alpha\}.$$

Then A_2 is a non-empty finite subset of A', $A_1\cap A_2=\phi$, and $z_1\neq z_2$. By induction principle we can construct a subset $\{z_n: n\in N\}$ of X and a non-empty sequence $\{A_n\}$ of finite subsets of A' with $z_n\in\bigcap\{Q_\alpha: \alpha\in A_n\}$ for each $n\in N$, where $A_n\cap A_m=\phi$ and $z_n\neq z_m$ when $n\neq m$. Take $\alpha_n\in A_n$ for each $n\in N$. Then $z_n\in Q_{\alpha_n}\subset P_{\alpha_n}$, and so $\{z_n: n\in N\}$ is an infinite discrete subset of the compact subset K of X for $\{P_{\alpha_n}: n\in N\}$ is HCP, a contradiction. Therefore $\mathscr{P}$ is an HCP collection of subset of X.

Corollary 2. 4 A space is a metrizable space if and only if it is a k-space with a σ-CHCP base.

Proof Necessity comes from the classic Nagata-Smirnov metrization theorem. By Lemma 2. 3 and Burke-Engelking-Lutzer metrization theorem, sufficiency is obvious.

§ 3. Characterizations of Paracompact p-spaces

Theorem 3. 1 The following properties of a space are equivalent:

(1) X is a paracompact p-space;

(2) X is a space with a σ-LF (mod$K$)-base;

(3) X is a k-space and it satisfies any of the following conditions:

(a) X has a σ-HCP (mod$K$)-base;

(b) X has a σ-LHCP (mod$K$)-base;

(c) X has a σ-CHCP (mod$K$)-base.

Proof In [5] Michael related that conditions (1) and (2) are equivalent, but he did not give proof.

(1)$\Rightarrow$(2): Suppose X is a paracompact p-space. Then there are a metric space Y and a perfect mapping $f: X\to Y$. Y has a σ-LF base $\mathscr{B}$ for Y is a metric space. Put

$$\mathscr{K} = \{f^{-1}(y): y \in Y\},$$
$$\mathscr{P} = \{f^{-1}(B): B \in \mathscr{B}\}.$$

Then $\mathscr{K}$ is a covering of X and is composed of compact subsets, and $\mathscr{P}$ is a σ-LF collection of open subsets of X. If $f^{-1}(y)\subset U$ with $y\in Y$ and U open in X, then $y\in Y-f(X-U)$, and so $y\in B\subset Y-f(X-U)$ for some $B\in\mathscr{B}$, hence $f^{-1}(y)\subset f^{-1}(B)\subset U$. Thus $\mathscr{P}$ is a (mod$K$)-base of X with respect to $\mathscr{K}$.

(2)⇒(3): Suppose X has a σ-LF (mod$K$)-base. It is sufficient to show that X is a k-space. Let $\mathscr{P}$ be a σ-LF (mod K)-base of X with respect to $\mathscr{K}$ which is a covering of X and is composed of compact subsets. For each $x\in X$, there exists $K\in\mathscr{K}$ such that $x\in K$. Put

$$\mathscr{P}(K)=\{P\in\mathscr{P}:\ K\subset P\}.$$

Then $\mathscr{P}(K)$ is countable. If $K\subset U$ with U open in X, then $K\subset P\subset U$ for some $P\in\mathscr{P}$; i. e., $K\subset P\subset U$ for some $P\in\mathscr{P}(K)$. So K has countable character. Thus we have shown that if $x\in X$, there exists a compact subset K of X such that $x\in K$ and K has countable character in X; i. e., X is of pointwise countable type. Therefore X is a k-space by 3. 3. I of [9].

(3)⇒(1): By Lemma 2. 3, it is sufficient to show that if X is a k-space with a σ-HCP (mod$K$)-base, then X is a paracompact p-space. Since paracompact p-spaces are equivalent to paracompact $\omega\Delta$-spaces by [10], it is sufficient to show that X is a $\omega\Delta$-space by Lemma 2. 1; i. e., X has a sequence $\{\mathscr{U}_n\}$ of open covers such that if $x_n\in st(x,\mathscr{U}_n)$ for each $n\in N$, then the sequence $\{x_n\}$ has a cluster point in X.

Let $\mathscr{P}$ be a σ-HCP (mod$K$)-base of X with respect to $\mathscr{K}$ which is a covering of X and is composed of compact subsets. Denote $\mathscr{P}$ by $\bigcup\{\mathscr{P}_n:\ n\in N\}$, where each $\mathscr{P}_n$ is an HCP collection of open subsets of X. We can assume that $X\in\mathscr{P}_n$ for each $n\in N$. For each $m\in N$, $P\in\mathscr{P}$, put

$$F_m(P)=\bigcup\{\overline{Q}:\ Q\in\mathscr{P}_m,\ \overline{Q}\subset P\}.$$

Then $F_m(P)$ is closed in X and $F_m(P)\subset P$. For each $m,k\in N$, put

$$\mathscr{F}_{m,k}=\{F_m(P):\ P\in\mathscr{P}_k\},$$
$$\mathscr{R}_{m,k}=\{R_{m,k}(x):\ x\in X\},$$

where

$$R_{m,k}(x)=\bigcap\{P\in\mathscr{P}_k:\ x\in F_m(P)\}$$
$$-\bigcup\{F_m(P):\ P\in\mathscr{P}_k,\ x\notin F_m(P)\}.$$

Then $\mathscr{F}_{m,k}$ is a CP collection of closed subsets of X, $\mathscr{R}_{m,k}$ is a covering of open subsets of X, and if $P\in\mathscr{P}_k$ and $x\in F_m(P)$, then $st(x,\ \mathscr{R}_{m,k})\subset P$. In fact, it is obvious that $\mathscr{F}_{m,k}$ is a CP collection of closed subsets of X. Since X is a k-space, the intersection of an HCP collection of open subsets of X is still open in X by Proposition 7 of [3]. Clearly $x\in R_{m,k}(x)$. So $\mathscr{R}_{m,k}$ is an open covering of X. Suppose $P\in\mathscr{P}_k$ and $x\in F_m(P)$. If $x\in R_{m,k}(y)$, then $y\in F_m(P)$ (otherwise, $x\in R_{m,k}(y)\cap F_m(P)\subset(X-F_m(P))\cap F_m(P)=\phi$, a contradiction), and so $R_{m,k}(y)\subset P$, $st(x,\mathscr{R}_{m,k})\subset P$.

Now, for each $n\in N$, put

$$\mathscr{U}_n=\bigwedge\{\mathscr{R}_{m,k}:\ m,k\leqslant n\}.$$

Then $\{\mathscr{U}_n\}$ is a sequence of open coverings of X. Suppose $x_n\in st(x,\mathscr{U}_n)$ for each $n\in N$, and the sequence $\{x_n\}$ has no cluster point in X. Then $\{x_n:\ n\in N\}$ is a closed discrete subset of X. Take $K\in\mathscr{K}$ with $x\in K$. Thus $K\cap\{x_n:\ n\in N\}$ is finite, and so there is $i\in N$ such that $K\cap\{x_n:\ n\geqslant i\}=\phi$; i. e. , $K\subset X-\{x_n:\ n\geqslant i\}$. Hence there exist $k\in N$ and $P\in\mathscr{P}_k$ such that $K\subset P\subset X-\{x_n:\ n\geqslant i\}$ because $X-\{x_n:\ n\geqslant i\}$ is open in X. Since X is a regular space, $K\subset Q\subset\overline{Q}\subset P$ for some $m\in N$ and $Q\in\mathscr{P}_m$, and so $x\in K\subset F_m(P)$. Take $j\geqslant\max\{k,m,i\}$. Then

$$x_j\in st(x,\mathscr{U}_j)\subset st(x,\mathscr{R}_{m,k})\subset P\subset X-\{x_n:\ n\geqslant i\},$$

a contradiction. Therefore the sequence $\{x_n\}$ has a cluster point in X, and X is a $\omega\Delta$-space. This completes the proof of the Theorem.

References

[1] Nagata, J. , On a necessary and sufficient condition of metrizability, *J. Inst. Polytech. Osaka City Univ. Ser A, Math.* , **1**(1950), 93—100.

[2] Smirnov, Ju. M. , A necessary and sufficient condition for metrizability of a topological space (in Russian), *Dokl. Akad. Nauk. SSSR*, **77**(1951), 197—200.

[3] Burke, D. K. , Engelking, R. , Lutzer, D. , Hereditarily closure-preserving collections and metrization, *Proc. Amer. Math. Soc.* , **51**(1975), 483—488.

[4] Jiang Jiguang, On paracompactness and metrizability of topological spaces (in Chinese), *Acta Math. Sinica*, **29**(1986), 697—701.

[5] Michael, E. , On Nagami's Σ-spaces and some related matters, *Proc. Washington State Univ. Conf.* , (1969), 1—7.

[6] Gao Guoshi, On the closure-preserving sum theorems (in Chinese), *Acta Math. Sinica*, **29**(1986), 58—62.

[7] Junnila, H. J. K. , Metacompactness, paracompactness, and interior-preserving open covers, *Trans. Amer. Math. Soc.* , **249**(1979), 373—385.

[8] Lin Shou, On a problem of K. Tamano, *Questions and Answers Gen. Top.* , **6**(1988), 99—102.

[9] Engelking, R. , General Topology, PWN, Warszawa, 1977.

[10] Burke, D. K. , On p-spaces and $\omega\Delta$-spaces, *Pacific J. Math.* , **35**(1970), 285—296.

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