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Note on k_R-spaces

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Throughout this paper, all spaces are assumed to be at least T_1 and regular.

1 Spaces with σ-HCP (modK)-bases

Since metric spaces play an important role in a multitude of the fields of mathematics, one of questions for study of general topology is to search for a general metrizable condition for topological spaces. Where some important results are following: A space X is a metrizable space if and only if it satisfies any of the following:

(1) X has a σ-discrete base(Bing metrization theorem).

(2) X has a σ-locally finite base(Nagata-Smirnov metrization theorem).

(3) X has a σ-hereditarily closure-preserving base

(Burke-Engelking-Lutzer metrization theorem).

The perfect preimage of metric spaces can be characterized by paracompact p-spaces. Can the class of paracompact p-spaces be characterized by open subset families which has some particular properties? E. Michael[5] characterized the class of paracompact p-spaces by the that of spaces with a σ-locally finite (modK)-base. It suggests the following question: Is a space with a σ-hereditarily closure-preserving (modK)-base a paracompact p-space? In[4], we proved that if condition of k-space was added, its answer is affirmative. In this section, we prove that condition of k-space can be omitted.

Definition 1.1 Call a collection $\mathcal{P}=\{P_\alpha : \alpha\in A\}$ of subsets of a space X hereditarily closure-preserving(abbre. HCP) if, for every subset $Q_\alpha \subset P_\alpha$, $\{Q_\alpha : \alpha\in A\}$ is closure-preserving.

Definition 1.2 Call a collection $\mathcal{P}$ of open subsets of a space X a (modK)-base for X, if there is a covering $\mathcal{K}$ of X by compact subsets such that, whenever $K\subset U$ with $K\in\mathcal{K}$ and U open in X, then $K\subset P\subset U$ for some $P\in\mathcal{P}$.

Theorem 1.3 The following properties of a space are equivalent:

(1) X is a paracompact p-space.

(2) X has a σ-discrete (modK)-base.

(3) X has a σ-locally finite (modK)-base.

(4) X has a σ-HCP (modK)-base.

Proof. (1)$\Rightarrow$(2). Suppose X is a paracompact p-space, then there are a metric space Y and a perfect mapping f from X onto Y. Y has a σ-discrete base $\mathcal{B}$. Put

$$\mathcal{K} = \{f^{-1}(y) : y \in Y\},$$

$$\mathcal{P} = \{f^{-1}(B) : B \in \mathcal{B}\},$$

then $\mathcal{P}$ is a σ-discrete (modK)-base of X with respect to $\mathcal{K}$.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are obvious.

(4)$\Rightarrow$(1). Suppose X is a space with a σ-HCP (modK)-base. Let $\mathcal{P}$ be a σ-HCP (modK)-base of X with respect to $\mathcal{K}$ which is a covering of X by compact subsets. It is sufficient to show that X is a k-space because a k-space with a σ-HCP (modK)-base is a paracompact p-space by [4, Theorem 3.1] .

For each $K \in \mathcal{K}$, we prove that K has countable character in X. Suppose K is not an open subset in X.

Let $q: X \to X/K$ be a natural quotient mapping. Put

$\mathcal{P}' = \{P \in \mathcal{P} : K \subset P\}$,

$\mathcal{B} = \{q(P) : P \in \mathcal{P}'\}$,

then $\mathcal{B}$ is a σ-HCP base of neighborhood of point q(K) in X/K, so $\mathcal{B}$ is countable by [2, Theorem 6], and $\mathcal{P}'$ is countable; i.e., K has countable character in X. Hence X is a space of pointwise countable type. This implies that X is a k-space. This completes the proof of the theorem.

Corollary 1.4 A space X is metrizable if and only if it has a σ-HCP base.

Proof. That every metric space has a σ-HCP base follows directly from the Nagata-Smirnov metrization theorem. If a space X has a σ-HCP base, X is a paracompact p-space by Theorem 1.3. Since X is a σ-space, it has a G_δ-diagonal, hence X is metrizable.

2 k_R-spaces with star-countable compact-k-networks

Definition 2.1 Call a collection $\mathcal{P}$ of subsets of a space X star-countable if for any $P \in \mathcal{P}$ the collection $\{Q \in \mathcal{P} : Q \cap P \neq \emptyset\}$ is countable. $\mathcal{P}$ is said to be a k-network for X, if $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite

$\mathcal{P}' \subset \mathcal{P}$. Call $\mathcal{P}$ a compact-k-network for X, if $\mathcal{P}$ is a k-network for X and every element of $\mathcal{P}$ is compact in X.

Definition 2.2 A topological space X is called a k_R-space [6] if it is completely regular and if every f: $X \to R$, whose restriction to every compact $K \subset X$ is continuous, is continuous on X.

Clearly every completely regular k-space is a k_R-space. The converse is false [6]. E. Michael [6] proved that a k_R-space with countable compact-k-networks is a k-space.

In [3], we studied characterization of the images of locally compact metric spaces under R-quotient, ss-mappings, and obtained the following result.

Theorem 2.3 (1) A space X is a k_R-space with a locally countable compact-k-network if and only if X is the image of a locally compact metric space under a (compact-covering) R-quotient, ss-mapping.

(2) A space X is a k-space with a locally countable compact-k-network if and only if X is the image of a

locally compact metric space under a (compact-covering) quotient, ss-mapping.

Here a continuous mapping $f:X \to Y$ of X onto Y is called an R-quotient mapping if for every mapping $g:Y \to R$ the continuity of gf implies the continuity of g. A continuous mapping $f:X \to Y$ of X onto Y is called a ss-mapping if for every $y \in Y$, there is a neighborhood V of y in Y such that $f^{-1}(V)$ is a separable subspace of X.

In view of the above theorem, the following question is posed in[3]: Is a k_R-space with a locally countable compact-k-network a k-space? In this section, this question is answered affirmatively by the following theorem.

Theorem 2.4 If k_R-space X has a star-countable compact-k-network, then X is a k-space with a locally countable compact-k-network and a σ-discrete compact-k-network.

Proof. Suppose $\mathcal{P}$ is a star-countable compact-k-network for X, where X is a k_R-space. By [1, Lemma 3.10], $\mathcal{P}$ can be expressed as $\mathcal{P} = \bigcup \{\mathcal{P}_a : a \in A\}$ where each subcollection $\mathcal{P}_a$ is countable and $(\bigcup \mathcal{P}_a) \cap (\bigcup \mathcal{P}_b) = \emptyset$ whenever $a \neq b$. For each $a \in A$, put $X_a = \bigcup \mathcal{P}_a$. We assert that X_a is open and closed in X. In fact, let $f: X \to R$ with $f(X_a) = \{0\}$, $f(X \setminus X_a) = \{1\}$. For arbitrary non-empty compact subset K of X, there is a finite $\mathcal{P}' \subset \mathcal{P}$ with $K \subset \bigcup \mathcal{P}'$. For each $b \in A$, put

$$\mathcal{P}_b' = \mathcal{P}' \cap \mathcal{P}_b ,$$

$$K_b = \bigcup \mathcal{P}_b' ,$$

$$A' = \{b \in A : K_b \neq \emptyset\} .$$

Then K_b is compact in X, A' is finite and $K \subset \bigcup\{K_b : b \in A'\}$. By the definition of f, $f|_{K_b}$ is a constant mapping, so f is continuous on $\bigcup\{K_b : b \in A'\}$. Consequently f is continuous on K. Hence f is continuous on X because X is a k_R-space. Thus X_a is open and closed in X, and $X = \oplus\{X_a : a \in A\}$. As $\{X_a : a \in A\}$ is an open cover in X, and $\mathcal{P}_a$ is countable, $\mathcal{P}$ is a locally countable compact-k-network and a σ-discrete compact-k-network for X. Since X is a k_R-space, its open and closed subspace X_a is a k_R-space. By the construction of X_a , $\mathcal{P}_a$ is a countable compact-k-network of X_a, then X_a is a k-space because a k_R-space with a countable k-network if it is σ-compact, then it is a k-space[6]. And X is a k-space.

By Theorem 2.3 and Theorem 2.4, we obtain an interesting corollary as follows.

Corollary 2.5 The following properties of a space are equivalent:

(1) X is an image of a locally compact metric space under a (compact-covering) R-quotient, ss-mapping.

(2) X is an image of a locally compact metric space under a (compact-covering) quotient, ss-mapping.

(3) X is a k-space or a k_R-space satisfying any of the following:

(a) X has a σ-discrete compact-k-network.

(b) X has a σ-locally finite compact-k-network.

(c) X has a locally countable compact-k-network.

(d) X has a σ-locally countable compact-k-network.

(e) X has a star-countable compact-k-network.

The author thanks to Professor Y. Tanaka for his following result related with Theorem 2.4.

Result. The following are equivalent.

(1) X is a k_R-space having a star-countable cover $\mathcal{P}$ of compact subsets, where for each compact subset

K of X, $K \subset \bigcup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

(2) X is the topological sum of k_ω-spaces, here a space Z is k_ω if Z is the union of countably many compact subsets Z_n such that a subset $F \subset Z$ is closed in Z whenever $F \cap Z_n$ is closed in Z_n .

Proof. (2)$\Rightarrow$(1) is obvious. Suppose that (1) holds. Then, in view of the proof of Theorem 2.4, X is the topological sum of k_R-spaces X_a, $a \in A$, where each X_a has a countable cover $\mathcal{P}_a$ of compact subsets such that each compact subset of X_a is contained in a finite union of elements of $\mathcal{P}_a$. (Such a space X_a is sometimes called hemicompact). Then each X_a is k_ω-space by a comment (p. 76) after the proof of Theorem 5.2.1 in[7]. Thus X is the topological sum of k_ω-spaces.

References

1. D. Burke, Covering properties, In: K. Kunen, J. Vaughan, eds., Handbook of Set-Theoretic Topology, Amsterdam, 1984, 347-422.
2. D. Burke, R. Engelking, D. Lutzer, Hereditarily closure-preserving collections and metrization, Proc. Amer. Math. Soc., 51(1975), 483-488.

3. S. Lin, On R-quotient, ss-mappings, Acta Math. Sinica, 34(1991), 7-11(Chinese).

4. S. Lin, (modK)-bases and paracompact p-spaces, Northeastern Math. J.(P. R. C.), 7(1991)

5. E. Michael, On Nagami's Σ-spaces and some related matters, Proc. Washington State Univ. Conf., 1969, 1-7.

6. E. Michael, On k-spaces, k_R-spaces and k(X), Pacific J. Math., 47(1973), 487-498.

7. R. A. McCoy, I. Ntantu, Topological properties of spaces of continuous functions, Lecture notes in Mathematics, Springer-Verlag, 1315, 1988.

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