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COPIES OF SPECIAL SPACES IN FREE (ABELIAN) PARATOPOLOGICAL GROUPS

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Abstract. Let $FP(X)$ ($AP(X)$) denote the free paratopological group (free Abelian paratopological group) over a topological space X. For every non-negative integer n, denote by $FP_n(X)$ ($AP_n(X)$) the subspace of $FP(X)$ ($AP(X)$) that consists of all words of reduced length $\leq n$ with respect to the free basis X. In this paper, a homeomorphism theorem for the free Abelian paratopological group over a topological space X is established, i.e., the subspace $AP_n(X)\setminus AP_{n-1}(X)$ of $AP(X)$ is homeomorphic to a subspace of the n-th symmetric power $(X \oplus -X_d)^n/S_n$ for every positive integer n, which extends a result of A. Arhangel'skiĭ. As an application, it is shown that if X is a Tychonoff space and P is a densely self-embeddable prime space with a q-point, then $AP(X)$ contains a copy of P if and only if $FP(X)$ contains a copy of P if and only if X contains a copy of P, which generalizes a theorem of K. Eda, H. Ohta and K. Yamada. At last, it is shown that if the free paratopological group $FP(X)$ (the free Abelian paratopological group $AP(X)$) over a Tychonoff space X contains a non-trivial convergent sequence, then $FP(X)$ ($AP(X)$) contains a closed copy of S_2. Further, if the free paratopological group $FP(X)$ or the free Abelian paratopological group $AP(X)$ over a Tychonoff space X is a Fréchet space, then X is discrete, which gives an affirmative answer to a question in literature.

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1. Introduction

A topological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous and the inversion mapping of G on itself is also continuous. A paratopological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. The absence of the continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups [4]. In 1941, free topological groups in the sense of A. Markov were introduced [14]. Copies of some special spaces, for instance, densely self-embeddable prime spaces, Arens' space S_2 and sequential spaces, in free topological groups were investigated in [7, 12, 16]. In 2002, S. Romaguera, M. Sanchis, and M. Tkachenko [18] introduced free (Abelian) paratopological groups on arbitrary topological spaces and discussed their topological properties. Our main motivation to do this work arises from [4, Open Problem 7.4.4], posed by A. Arhangel'skiĭ and M. Tkachenko. This guides us to discuss which important results on free topological groups can be generalized to free paratopological groups. Around this subject, some publications about free (Abelian) paratopological groups have emerged, for example, see [8, 9, 11, 13, 17] etc.

In [2, 3], A. Arhangel'skiĭ proved the following theorem.

Theorem 1.1. *Let $F(X)$ $(A(X))$ denote the free topological group (free Abelian topological group) over a Tychonoff space X. For every non-negative integer n, denote by $F_n(X)$ $(A_n(X))$ the subspace of $F(X)$ $(A(X))$ that consists of all words of reduced length $\leq n$ with respect to the free basis X. Then for every positive integer n,*

(1) the subspace $A_n(X) \setminus A_{n-1}(X)$ of $A(X)$ is homeomorphic to a subspace of the n-th symmetric power $(X \oplus -X)^n/S_n$ [2];

(2) the subspace $F_n(X) \setminus F_{n-1}(X)$ of $F(X)$ is homeomorphic to a subspace of $\widetilde{X}^n$, where $\widetilde{X}$ is the topological sum $X \oplus \{e\} \oplus X^{-1}$ [3].

Not long ago, item (2) of Theorem 1.1 was generalized to free paratopological groups, i.e., the following homeomorphism theorem for free paratopological groups was proved in [9, 13].

Theorem 1.2. *Let X be a topological space and $FP(X)$ be the free paratopological group over X. Then for every positive integer n, the subspace $FP_n(X) \setminus FP_{n-1}(X)$ of $FP(X)$ is homeomorphic to a subspace of $\widetilde{X}^n$, where $\widetilde{X}$ is the topological sum $X \oplus \{e\} \oplus X_d^{-1}$ and $FP_n(X)$ is the subspace of $FP(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X.*

Therefore, it is very natural to ask whether item (1) of Theorem 1.1 can be generalized to free Abelian paratopological groups. In this paper, we shall give an affirmative answer to this question, i.e., we shall show that the subspace $AP_n(X)\setminus AP_{n-1}(X)$ of $AP(X)$ is homeomorphic to a subspace of the n-th symmetric power $(X \oplus -X_d)^n/S_n$ for every positive integer n, where $AP_n(X)$ is the subspace of $AP(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X. As an application, it is shown that if X is a Tychonoff space and P is a densely self-embeddable prime space with a q-point, then $AP(X)$ contains a copy of P if and only if $FP(X)$ contains a copy of P if and only if X contains a copy of P, which extends [7, Theorem 2.6] to free (Abelian) paratopological groups.

It is well known that if the free topological group $F(X)$ over a Tychonoff space X is Fréchet, then the space X is discrete [12, 16]. The authors of [6] asked whether the space X is discrete or not if the free paratopological group $FP(X)$ or the free Abelian paratopological group $AP(X)$ over a Tychonoff space X is Fréchet [6, Question 5.9]. In this paper, we shall prove that if the free paratopological group $FP(X)$ (the free Abelian paratopological group $AP(X)$) over a Tychonoff space X contains a non-trivial convergent sequence, then $FP(X)$ ($AP(X)$) contains a closed copy of S_2. Further, if the free paratopological group $FP(X)$ (the free Abelian paratopological group $AP(X)$) over a Tychonoff space X is a sequential space and contains no closed copy of S_2, more generally, a Fréchet space, then X is discrete, which gives an affirmative answer to [6, Question 5.9].

2. Preliminary facts about free (Abelian) paratopological groups

Definition 2.1. (See [18]) Let X be a subspace of a paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X\rangle = G$; and

(2) every continuous mapping $f : X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group* (briefly, *free paratopological group*) on X and is denoted by $FP(X)$.

If all groups in the above definition are Abelian, we obtain the definition of *Markov free Abelian paratopological group* (briefly, *free Abelian paratopological group*) on X, which is denoted by $AP(X)$.

In the paper, $F_a(X)$ ($A_a(X)$) algebraically denotes the free group (free Abelian group) on a non-empty set X and e (0) is the identity of $F_a(X)$ ($A_a(X)$). The set X is called the free basis of $F_a(X)$ ($A_a(X)$). Here are some details, for instance, see [4]. Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where

$x_1, ..., x_n \in X$ and $\epsilon_1, ..., \epsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say in this case that the length $l(g)$ of g equals n. Every element $g \in F_a(X)$ distinct from the identity e can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$ and $x_i \neq x_{i+1}$ for every $i = 1, ..., n-1$. Similar assertions (with the obvious changes for commutativity) are valid for $A_a(X)$.

Remark 2.2. It is known that the topology of $FP(X)$ ($AP(X)$) is the finest paratopological group topology on the group $F_a(X)$ ($A_a(X)$) which induces the original topology on X [18].

For every non-negative integer n, denote by $FP_n(X)$ ($AP_n(X)$) the subspace of the free paratopological group $FP(X)$ ($AP(X)$) that consists of all words of reduced length $\leq n$ with respect to the free basis X.

Remark 2.3. If X is a T_1-space, then $FP(X)$ is also T_1, X^{-1} is closed and discrete, which is denoted by X_d^{-1}, and the subspaces X and $FP_n(X)$ of $FP(X)$ are all closed in $FP(X)$ for every non-negative integer n [8]. The same is true for $AP(X)$ [17].

In what follows, all topological spaces are assumed to be T_1. The set of positive integers is denoted by $\mathbb{N}$. For unexplained terminology the reader may consult [4, 10].

3. A homeomorphism theorem for free Abelian paratopological groups and its applications

First of all, let us recall the definition of a symmetric product space (see [5]).

Definition 3.1. Let X be a topological space and S_n be the set of all permutations on $n = \{0, ..., n-1\}$, where n is an arbitrary positive integer. Define an equivalence relation $\sim$ on the set X^n as follows.

For arbitrary $u = (u(0), ..., u(n-1)), v = (v(0), ..., v(n-1)) \in X^n$, let $u \sim v$ if and only if there exists a permutation $\varphi \in S_n$ such that $u(k) = v(\varphi(k))$ for every $k < n$.

Denote by X^n/S_n the set of all equivalence classes of $\sim$ and by q the mapping of X^n to X^n/S_n assigning to the point $u \in X^n$ the equivalence class $[u] \in X^n/S_n$. Put

$$\tau = \{O \subset X^n/S_n : q^{-1}(O) \text{ is open in } X^n\}.$$

The quotient space $(X^n/S_n, \tau)$ is called the *n-th symmetric power* of X.

It is not difficult to check from Definition 3.1 that for every point $[(x_0, ..., x_{n-1})] \in X^n/S_n$,

$$\{[U_0 \times \cdots \times U_{n-1}] : U_i \text{ is a neighbourhood of } x_i \text{ in } X, 0 \leq i \leq n-1\}$$

is a neighbourhood base at the point $[(x_0, ..., x_{n-1})]$ in the space X^n/S_n, where

$$[U_0 \times \cdots \times U_{n-1}] = \{[(y_0, ..., y_{n-1})] : y_i \in U_i, 0 \leq i \leq n-1\}.$$

Lemma 3.2. [8, Theorem 4.11] *Let X be a topological space and $w = \epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$ be a reduced word in $AP_n(X)$, where $x_i \in X$, $\epsilon_i = \pm 1$ for $i = 1, 2, ..., n$. Let $\mathcal{B}$ denote the collection of all sets of the form $\epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n$, where for $i = 1, 2, ..., n$, the set U_i is a neighbourhood of x_i in X when $\epsilon_i = 1$ and $U_i = \{x_i\}$ when $\epsilon_i = -1$. Then $\mathcal{B}$ is a neighbourhood base at the point w in the subspace $AP_n(X)$ of $AP(X)$.*

Now, we can establish a homeomorphism theorem for free Abelian paratopological groups to generalize item (1) of Theorem 1.1.

Theorem 3.3. *Let X be a topological space. Then for every positive integer n, the subspace $AP_n(X) \setminus AP_{n-1}(X)$ of $AP(X)$ is homeomorphic to a subspace of the n-th symmetric power $(X \oplus -X_d)^n/S_n$.*

PROOF. Define $f : (X \oplus -X_d)^n/S_n \to AP_n(X)$ by

$$f([(\epsilon_0 x_0, ..., \epsilon_{n-1} x_{n-1})]) = \epsilon_0 x_0 + \cdots + \epsilon_{n-1} x_{n-1}$$

for every $[(\epsilon_0 x_0, ..., \epsilon_{n-1} x_{n-1})] \in (X \oplus -X_d)^n/S_n$, where $x_i \in X$, $\epsilon_i = \pm 1$ for $i = 0, ..., n-1$. It is easy to see that f is well defined.

Let $C_n(X) = AP_n(X) \setminus AP_{n-1}(X)$.

Claim 1. The restriction mapping $f|_{f^{-1}(C_n(X))} : f^{-1}(C_n(X)) \to C_n(X)$ is continuous.

Let $[(\epsilon_0 x_0, ..., \epsilon_{n-1} x_{n-1})] \in f^{-1}(C_n(X))$ and $V \cap C_n(X)$ be an open neighbourhood of the point $\epsilon_0 x_0 + \cdots + \epsilon_{n-1} x_{n-1}$ in $C_n(X)$, where V is open in $AP_n(X)$. By Lemma 3.2, for $i = 0, ..., n-1$, there exists a set $U_i \subset X$ such that

$$\epsilon_0 U_0 + \cdots + \epsilon_{n-1} U_{n-1} \subset V,$$

where U_i is a neighbourhood of x_i in X when $\epsilon_i = 1$, and $U_i = \{x_i\}$ when $\epsilon_i = -1$. Thus

$$f([\epsilon_0 U_0 \times \cdots \times \epsilon_{n-1} U_{n-1}] \cap f^{-1}(C_n(X))) \subset V \cap C_n(X).$$

Since $[\epsilon_0 U_0 \times \cdots \times \epsilon_{n-1} U_{n-1}]$ is a neighbourhood of $[(\epsilon_0 x_0, ..., \epsilon_{n-1} x_{n-1})]$ in the space $(X \oplus -X_d)^n/S_n$, the mapping $f|_{f^{-1}(C_n(X))}$ is continuous.

Claim 2. The restriction mapping $f|_{f^{-1}(C_n(X))} : f^{-1}(C_n(X)) \to C_n(X)$ is a homeomorphism.

In order to prove Claim 2, it suffices to verify following subclaims 2.1, 2.2 and 2.3.

Subclaim 2.1. The restriction $f|_{f^{-1}(C_n(X))}$ is surjective.

For an arbitrary $w = \epsilon_0 x_0 + \cdots + \epsilon_{n-1}x_{n-1} \in C_n(X)$, it is obvious that

$$[(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})] \in f^{-1}(C_n(X))$$

and

$$f([(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})]) = w.$$

Hence $f|_{f^{-1}(C_n(X))}$ is surjective.

Subclaim 2.2. The restriction $f|_{f^{-1}(C_n(X))}$ is injective.

For arbitrary two points

$$[(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})], [(\eta_0 y_0, ..., \eta_{n-1}y_{n-1})] \in f^{-1}(C_n(X)),$$

if

$$f([(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})]) = f([(\eta_0 y_0, ..., \eta_{n-1}y_{n-1})]),$$

then

$$\epsilon_0 x_0 + \cdots + \epsilon_{n-1}x_{n-1} = \eta_0 y_0 + \cdots + \eta_{n-1}y_{n-1}.$$

Thus, by the construction of free Abelian paratopological group $AP(X)$, there exists a permutation $\varphi \in S_n$ such that $\epsilon_k x_k = \eta_{\varphi(k)} y_{\varphi(k)}$ for every $k < n$,

$$(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1}) \sim (\eta_0 y_0, ..., \eta_{n-1}y_{n-1}),$$

and

$$[(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})] = [(\eta_0 y_0, ..., \eta_{n-1}y_{n-1})].$$

This shows that $f|_{f^{-1}(C_n(X))}$ is injective.

Subclaim 2.3. The restriction $f|_{f^{-1}(C_n(X))}$ is an open mapping.

Let O be an arbitrary open subset of $f^{-1}(C_n(X))$. We shall show that $f(O)$ is open in $C_n(X)$. Suppose $[(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})] \in O$. Pick an open subset V of $(X \oplus -X_d)^n/S_n$ such that

$$O = V \cap f^{-1}(C_n(X)).$$

Then

$$f(O) = f(V) \cap C_n(X).$$

For $i = 0, ..., n-1$, there exists a set $U_i \subset X$ such that

$$[\epsilon_0 U_0 \times \cdots \times \epsilon_{n-1}U_{n-1}] \subset V,$$

where U_i is a neighbourhood of x_i in X when $\epsilon_i = 1$, and $U_i = \{x_i\}$ when $\epsilon_i = -1$. Since X is a T_1-space, we can assume that if $\epsilon_i = 1$ and $\epsilon_j = -1$, then $x_j \notin U_i$. Therefore,

$$\epsilon_0 x_0 + \cdots + \epsilon_{n-1}x_{n-1} \in \epsilon_0 U_0 + \cdots + \epsilon_{n-1}U_{n-1} \subset f(O).$$

By Lemma 3.2, $f(O)$ is a neighbourhood of $f([(\epsilon_0 x_0, ..., \epsilon_{n-1}x_{n-1})])$ in $C_n(X)$, i.e., $f(O)$ is open in $C_n(X)$. □

As an application of Theorem 3.3, we may obtain an embedding theorem for self-embeddable prime spaces in free Abelian paratopological groups on Tychonoff spaces. Let us recall some related concepts.

Definition 3.4. (See [4]) A topological space X is called *densely self-embeddable* if every non-empty open set in X contains a copy of X.

Obviously, if a space X contains at least two points and is densely self-embeddable, then X has no isolated points.

Definition 3.5. (See [4]) A topological space P is said to *prime* if $X \times Y$ contains a copy of P, then either X or Y contains a copy of P for any spaces X and Y.

Lemma 3.6. [7, Proposition 2.1] *Let X be a Tychonoff space. Assume that P is a densely self-embeddable prime space and some n-th symmetric power X^n/S_n contains a copy of P, where $n \geq 1$. Then X itself contains a copy of P.*

Lemma 3.7. [11, Theorem 3.12] *Let X be a Tychonoff space and K be a countably compact subset of $AP(X)$. Then $K \subset AP_n(X)$ for some positive integer n. The same is valid for $FP(X)$.*

A point x of a topological space X is called a *q-point* [15], provided that x has a sequence $\{U_m\}_{m<\omega}$ of open neighbourhoods with the property that if $x_m \in U_m$ for every $m < \omega$, then $\{x_m : m < \omega\}$ lies in some countably compact subset of X. Such a sequence $\{U_m\}_{m<\omega}$ is called a *q-sequence* of the q-point x. Obviously, every first-countable space or countably compact space has a q-point.

The following Corollary 3.8 extends [7, Theorem 2.6] to free (Abelian) paratopological groups.

Corollary 3.8. *Let X be a Tychonoff space, and P be a densely self-embeddable prime space with a q-point. Then the following are equivalent.*

(1) $AP(X)$ contains a copy of P;
(2) $FP(X)$ contains a copy of P;
(3) X contains a copy of P.

PROOF. (3) $\Rightarrow$ (1), (2). This is obvious because each of the groups $FP(X)$, $AP(X)$ contains a (closed) copy of X.

(1) $\Rightarrow$ (3). Suppose $AP(X)$ contains a copy of P. Without loss of generality, we assume that P contains at least two points and is a subspace of $AP(X)$. Let $\{U_m\}_{m<\omega}$ be a q-sequence of a q-point of P. Then there exist $m_0, n_0 < \omega$ such that $U_{m_0} \subset AP_{n_0}(X)$. Otherwise, for any $m, n < \omega$, we have $U_m \not\subset AP_n(X)$. In particular, for every $m < \omega$, pick $x_m \in U_m \setminus AP_m(X)$. Thus $\{x_m : m < \omega\}$ lies in some countably compact subset of P. By Lemma 3.7, $\{x_m : m < \omega\} \subset AP_k(X)$ for some positive integer k, which is a contradiction. The space P being a densely self-embeddable, U_{m_0} contains a copy of P, so does $AP_{n_0}(X)$.

Now, let n be the least positive integer such that $AP_n(X)$ contains a copy of P. Since $AP_{n-1}(X)$ is closed in $AP(X)$ and P is a densely self-embeddable, by the choice of n, $AP_n(X) \setminus AP_{n-1}(X)$ contains a copy of P. By Theorem 3.3, $(X \oplus -X_d)^n/S_n$ contains a copy of P. According to Lemma 3.6, $X \oplus -X_d$ contains a copy of P. The space P has no isolated points, so X contains a copy of P.

(2) $\Rightarrow$ (3). This can be proved, making use of Theorem 1.2, in a fashion similar to "(1) $\Rightarrow$ (3)". □

It is well known, for instance, see [4, 7], that the spaces $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\beta\omega \setminus \omega$, 2^κ are all densely self-embeddable and prime, and $\beta\omega \setminus \omega$ contains a copy of $\beta\omega$, where each of them carries its usual topology and κ is an arbitrary infinite cardinal.

Corollary 3.9. *Let X be a Tychonoff space, and P be one of the spaces $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\beta\omega \setminus \omega$, 2^κ, $\beta\omega$. Then the following are equivalent.*
(1) $AP(X)$ contains a copy of P;
(2) $FP(X)$ contains a copy of P;
(3) X contains a copy of P.

4. Fréchetness of free (Abelian) paratopological groups

We recall that a topological space X is called a *Fréchet* space [10] if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence x_1, x_2, ... of points of A converging to x. A topological space X is called a *sequential* space [10] if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits. Obviously, every Fréchet space is a sequential space.

Definition 4.1. (See [1]) Let $X = \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$. Let also $\mathbb{N}^\mathbb{N}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. For every $n, m, k \in \mathbb{N}$, put $V(n, m) = \{n\} \cup \{(n, k) : k \geq m\}$. For every $x \in \mathbb{N}^2$, let $\mathcal{B}(x) = \{\{x\}\}$. For every $n \in \mathbb{N}$, let $\mathcal{B}(n) = \{V(n, m) : m \in \mathbb{N}\}$. Let $\mathcal{B}(0) = \{\{0\} \cup \bigcup_{n \geq i} V(n, f(n)) : i \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N}\}$. The topological space X, whose topology is generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x\in X}$, is called the *Arens'* space and denoted by S_2.

It is not difficult to check that S_2 is a sequential space, but not a Fréchet space.

Lemma 4.2. *Suppose X is a Tychonoff space.*

(1) Let $L = \{x_n\}_{n\in\mathbb{N}}$ be a non-trivial sequence in $FP(X)$ converging to the identity e. For every $p \in \mathbb{N}$, there exist $q \in \mathbb{N}, y \in X \cup X^{-1}$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{y^q x_{n_k} y^{-q}\}_{k\in\mathbb{N}}$ converges to e, and $l(y^q x_{n_k} y^{-q}) > p$ for every $k \in \mathbb{N}$.

(2) Let $L = \{x_n\}_{n\in\mathbb{N}}$ be a non-trivial sequence in $AP(X)$ converging to the identity 0. For every $p \in \mathbb{N}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ converging to the identity 0 such that $l(y_n) > p$ for every $n \in \mathbb{N}$.

Proof. (1) By Lemma 3.7, $L \subset FP_m(X)$ for some $m \in \mathbb{N}$. Without loss of generality, we assume, for every $n \in \mathbb{N}$, $l(x_n) = s$ for some $s \leq m$. We write $x_n = x_{n,1} \cdots x_{n,s}$ for every $n \in \mathbb{N}$, where $x_{n,1}, ..., x_{n,s} \in X \cup X^{-1}$. It is easy to see that either $|\{n : x_{n,1} \in X\}| = \omega$ or $|\{n : x_{n,1} \in X^{-1}\}| = \omega$.

Without loss of generality, we may assume $x_{n,1} \in X$ for every $n \in \mathbb{N}$. If $|\{n : x_{n,s} = x_0, n \in \mathbb{N}\}| = \omega$ for some $x_0 \in X \cup X^{-1}$, we pick $y \in X$ such that $y \neq x_0$ if $x_0 \in X$; and $y = x_0^{-1}$ if $x_0 \in X^{-1}$. Let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of L with $x_{n_k,s} = x_0$ for every $k \in \mathbb{N}$. Choose $q > p$, then $\{y^q x_{n_k} y^{-q}\}_{k\in\mathbb{N}}$ converges to e and $l(y^q x_{n_k} y^{-q}) = 2q + s > p$ for every $k \in \mathbb{N}$. Otherwise, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of L such that $x_{n_i,s} \neq x_{n_j,s}$ if $i \neq j$. Let $y = x_{n_1,s}$ if $x_{n_1,s} \in X$, and $y = x_{n_1,s}^{-1}$ if $x_{n_1,s} \in X^{-1}$. Choose $q > p$, then there exists a subsequence $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\{y^q x_{n_{k_j}} y^{-q}\}_{k\in\mathbb{N}}$ converges to e and $l(y^q x_{n_{k_j}} y^{-q}) = 2q + s > p$ for every $j \in \mathbb{N}$. This completes the proof of item (1) of Lemma 4.2.

(2) By Lemma 3.7, $L \subset AP_m(X)$ for some $m \in \mathbb{N}$. Without loss of generality, we can assume that there exists $s \leq m$ such that $l(x_n) = s$ for every $n \in \mathbb{N}$. For a given $p \in \mathbb{N}$, pick $k \in \mathbb{N}$ such that $k \times s > p$. For every $n \in \mathbb{N}$, put $y_n = kx_n$, whence $l(y_n) > p$. By the joint continuity of multiplication in $AP(X)$, the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges to the identity 0. □

Lemma 4.3. [11, Theorem 3.11] *Let X be a Tychonoff space and C be any subset of $FP(X)$. If $C \cap FP_n(X)$ is finite for every $n \in \mathbb{N}$, then C is closed and discrete in $FP(X)$. The same is valid for $AP(X)$.*

Theorem 4.4. *Let X be a Tychonoff space. If $FP(X)$ contains a non-trivial convergent sequence, then $FP(X)$ contains a closed copy of S_2. The same is valid for $AP(X)$.*

PROOF. If $FP(X)$ contains a non-trivial convergent sequence, then there is a non-trivial sequence $L = \{x_n\}_{n\in\mathbb{N}}$ converging to the identity e. By Lemma 3.7, $L \subset FP_{n_0}(X)$ for some $n_0 \in \mathbb{N}$, which implies $l(x_n) \leq n_0$ for every $n \in \mathbb{N}$. By Lemma 4.2, there is a sequence $\{t_k\}_{k\in\mathbb{N}}$ converging to e such that the length of every t_k is greater than $2n_0$. Thus $\{x_1 t_k\}_{k\in\mathbb{N}}$ converges to x_1, and $l(x_1 t_k) > n_0$ for every $k \in \mathbb{N}$. Put $y_{1,k} = x_1 t_k$ for every $k \in \mathbb{N}$ and $L_1 = \{y_{1,k}\}_{k\in\mathbb{N}}$. Using again Lemma 3.7, we can choose $n_1 \in \mathbb{N}$ such that the length of every element in L_1 is less than n_1. By induction, we can choose a sequence $\{n_i\}_{i\in\mathbb{N}}$ with $n_1 < n_2 < ...$, and a sequence $\{L_i\}_{i\in\mathbb{N}}$ with $L_i = \{y_{i,k}\}_{k\in\mathbb{N}}$ converging to x_i and satisfying $n_{i-1} < l(y_{i,k}) < n_i$ for all $i, k \in \mathbb{N}$. Put

$$S = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_{n,k} : n, k \in \mathbb{N}\}.$$

Claim. S is closed in $FP(X)$ and is homeomorphic to S_2.

Let $f \in \mathbb{N}^{\mathbb{N}}$. The set $\bigcup_{n\in\mathbb{N}}\{y_{n,k} : k < f(n)\}$ is closed and discrete in $FP(X)$ by Lemma 4.3. Then the set

$$\{e\} \cup \bigcup_{n\geq i}\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$$

is an open neighborhood of e in S for every $i \in \mathbb{N}$. It is also easy to see that $\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$ is open in S for every $n \in \mathbb{N}$, and $\{y_{n,k}\}$ is open in S for every $n, k \in \mathbb{N}$. Hence the space S is a copy of S_2.

Now we show that S is closed in $FP(X)$. Suppose $p \notin S$. The space X being Tychonoff, $FP(X)$ is Hausdorff [17, Proposition 3.8]. Since $\{e\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, there exist open subsets U and V of $FP(X)$ such that

$$p \in U, \{e\} \cup \{x_n : n \in \mathbb{N}\} \subset V \text{ and } U \cap V = \emptyset.$$

Thus there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{e\} \cup \bigcup_{n\in\mathbb{N}}\{x_n\} \cup \{y_{n,k} : k \geq f(n)\} \subset V.$$

Let

$$W = U \setminus \bigcup_{n\in\mathbb{N}}\{y_{n,k} : k < f(n)\}.$$

The set W is an open neighbourhood of p in $FP(X)$ by Lemma 4.3 and $W \cap S = \emptyset$, whence S is closed in $FP(X)$. The case of $AP(X)$ can be proved in a similar fashion. □

Corollary 4.5. *Let X be a Tychonoff space. If $FP(X)$ is a sequential space, then either X is discrete or $FP(X)$ contains a closed copy of S_2. The same is valid for $AP(X)$.*

PROOF. If X is not discrete, then $FP(X)$ is also not discrete. $FP(X)$ contains a non-trivial convergent sequence, since $FP(X)$ is a sequential space. By Theorem 4.4, $FP(X)$ contains a closed copy of S_2. The argument in the case of $AP(X)$ is exactly the same. □

The following corollary 4.6 gives an affirmative answer to [6, Question 5.9].

Corollary 4.6. *Let X be a Tychonoff space. If $FP(X)$ or $AP(X)$ is a Fréchet space, then the space X is discrete.*

We conclude this paper with a question.

Question 4.7. *Let X be a Tychonoff space. Fix $n \in \mathbb{N}$. How to characterize the spaces X such that $FP_n(X)$ ($AP_n(X)$) is Fréchet?*

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