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snf-Countability and csf-countability in $F_4(X)$ [☆]

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ABSTRACT

Let $F(X)$ be the free topological group on a Tychonoff space X, and $F_n(X)$ the subspace of $F(X)$ consisting of all words of reduced length at most n for each $n \in \mathbb{N}$. In this paper conditions under which the subspace $F_4(X)$ of the free topological group $F(X)$ on a generalized metric space X contains no closed copy of S_ω are obtained and used to discuss countability axioms in free topological groups. It is proved that for a k-semistratifiable k-space X the subspace $F_4(X)$ is snf-countable if and only if X is compact or discrete; for a normal k- and $\aleph$-space X $F_4(X)$ is csf-countable if and only if X is an $\aleph_0$-space or discrete; and for a k^*-metrizable space X $F_5(X)$ is a k-space and $F_4(X)$ is csf-countable if and only if X is a k_ω-space or discrete. Some results of K. Yamada, and F. Lin, C. Liu and J. Cao are improved.

1. Introduction

The symbols $F(X)$ and $A(X)$ denote respectively the *free topological group* and the *free Abelian topological group* on a Tychonoff space X in the sense of Markov [25]. Free topological groups have become a powerful tool of investigation in the theory of topological groups that serve as a source of various examples and as

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an instrument for proving new theorems [3]. We use $G(X)$ to denote either $F(X)$ or $A(X)$. It is a natural question whether there is a topological property P of a space X which characterizes a topological property Q of $G(X)$. For example, the question of on what space X the free topological group $G(X)$ is a k-space has been studied by several topologists. It is a classic result that a space X is a k_ω-space if and only if so is the group $G(X)$ [24]. Arhangel'skiĭ, Okunev and Pestov [2] proved that the topological group $F(X)$ on a metrizable space X is a k-space if and only if X is locally compact and separable or discrete.

Let $\mathbb{N}$ be the set of all positive integers. In what follows, for each $n \in \mathbb{N}$ $F_n(X)$ and $A_n(X)$ stand for the subset of $F(X)$ and $A(X)$ formed by all words whose length is less than or equal to n, respectively. Thus, any statement about $G_n(X)$ applies to both $F_n(X)$ and $A_n(X)$. It is well known that if a space X is not discrete, then neither $A(X)$ is first-countable nor $F(X)$ is Fréchet–Urysohn (see [3, Theorem 7.1.20] and [13, Corollary 4.17]). However, $F_n(X)$ and $A_n(X)$ have a chance to be first-countable or Fréchet–Urysohn for a non-discrete space X. These facts motivate researchers to investigate the countability axioms of free topological groups in the following two directions [15]: one is to study some weak forms of countability axioms in $F(X)$ or $A(X)$ over certain classes of spaces X; another is to study some weak forms of countability axioms in $F_n(X)$ or $A_n(X)$ over certain classes of spaces X.

The set of all non-isolated points of a space X is denoted by $NI(X)$ in this paper. Let X be a Tychonoff space. Denote by X^{-1} a copy of a space X and by e the identity of the free group $G(X)$. The mapping $i_n : (X \oplus \{e\} \oplus X^{-1})^n \to G_n(X)$ is defined by $i_n((x_1, x_2, \cdots, x_n)) = x_1 x_2 \cdots x_n$ for each $n \in \mathbb{N}$.

K. Yamada [33–35] made a systematic and outstanding work in the two research directions over metrizable spaces. The following results were obtained.

Theorem 1.1. *([33,34]) The following are equivalent for a metrizable space X:*

(1) *$A_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$A_3(X)$ is Fréchet–Urysohn;*
(3) *$F_3(X)$ is metrizable;*
(4) *$G_2(X)$ is first-countable;*
(5) *i_2 is a closed mapping;*
(6) *$NI(X)$ is compact.*

Theorem 1.2. *([33,34]) The following are equivalent for a metrizable space X:*

(1) *$F_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$F_5(X)$ is Fréchet–Urysohn;*
(3) *$F_4(X)$ is first-countable;*
(4) *i_n is a closed mapping for each $n \in \mathbb{N}$;*
(5) *i_4 is a closed mapping;*
(6) *X is compact or discrete.*

Theorem 1.3. *([35]) The following are equivalent for a metrizable space X:*

(1) *$F(X)$ is a k-space;*
(2) *$F_n(X)$ is a k-space for each $n \in \mathbb{N}$;*
(3) *X is locally compact separable or discrete.*

These results are beautiful, but slightly incomplete, which leaves a space for further research. For example, when, in terms of the space X, is the subspace $F_4(X)$ or $F_3(X)$ Fréchet–Urysohn? Is the mapping i_3 closed? Is $F_n(X)$ a k-space for some $k \in \mathbb{N}$? Recently, F. Lin, C. Liu et al. [12,13,15] attempted to extend Yamada's results to generalized metric spaces and study weak forms of countability axioms in $F(X)$ and $A(X)$ over generalized metric spaces. For example, the following results were proved.

Theorem 1.4. *([13]) Let X be a paracompact space with a point-countable k-network. Then $F_5(X)$ is Fréchet–Urysohn if and only if X is compact or discrete.*

Theorem 1.5. *([12]) The following are equivalent for a k^*-metrizable μ-space X:*

(1) *$F(X)$ is a k-space;*
(2) *$F_{10}(X)$ is a k-space;*
(3) *X is a k_ω-space or discrete.*

Theorem 1.6. *([15]) The following are equivalent for a k-space X with a regular G_δ-diagonal:*

(1) *$F_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$F_4(X)$ is an snf-countable space;*
(3) *X is compact or discrete.*

Theorem 1.7. *([15]) The following are equivalent for a Lašnev space X:*

(1) *$F(X)$ is a csf-countable space;*
(2) *$F_4(X)$ is a csf-countable space;*
(3) *X is an $\aleph_0$-space or discrete.*

Recently, some researchers in topological algebra have been interested in the free topological groups over generalized metric spaces. Some questions are similar to those for free topological groups over metric spaces. For example, when, in terms of the space X, is the subspace $G_3(X)$ Fréchet–Urysohn? Is $F_3(X)$ or $F_2(X)$ snf-countable? Is $F_3(X)$ csf-countable? These results and questions inspire us to study the countability axioms in free topological groups. We discussed the countable tightness and the k-property of free topological groups over generalized metric spaces in the above-mentioned first direction in [32]. The present paper contributes to characterizing some weak forms of countability axioms of the subspace $G_n(X)$ over certain classes of spaces X in the above-mentioned second direction; and we show that A. Arhangel'skiĭ, K. Yamada and F. Lin's results hold for a broader class of generalized metric spaces, including normal spaces with point-countable k-networks, k^*-metrizable spaces and k-semistratifiable spaces.

The paper is organized as follows. In Section 2, the necessary notation and terminology are introduced; in particular, weak forms of countability axioms, generalized metric spaces and free topological groups are defined. In Section 3, under certain assumptions, generalized metric spaces X such that the subspace $G_2(X)$ or $F_4(X)$ contains no closed copy of S_ω are characterized. It is shown that the set $NI(X)$ of all non-isolated points of a space X is countably compact for a sequential normal space X if $G_2(X)$ contains no closed copy of S_ω (see Theorem 3.5). In Section 4 we show that the subspace $F_3(X)$ is Fréchet–Urysohn if and only if $NI(X)$ is compact and X is first-countable for a normal space X with a point-countable k-network (see Theorem 4.3), and the subspace $F_4(X)$ is snf-countable if and only if X is compact or discrete for a k-semistratifiable k-space X (see Theorem 4.5). In Section 5, it is proved that the subspace $F_4(X)$ is csf-countable if and only if X is an $\aleph_0$-space or discrete for a normal k-and $\aleph$-space X (see Theorem 5.2), and $F_5(X)$ is a k-space and $F_4(X)$ is csf-countable if and only if X is a k_ω-space or discrete for a k^*-metrizable space X (see Theorem 5.4). These facts refine results in [12,13,15,33,34].

2. Notation and terminology

In this section we introduce the necessary notation and terminology; in particular, we define weak forms of countability axioms, generalized metric spaces and free topological groups.

Recall that a space X is a *k-space* provided that a subset $C \subseteq X$ is closed in X if $C \cap K$ is closed in K for each compact subset K of X. A space X is a *sequential space* if for any non-closed set A of X there is a sequence in A converging to some point in $X \setminus A$. A space X is *Fréchet–Urysohn* if for any set $A \subseteq X$ and a point $x \in \overline{A}$ there is a sequence in A converging to x in X. A space X is called a *k_ω-space* if $X = \bigcup_{i\in\omega} X_i$, where each X_i is compact, and each set $E \subseteq X$ such that every $E \cap X_i$ closed in X_i is closed in X. Obviously, every k_ω-space is a k-space. It is known that every first-countable space is a Fréchet–Urysohn space, every Fréchet–Urysohn space is a sequential space and every sequential space is a k-space.

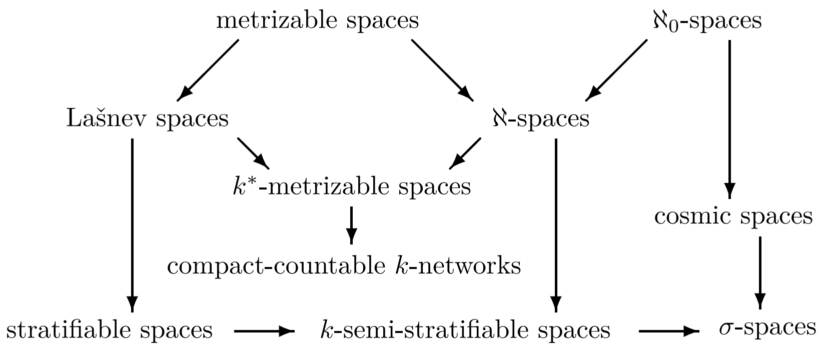
Let $\mathscr{P}_x$ be a family of subsets of a space X, where $x \in X$. The family $\mathscr{P}_x$ is called a *cs-network* at x [11] if, for every sequence $\{x_n\}$ converging to x and an arbitrary neighborhood U of x in X, there exist $m \in \mathbb{N}$ and $P \in \mathscr{P}_x$ such that $\{x\} \cup \{x_n : n > m\} \subseteq P \subseteq U$. $\mathscr{P}_x$ is called an *sn-network* [17] (or a *sequential barrier* [18]) at x if the following conditions are satisfied: (1) every $P \in \mathscr{P}_x$ is a *sequential neighborhood* of x in X, i.e., each sequence $\{x_n\}$ in X converging to x is eventually in P; (2) if $x \in U$ with U open in X, then there is $P \in \mathscr{P}_x$ such that $x \in P \subseteq U$; (3) if $U, V \in \mathscr{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathscr{P}_x$. A space X is called *csf-countable* [18, Definition 2.7] (resp., *snf-countable* [20, Definition 3], i.e., *universally csf-countable* [18, Definition 2.7]) if X has a countable *cs*-network (resp., snf-network) at each point $x \in X$. It is obvious that every first-countable space is snf-countable, and every snf-countable space is csf-countable.

Given an infinite cardinal κ, the *fan space* S_κ is the quotient space obtained by identifying all the limit points of the topological sum of κ many non-trivial convergent sequences. A space X is called an *S_2 space*, i.e., Arens' space, if $X = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{mn} : n, m \in \mathbb{N}\}$ and the topology is defined as follows: each x_{mn} is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_{nm} : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of x is $\{x\} \cup \bigcup_{n>k} V_n$ for some $k \in \mathbb{N}$, where V_n is a neighborhood of x_n. It is easy to see that every fan space S_κ is a Fréchet–Urysohn space, S_{ω_1} is not csf-countable (see [20, Remark 3]), S_ω is csf-countable but not snf-countable, and S_2 is sequential and snf-countable but not Fréchet–Urysohn.

Let $\mathcal{P}$ be a cover of a space X. The family $\mathcal{P}$ is a *network* for X [1] if, for each U open in X and $x \in U$, there is $P \in \mathcal{P}$ such that $x \in P \subseteq U$. $\mathcal{P}$ is called a *k-network* for X [29] if, for each U open in X and each compact set $K \subseteq U$, there is a finite subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \cup\mathcal{P}' \subseteq U$. A regular space X is called a *σ-space* [28] (resp. an *$\aleph$-space* [29]) if it has a σ-locally finite network (resp. a σ-locally finite k-network). A regular space X with a countable network (resp. a countable k-network) is called a *cosmic space* (resp. an *$\aleph_0$-space* [26]).

We shall concern ourselves with three classes of generalized metric spaces: normal spaces with a point-countable k-network, k^*-metrizable spaces and k-semistratifiable spaces. Let $\mathscr{P}$ be a family of subsets of a space X. $\mathscr{P}$ is called *point-countable* if every point of X only belongs to at most countably many elements of $\mathscr{P}$. *k^*-Metrizable spaces* [4] are defined as the images of metric spaces under certain mappings; they can be characterized as regular spaces with a σ-compact-finite k-network (see [4, Theorem 6.4]). Recall that a family $\mathscr{P}$ of subsets of a space X is *compact-finite* (resp. *compact-countable*) if every compact subset of X meets at most finitely (resp. countably) many $P \in \mathscr{P}$. A regular space X is said to be a *k-semistratifiable space* [23] if there is an operator U assigning to each closed set F a sequence $U(F) = \{U(n, F)\}_{n\in\mathbb{N}}$ of open sets in X such that (1) $\bigcap_{n\in\mathbb{N}} U(n, F) = F$; (2) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$; (3) if K is compact in X and $K \cap F = \varnothing$, then $K \cap U(m, F) = \varnothing$ for some $m \in \mathbb{N}$. If, instead of the above conditions (1) and (3), the condition (1′) $\bigcap_{n\in\mathbb{N}} U(n, F) = \bigcap_{n\in\mathbb{N}} \overline{U(n, F)} = F$ holds, then X is said to be a *stratifiable space* [5]. A space X is called a *Lašnev space* if X is a closed image of a metric space.

We summarize some relations between the above-mentioned generalized metric spaces as follows [4,9,19].



Let X be a Tychonoff space. As an abstract group, $F(X)$ (resp. $A(X)$) is the free (resp. free Abelian) group on X having the finest group topology among those inducing the original topology of X, so that every continuous map from X to an arbitrary (resp. Abelian) topological group G extends to a unique continuous homomorphism from $F(X)$ (resp. $A(X)$) to G. We always use $G(X)$ to denote topological groups $F(X)$ and $A(X)$. For each $n \in \mathbb{N}$, every $F_n(X)$ contains a closed copy of X^n (see [3, Theorem 7.1.13]), and every $G_n(X)$ is a closed subspace of $G(X)$. For a subset Y of X, the symbol $G(Y,X)$ denotes the subgroup of $G(X)$ generated by Y. If Y is a closed subspace of X, the subgroup $G(Y,X)$ is closed in $G(X)$ (see [3, Theorem 7.4.5]). Denote by X^{-1} a copy of a space X and by e the identity of the free group $G(X)$. The mapping $i_n : (X \oplus \{e\} \oplus X^{-1})^n \to G_n(X)$ is defined as $i_n((x_1, x_2, \cdots, x_n)) = x_1x_2\cdots x_n$ for each $n \in \mathbb{N}$. Clearly, each i_n is continuous and onto. The *support* of a reduced word $g = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n} \in G(X)$, where $\varepsilon_i = \pm 1$ and $x_i \in X$, is defined as the set $\operatorname{supp}(g) = \{x_1, x_2, \cdots, x_n\}$. Given a subset K of $G(X)$, we put

$$\operatorname{supp}(K) = \bigcup_{g\in K} \operatorname{supp}(g).$$

For notation and terminology not given here the reader is referred to [3,6,9].

3. When $G_2(X)$ or $F_4(X)$ contains no closed copy of S_ω

In this section, we shall characterize, under certain additional assumptions, generalized metric spaces X such that $G_2(X)$ or $F_4(X)$ contains no closed copy of S_ω: this characterization plays an important role in Sections 4 and 5.

Firstly, we discuss when $G_2(X)$ contains no closed copy of S_ω. A subspace Y of a space X is said to be *C-embedded* if every continuous real-valued function on Y has a continuous extension to X. A subspace Y of X is said to be *F-embedded* if $F(Y)$ is a topological subgroup of $F(X)$, i.e., $F(Y) \cong F(Y,X)$.

Lemma 3.1. *([31, Theorem 2]) Suppose that Y is a Lindelöf subspace of a space X. Then Y is F-embedded in X if and only if Y is C-embedded in X.*

Lemma 3.2. *Let Y be the topological sum of countably many non-trivial convergent sequences together with their limits. Then $G_2(Y)$ contains a closed copy of S_ω.*

Proof. Let $Y = \bigoplus_{n\in\mathbb{N}} C_n$, where each C_n is a non-trivial convergent sequence together with its limit point x_{n0}. Put $A_n = x_{n0}^{-1}C_n$ for each $n \in \mathbb{N}$. It is obvious that A_n is homeomorphic to C_n and $A_n \cap A_m = \{e\}$ whenever $n \neq m \in \mathbb{N}$. Put $A = \bigcup_{n\in\mathbb{N}} A_n$. Then $A \subseteq G_2(Y)$. In the following, we shall prove that A is a closed copy of S_ω in $G_2(Y)$.

Let E be a subset of A such that $E \cap A_n$ is closed in A_n for each $n \in \mathbb{N}$. Suppose K is any compact set in $G_2(Y)$. From [3, Corollary 7.5.6] it follows that $\overline{\text{supp}(K)}$ is compact in Y. Hence, there is $m \in \mathbb{N}$ such that $\text{supp}(K) \subseteq \bigcup_{i\le m} C_i$, and, therefore, $K \subseteq G(\bigcup_{i\le m} C_i, Y)$. It follows that

$$E \cap K = (E \cap A) \cap (K \cap G(\bigcup_{i\le m} C_i, Y)) = E \cap K \cap \bigcup_{i\le m} A_i = K \cap \bigcup_{i\le m} (E \cap A_i).$$

Then $E \cap K$ is closed in K. Observe that Y is a k_ω-space, so is $G(Y)$ by [3, Theorem 7.4.1]. Thus $G_2(Y)$ is a k-space, so E is closed in $G_2(Y)$. This implies that the subspace A is a closed copy of S_ω in $G_2(Y)$. □

Lemma 3.3. *Suppose that a space X contains a closed copy of S_2. Then $G_2(X)$ contains a closed copy of S_ω.*

Proof. Let $B = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{nm} : n, m \in \mathbb{N}\}$ be a closed copy of S_2 in X, where the sequence $\{x_n\}$ converges to x and the sequence $\{x_{nm}\}_{m\in\mathbb{N}}$ converges to x_n for each $n \in \mathbb{N}$. Put $C_n = \{x_n^{-1}x_{nm} : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. Then the sequence C_n converges to the identity e of $G(X)$. We set $C = \{e\} \cup \bigcup_{n\in\mathbb{N}} C_n \subseteq G_2(X)$. We shall prove that C is a closed copy of S_ω in $G_2(X)$.

Firstly, we shall show that the set $F = \{x_n^{-1}x_{nm} \in C : m \le f(n), n \in \mathbb{N}\}$ is closed in C for any function $f : \mathbb{N} \to \mathbb{N}$, which implies that C is a copy of S_ω.

Suppose there exists $y \in C \cap \overline{F} \setminus F$. Put $A = \{x_{nm} \in B : m \le f(n), n \in \mathbb{N}\}$. Since B is closed in X, A is discrete and closed in X; thus A is discrete and closed in $G(X)$ [3, Theorem 7.1.13], and $xy \notin \overline{A \setminus \{xy\}}$. Let U be an open neighborhood U of e in $G(X)$ with $Uxy \cap (A \setminus \{xy\}) = \varnothing$. There is an open neighborhood V of e such that $VV \subseteq U$. Then $x^{-1}Vxy \cap F \ne \varnothing$, because $y \in \overline{F}$. Since the sequence $\{x_n\}$ converges to x, there is $n_0 \in \mathbb{N}$ such that $x_n \in Vx$ whenever $n > n_0$. Moreover, the set $M_1 = \{n \in \mathbb{N} : x^{-1}Vxy \cap \{x_n^{-1}x_{nm} : m \le f(n)\} \ne \varnothing\}$ is infinite. Thus there are $j_0 > n_0$ and $m_0 \le f(j_0)$ such that $x_{j_0}^{-1}x_{j_0m_0} \in x^{-1}Vxy \cap F$ and $x_{j_0m_0} \ne xy$. It follows from $x_{j_0} \in Vx$ that

$$x_{j_0m_0} = x_{j_0}x_{j_0}^{-1}x_{j_0m_0} \in Vxx^{-1}Vxy = VVxy \subseteq Uxy.$$

This implies that $Uxy \cap (A \setminus \{xy\}) \ne \varnothing$. This is a contradiction, which shows that the set C is a copy of S_ω.

Secondly, if C is not closed in $G_2(X)$, then there exists a point $z \in \overline{C} \setminus C$. Since B is closed in $G(X)$ and $xz \ne x$, there is an open neighborhood O of e such that

$$|\{n \in \mathbb{N} : Oxz \cap \{x_{nm} : m \in \mathbb{N}\} \ne \varnothing\}| \le 1.$$

Take an open neighborhood W of e such that $W^2 \subseteq O$. There is $i_0 \in \mathbb{N}$ such that $x_n \in Wx$ whenever $n > i_0$. Since $z \in \overline{C} \setminus C$, the set $M_2 = \{n \in \mathbb{N} : n > i_0, x^{-1}Wxz \cap C_n \ne \varnothing\}$ is infinite. Let $n \in M_2$ and take $m \in \mathbb{N}$ such that $x_n^{-1}x_{nm} \in x^{-1}Wxz$. Then

$$x_{nm} \in x_nx^{-1}Wxz \subseteq Wxx^{-1}Wxz = WWxz \subseteq Oxz.$$

This implies that the set $\{n \in \mathbb{N} : Oxz \cap \{x_{nm} : m \in \mathbb{N}\} \ne \varnothing\}$ is infinite. This is a contradiction, and C is closed in $G_2(X)$. □

Lemma 3.4. *Suppose one of the following conditions is satisfied for a space X:*

(a) *X has a point-countable k-network;*
(b) *X is a σ-space.*

Then

(1) *every countably compact subset of X is compact metrizable;*
(2) *X is sequential if X is a k-space.*

Proof. Let X be a space with a point-countable k-network (resp. a σ-space).

(1) Clearly, the subspace A has also a point-countable k-network (resp. is a σ-space). Since A is a countably compact subset of X, by [10, Theorem 4.1] (resp. [9, Corollary 4.7]), A is compact metrizable.

(2) If X is a k-space, then X is a quotient image of a topological sum of compact subsets of X [27, Theorem 6.E.3(c)]. By (1), X is a quotient image of a metrizable space; hence X is a sequential space [27, Theorem 6.D.2]. □

Theorem 3.5. *If one of the following conditions is satisfied and $G_2(X)$ contains no closed copy of S_ω, then $NI(X)$ is countably compact.*

(a) *X is a sequential normal space.*
(b) *X is a k^*-metrizable k-space.*
(c) *X is a k-semistratifiable k-space.*

Proof. (a) Suppose X is a sequential normal space and $NI(X)$ is not countably compact. Choose an infinite discrete closed set $\{x_{n0} : n \in \omega\} \subseteq NI(X)$. There are a disjoint family $\{A_n\}$ such that each A_n is a non-trivial convergent sequence in X together with its limit point x_{n0} and a discrete family $\{U_n\}$ of open sets in X such that each $A_n \subseteq U_n$. Clearly, the subspace $\bigcup_{n\in\omega} A_n$ of X is homeomorphic to $\bigoplus_{n\in\omega} A_n$ and closed in X. Hence $\bigcup_{n\in\omega} A_n$ is C-embedded in X and therefore, by Lemmas 3.1 and 3.2, $G_2(X)$ contains a closed copy of S_ω. This is a contradiction; thus $NI(X)$ is not countably compact.

Next, we consider conditions (b) and (c). To complete the proof, by Lemma 3.4 and already proved assertion (a), we only need to show that the space X is normal. By Lemma 3.3, X contains no closed copy of S_2.

(b) Suppose X is a k^*-metrizable k-space. From the fact that every sequential space with a point-countable k-network containing no closed copy of S_2 is a Fréchet–Urysohn space [18, Theorem 2.12] it follows that X is Fréchet–Urysohn and, therefore, X is a Lašnev space by the fact that every Fréchet–Urysohn space with a σ-compact-finite k-network is a Lašnev space [21]. Thus X is a normal space.

(c) Suppose X is a k-semistratifiable k-space. Observe the fact that (1) every regular sequential space with every point being a G_δ-set is Fréchet–Urysohn if it contains no closed copy of S_2 [18, Lemma 2.5]; and (2) every Fréchet–Urysohn k-semistratifiable space is stratifiable [8, Theorem 1]. Therefore the space X is stratifiable, and thus it is normal. □

Since every weakly first-countable space is snf-countable (= universally csf-countable space [18, Definition 2.7 and Lemma 3.12]), and every snf-countable space contains no closed copy of S_ω [18, Theorem 3.13], Theorem 3.5 improves Theorem 1.1, [12, Theorem 4.10] and [15, Theorem 3.7].

Secondly, we consider when the subspace $F_4(X)$ contains no closed copy of S_ω.

Lemma 3.6. *([31, Theorem 1]) Suppose that Y is a closed subspace of a metrizable space X. Then the free topological group $F(Y)$ is a topological subgroup of $F(X)$.*

A subset A of a space X is called *sequentially closed* if A contains the limits of all sequences in A convergent in X. It is obvious that a space X is sequential if and only if every sequentially closed subset of X is closed. Let $C \oplus D$ be the topological sum of topological spaces C and D, where C is a non-trivial convergent sequence together with its limit point x_0 and D is a discrete space, and consider the free topological group $F(C \oplus D)$ on $C \oplus D$. Put $C_a = \{ax_0x^{-1}a^{-1} : x \in C\} \subset F(C \oplus D)$ for each $a \in D$. It is easy to check that each C_a is homeomorphic to C and $C_a \cap C_b = \{e\}$ for any distinct elements $a, b \in D$. Put $S_{C\oplus D} = \bigcup_{a\in D} C_a$. It is clear that $S_{C\oplus D} \subseteq F_4(C \oplus D)$.

Lemma 3.7. *Let C be a non-trivial convergent sequence together with its limit point and D a discrete space. Then*

(1) *$S_{C\oplus D}$ is a closed copy of S_ω in $F_4(C \oplus D)$ if D is countably infinite;*
(2) *$S_{C\oplus D}$ is non-csf-countable and sequentially closed in $F_4(C \oplus D)$ if D is uncountable.*

Proof. Firstly, we show the following fact $(*)$: A countable subset Y of $S_{C\oplus D}$ is closed in $F(C\oplus D)$ if $Y \cap C_a$ is finite for each $a \in D$.

Without loss of generality, we can assume that Y is infinite. Put $G = \mathrm{supp}(Y) \cup \{x_0\}$, where x_0 is the limit point of C. Then G is a locally compact closed subspace in $C \oplus D$. Hence, $F(G)$ is a k-space by Theorem 1.3 and, therefore, is a sequential space, because $F(G)$ is countable. Since $C \oplus D$ is metrizable, it follows from Lemma 3.6 that $F(G, C\oplus D)$ is a closed copy of $F(G)$. Thus $F(G, C\oplus D)$ is a closed and sequential subspace of $F(C \oplus D)$. To show that Y is closed in $F(C \oplus D)$ it is enough to show that Y is closed in $F(G, C \oplus D)$. If not, then there is a non-trivial sequence $l \subseteq Y$ converging to some point $y \in F(G, C \oplus D) \setminus Y$. Note that $Y \cap C_a$ is finite for each $a \in D$, so $\mathrm{supp}(l) \cap D$ is an infinite closed discrete set in $C \oplus D$. However, from [3, Corollary 7.5.6] it follows that $Q = \overline{\mathrm{supp}(l \cup \{x_0\})}$ is compact. On the other hand, Q contains the infinite discrete closed set $\mathrm{supp}(l) \cap D$. This contradiction completes the proof of $(*)$.

According to $(*)$, $S_{C\oplus D}$ is sequentially closed in $F_4(C \oplus D)$.

(1) Assume D is countable infinite. According to $(*)$, $S_{C\oplus D}$ is a copy of S_ω. Since $C \oplus D$ is a locally compact separable metrizable space, from Theorem 1.3 it follows that $F(C\oplus D)$ is a k-space. Since $F(C\oplus D)$ is countable, $F(C \oplus D)$ is a σ-space; so $F(C \oplus D)$ is a sequential space by Lemma 3.4. Thus $S_{C\oplus D}$ is closed in $F_4(C \oplus D)$.

(2) Assume D is uncountable and $S_{C\oplus D}$ is csf-countable. Let $\mathscr{P}$ be a countable cs-network at e in $S_{C\oplus D}$. Denoting

$$\{P \in \mathscr{P} : \text{there are infinitely many } a \in D \text{ such that } P \cap (C_\alpha \setminus \{e\}) \neq \varnothing\}$$

by $\{P_n\}_{n\in\mathbb{N}}$, we can inductively choose a subset $\{y_n : n \in \mathbb{N}\}$ of $S_{C\oplus D}$ such that $y_n \in P_n \setminus \{e\}$ and different y_n belong to different C_a. Then $\{y_n : n \in \mathbb{N}\}$ is a closed set in $S_{C\oplus D}$ by $(*)$. Let

$$V = S_{C\oplus D} \setminus \{y_n : n \in \mathbb{N}\}, \quad \mathscr{F} = \{P \in \mathscr{P} : P \subset V\}.$$

If $P \in \mathscr{F}$, then $P \notin \{P_n : n \in \mathbb{N}\}$, so P only meets finitely many $C_a \setminus \{e\}$, and hence $\cup\mathscr{F}$ only meets countably many $C_a\setminus\{e\}$. As a consequence, there is $b \in D$ such that $(C_b\setminus\{e\})\cap\bigcup\mathscr{F} = \varnothing$. Let $V\cap(C_b\setminus\{e\}) = \{c_n : n \in \mathbb{N}\}$. Then the sequence $\{c_n\}$ converges to $e \in V$, so there exists $P \in \mathscr{F}$ such that $\{c_n\}$ is eventually in P, and hence $(C_b \setminus \{e\}) \cap \bigcup\mathscr{F} \neq \varnothing$, which is a contradiction. Therefore, $S_{C\oplus D}$ is not csf-countable. □

Theorem 3.8. *If one of the following conditions is satisfied, then $F_4(X)$ contains no closed copy of S_ω if and only if X is countably compact or discrete.*

(a) *X is a sequential normal space.*
(b) *X is a k^*-metrizable k-space.*
(c) *X is a k-semistratifiable k-space.*

Proof. If X is a k^*-metrizable k-space, by [4, Theorem 3.5(6) and Proposition 3.7], X is a sequential space in which every countably compact subset is compact metrizable. If X is a k-semistratifiable k-space, by

[16, Theorem 2.3], X is a σ-space; and by Lemma 3.4, X is a sequential space in which every countably compact subset is compact metrizable.

Sufficiency. Without loss of generality, we can assume that X is a countably compact sequential space. Then X is sequentially compact, and the spaces $X \oplus \{e\} \oplus X^{-1}$ and $(X \oplus \{e\} \oplus X^{-1})^4$ are also sequentially compact. Therefore, the subspace $F_4(X)$ is sequentially compact as the continuous image of $(X \oplus \{e\} \oplus X^{-1})^4$ under the mapping i_4, and $F_4(X)$ contains no closed copy of S_ω.

Necessity. Assume that a space X is neither countably compact nor discrete. We shall show that $F_4(X)$ contains a closed copy of S_ω if one of conditions (a), (b) and (c) is satisfied. Take an infinite countable discrete closed set D in X, which exists because X is not countably compact.

(1) Suppose X is a sequential normal space. Then there is a non-trivial convergent sequence C (including the limit point) in X with $C \cap D = \varnothing$, because X is sequential. It is obvious that the set $Y = C \cup D$ is Lindelöf and C-embedded in X, because X is normal. Therefore, by Lemma 3.1, the subgroup $F(Y, X)$ of $F(X)$ generated by Y is topologically isomorphic to $F(C \oplus D)$. Hence, $F_4(X)$ contains a closed copy of S_ω by Lemma 3.7.

(2) Suppose X is a k^*-metrizable k-space or k-semistratifiable k-space. Put $I(X) = X \setminus NI(X)$. By Theorem 3.5 we can assume that $NI(X)$ is countably compact and $D \subseteq I(X)$. Since X is not countably compact, $I(X)$ is not countably compact.

Case 1. $I(X)$ is closed in X.

Clearly, X is homeomorphic to $NI(X) \oplus I(X)$, so X is metrizable. Choose a non-trivial sequence $l \subseteq NI(X)$ with the limit point included. By Lemma 3.6, $F(l \cup D) \cong F(l \oplus D) \cong F(l \cup D, X)$. Hence $F_4(X)$ contains a closed copy of S_ω by Lemma 3.7.

Case 2. $I(X)$ is not closed in X.

Since the space X is sequential, there is a non-trivial sequence $l \subsetneq I(X)$ converging to some point $x \in NI(X)$. Put $\bar{l} = l \cup \{x\}$. Without loss of generality, we can assume that $\bar{l} \cap D = \varnothing$. Put $F = X \setminus (l \cup D)$ and $Z = X/F$, and let $p : X \to Z$ be the natural quotient mapping. Observing that every point in $l \cup D$ is an isolated point in X and D is both closed and open in X, one can easily check that the space Z is homeomorphic to $D \oplus \bar{l}$. The homomorphism $\widetilde{p} : F(X) \to F(Z)$ extending the quotient mapping p is open by [3, Corollary 7.1.9]. Clearly, $\widetilde{p}_{|F(\bar{l} \cup D, X)} : F(\bar{l} \cup D, X) \to F(Z)$ is a topological isomorphism. Thus $F(\bar{l} \cup D, X) \cong F(Z)$. Note that $F(\bar{l} \cup D, X)$ is closed in $F(X)$. Hence $F_4(X)$ contains a closed copy of S_ω by Lemma 3.7. □

4. snf-Countability in free topological groups

In this section the Fréchet–Urysohn property and snf-countability of $F_n(X)$ are discussed to improve Theorems 1.1, 1.2 and 1.4.

It is known that the subspace $F_{k+1}(X)$ contains a copy of S_2 if X is a sequential space and $F_k(X)$ contains a closed copy of S_ω for some $k \in \mathbb{N}$ [13, Proposition 3.3], of which the proof was omitted. However, we have the following result.

Lemma 4.1. *Suppose that X contains a non-trivial convergent sequence and $G_k(X)$ contains a closed copy of S_ω for some $k \in \mathbb{N}$, then $G_{k+1}(X)$ contains a closed copy of S_2.*

Proof. Suppose the space X contains a non-trivial sequence $\{x_n\}$ converging to x. Let $B = \{y_{nm} : n, m \in \mathbb{N}\} \cup \{y\}$ be a closed copy of S_ω in $G_k(X)$ such that every sequence $\{y_{nm}\}_{m \in \mathbb{N}}$ converges to y. Put $C = \{x_n y_{nm} : n, m \in \mathbb{N}\} \cup \{x_n y : n \in \mathbb{N}\} \cup \{xy\}$. Clearly, $C \subseteq G_{k+1}(X)$.

For any function $f : \mathbb{N} \to \mathbb{N}$, we shall show that the set $F = \{x_n y_{nm} : m \leq f(n), n \in \mathbb{N}\}$ is closed in $G_{k+1}(X)$, which implies that C is a closed copy of S_2.

If not, there is a point $a \in \overline{F} \setminus F$. Since B is a closed copy of S_ω, there is an open neighborhood V of e such that $Va \cap \{x^{-1}y_{nm} : m \leq f(n), n \in \mathbb{N}\} \setminus \{a\} = \varnothing$. On the other hand, take an open neighborhood U of e such that $U^2 \subseteq V$. Then $\{x^{-1}x_n^{-1} : n > i\} \subseteq U$ for some $i \in \mathbb{N}$. Since $Ua \cap F$ is infinite, the set $UUa \cap \{x^{-1}y_{nm} : m \subseteq f(n), n \in \mathbb{N}\}$ is infinite as well. This implies that $Va \cap \{x^{-1}y_{nm} : m \leq f(n), n \in \mathbb{N}\}$ is infinite. This is a contradiction. $\square$

According to Theorem 3.5 and Lemma 4.1 we obtain the following results, which improve Theorem 1.1.

Corollary 4.2. *Let $G_3(X)$ be Fréchet–Urysohn. Then $NI(X)$ is countably compact if one of the following conditions is satisfied.*

(a) *X is a normal space.*
(b) *X is a k^*-metrizable space.*
(c) *X is a k-semistratifiable space.*

Theorem 4.3. *Suppose one of the following conditions is satisfied for a k-space X:*

(a) *X is a normal space with a point-countable k-network;*
(b) *X is a k^*-metrizable space.*

Then the following are equivalent:

(1) *$F_3(X)$ is metrizable;*
(2) *$F_3(X)$ is Fréchet–Urysohn;*
(3) *$F_2(X)$ is snf-countable;*
(4) *$NI(X)$ is compact and X is first-countable.*

Proof. By Lemma 3.4 every countably compact subset of the space X is compact metrizable, and X is sequential.

Obviously, (1) implies (2) and (3).

(3) ⇒ (4). Since S_ω is not snf-countable, it follows from Theorem 3.5 that $NI(X)$ is countably compact, and hence $NI(X)$ is compact. Since $F_2(X)$ is snf-countable, X is snf-countable; thus X contains no closed copy of S_ω. On the other hand, by Lemma 3.3, X contains no closed copy of S_2, and then X is first-countable [18, Corollary 3.9].

(2) ⇒ (4). It follows from Corollary 4.2 that $NI(X)$ is countably compact, and $NI(X)$ is compact. By Lemma 4.1, X contains no closed copy of S_ω. Hence, it follows that X is first-countable from the fact that a Fréchet–Urysohn space with a point-countable k-network is first-countable if it contains no closed copy of S_ω [30, Lemma 2.2].

(4) ⇒ (1). Since X is a first-countable space with a point-countable k-network, X has a point-countable base [9]. Since $NI(X)$ is compact, X is metrizable [13, Lemma 4.3] and therefore, $F_3(X)$ is metrizable by Theorem 1.1. $\square$

Next, we consider improvement of Theorems 1.2 and 1.4.

Lemma 4.4. *([33, Lemma 4.7]) If there are a non-closed subset Y and a closed discrete subset D of a space X with $|Y| \leq |D|$, then the mapping $i_n : (X \oplus \{e\} \oplus X^{-1})^n \to G_n(X)$ is not closed for each $n \geq 3$.*

Theorem 4.5. *Suppose one of the following conditions is satisfied for a k-space X:*

(a) *X is a normal space in which every countably compact subset is metrizable;*
(b) *X is a k^*-metrizable space;*
(c) *X is a k-semistratifiable space.*

Then the following are equivalent:

(1) *$F_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$F_5(X)$ is Fréchet–Urysohn;*
(3) *$F_4(X)$ is snf-countable;*
(4) *X is compact or discrete;*
(5) *i_n is a closed mapping for some $n \geq 3$;*
(6) *i_3 is a closed mapping.*

Proof. By Lemma 3.4 every countably compact subset of the space X is compact metrizable, and X is sequential.

Obviously, (6) $\Rightarrow$ (5), and (4) $\Rightarrow$ (1) $\Rightarrow$ (2) and (3). (3) $\Rightarrow$ (4) $\Rightarrow$ (6) by Theorems 3.8 and 1.1. Next, we will prove that (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (4).

(5) $\Rightarrow$ (4). Suppose the space X is neither compact nor discrete. Then X is a non-countably compact sequential space; thus X contains a countably infinite closed discrete set and a non-trivial convergent sequence. Therefore, i_n is not closed for each $n \geq 3$ by Lemma 4.4.

(2) $\Rightarrow$ (4). Suppose $F_5(X)$ is Fréchet–Urysohn. Then $F_5(X)$ contains no copy of S_2. By Lemma 4.1, $F_4(X)$ contains no closed copy of S_ω. By Theorem 3.8, X is countably compact or discrete, hence X is compact or discrete. □

5. csf-Countability in free topological groups

It is known that the group $G(X)$ is csf-countable if and only if $G_n(X)$ is csf-countable for each $n \in \mathbb{N}$ [15]. When, in terms of the space X, is the subspace $G_n(X)$ csf-countable for some $n \in \mathbb{N}$? In this section we shall characterize k^*-metrizable spaces X such that $F_4(X)$ is csf-countable.

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on a set X such that $\mathscr{T}_1$ is finer than $\mathscr{T}_2$. If the space $(X, \mathscr{T}_2)$ has a σ-discrete family of subsets which is a network for $(X, \mathscr{T}_1)$, then the topology $\mathscr{T}_2$ is called an *s-approximation* for $\mathscr{T}_1$ [3]. The space $(X, \mathscr{T}_2)$ is a σ-space. A space X is *ω_1-compact* if every closed discrete subset of X is countable.

Proposition 5.1. Let X be a non-discrete paracompact k- and σ-space. If $F_4(X)$ is csf-countable, then X is a cosmic space.

Proof. It is enough to show that the space $(X, \mathscr{T}_1)$ is a ω_1-compact space, because every ω_1-compact σ-space has a countable network.

Suppose that X is not ω_1-compact. Since X is a k- and σ-space, X is a sequential space; thus X contains a non-trivial convergent sequence S (with the limit point included). Since X is a paracompact σ-space, $\mathscr{T}_1$ admits a metrizable s-approximation $\mathscr{T}_2$ on X [3, Theorem 7.6.6]. Put $Y = (X, \mathscr{T}_2)$. Since $(X, \mathscr{T}_1)$ has no countable network, Y is not ω_1-compact; thus Y contains an uncountable closed discrete subset D with $D \cap S = \varnothing$, so $\mathscr{T}_1|_{S\cup D} = \mathscr{T}_2|_{S\cup D}$, and the subspace $S \cup D$ is homeomorphic to $S \oplus D$. Since Y is metrizable, according to Lemma 3.6 it follows that $F(S \cup D, Y)$ is a copy of $F(S \oplus D)$. Thus, from the fact that the topology of the group $F(T)$ is the finest topological group topology on $F_a(T)$ that generates on T

its original topology [3, Corollary 7.1.8] it follows that $F(S \cup D, X)$ is a copy of $F(S \oplus D)$ as well. Since $F_4(S \oplus D) \subseteq F_4(X)$ and $F_4(X)$ is csf-countable, $F_4(S \oplus D)$ is csf-countable as well. This is a contradiction to Lemma 3.7. □

We have the following result, which improves Theorem 1.7. Let $\mathscr{P}$ be a family of subsets in a space X. The family $\mathscr{P}$ is called *star-countable* if each $P \in \mathscr{P}$ meets at most countably many elements of $\mathscr{P}$. Every space with a star-countable k-network has a σ-compact-finite k-network [22, Lemma 2.3].

Theorem 5.2. *Suppose one of the following conditions is satisfied for a space X:*

(a) *X is a normal k- and $\aleph$-space;*
(b) *X has a compact-countable k-network and X^2 is a k-space.*

Then the following are equivalent:

(1) *$F(X)$ is csf-countable;*
(2) *$F_4(X)$ is csf-countable;*
(3) *X is an $\aleph_0$-space or discrete;*
(4) *X is separable or discrete.*

Proof. Firstly, we prove (b) ⇒ (a). Suppose X is a space with a compact-countable k-network and X^2 is a k-space. It follows from [22, Lemmas 3.2 and 3.3] that X is either first-countable or a local k_ω-space. If X is first-countable, then X is metrizable and thus a paracompact $\aleph$-space. If X is a local k_ω-space, then X is a locally ω_1-compact space having a compact-countable k-network $\mathscr{P}$ such that the closure of every element in $\mathscr{P}$ is σ-compact. Hence, X has a star-countable k-network; therefore X is a topological sum of $\aleph_0$-spaces by [22, Theorem 2.13], and X is a paracompact $\aleph$-space.

Secondly, assume a space X satisfies condition (a). Let us show that (1) ⇔ (2) ⇔ (3) ⇔ (4). Obviously, (3) ⇒ (4), and (1) ⇒ (2). It is well known that every normal k- and $\aleph$-space is paracompact [7]. Without loss of generality, we can assume that X is a non-discrete paracompact k- and $\aleph$-space.

(4) ⇒ (1). Since X is paracompact and separable, X is Lindelöf. Thus X has a countable k-network, i.e., X is an $\aleph_0$-space. From [2, Theorem 4.1] it follows that $F(X)$ is an $\aleph_0$-space as well. Thus $F(X)$ is csf-countable.

(2) ⇒ (3). By Proposition 5.1 we can obtain that X is Lindelöf. Thus X has a countable k-network. □

Lemma 5.3. *([13, Lemma 4.9]) If $F_{k+1}(X)$ is a sequential space for some $k \in \mathbb{N}$, then either $F_k(X)$ contains no closed copy of S_ω or every first-countable closed subspace of X is locally countably compact.*

Recently, F. Lin and C. Liu proved that, for a k^*-metrizable μ-space X, $F(X)$ is k-space if and only if $F_5(X)$ is a k-space [13, Theorem 1.4]. However, there is a gap in the proof of this result [14]. We have the following result, which improves Theorem 1.5.

Theorem 5.4. *The following are equivalent for a k^*-metrizable space X:*

(1) *$F(X)$ is a k_ω-space or discrete;*
(2) *$F(X)$ is a k-space;*
(3) *$F_n(X)$ is a k-space and csf-countable for some $n \geq 5$;*
(4) *$F_5(X)$ is a k-space and $F_4(X)$ is csf-countable;*
(5) *X is a k_ω-space or discrete.*

Proof. It is shown that (1) ⇔ (2) ⇔ (5) in [32, Theorem 5.3]. (5) ⇒ (3) by Theorem 5.2. Obviously, (3) ⇒ (4). Next, we shall show that (4) ⇒ (5). Without loss of generality we can assume that X is a non-discrete k^*-metrizable space.

(4) ⇒ (5). It follows from Theorem 5.2 that X is an $\aleph_0$-space. So $F_5(X)$ is an $\aleph_0$-space by [2, Theorem 4.1]. Thus, by Lemma 3.4, $F_5(X)$ is a sequential space, because $F_5(X)$ is a k- and σ-space. If $F_4(X)$ contains no closed copy of S_ω, X is compact by Theorem 3.8. If $F_4(X)$ contains a closed copy of S_ω, every first-countable closed subspace of X is locally countably compact by Lemma 5.3. Since X is k^*-metrizable, X has a point-countable k-network; therefore every first-countable closed subspace of X is locally metrizable [10], and hence locally compact. Thus X has a countable k-network $\mathcal{B}$ such that the closure of every element of $\mathcal{B}$ is compact. This implies that X is a k_ω-space. □

Corollary 5.5. *Let X be a metrizable space. Then $F(X)$ is a k-space if and only if $F_5(X)$ is a k-space and csf-countable*

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