

A DECOMPOSITION THEOREM FOR Σ^* -SPACES

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ABSTRACT. In this note it is shown that if f is a continuous closed mapping from a T_1, Σ^* -space X onto Y , then there is a σ -closed discrete subspace Z of Y such that $f^{-1}(y)$ is an ω_1 -compact subspace of X for each $y \in Y \setminus Z$.

We assume that all spaces are T_1 , and all mappings are continuous and onto.

In 1985, Y. Tanaka and Y. Yajima [4] obtained a decomposition theorem for Σ -spaces that every Σ -space X satisfies the following condition(*).

(*) If f is a closed mapping from X onto Y , then there is a σ -closed discrete subspace Z of Y such that $f^{-1}(y)$ is an ω_1 -compact subspace of X for each $y \in Y \setminus Z$.

J. Chaber [1] constructed a counterexample to show that $\Sigma^\#$ -spaces are not always satisfying the above condition (*). Since Σ -spaces are Σ^* -spaces, and Σ^* -spaces are $\Sigma^\#$ -spaces, it is a natural question whether Σ^* -spaces satisfy(*). Y. Tanaka and Y. Yajima [4] obtained only a weak form of decomposition theorem for Σ^* -spaces. The purpose of this note is to prove that Σ^* -spaces satisfy(*).

Recall basic definitions concerning Σ^* -spaces. Suppose that \mathcal{P} is a collection of subsets of a space X . \mathcal{P} is called hereditarily closure-preserving (abbrev. HCP) if $\{H(P); P \in \mathcal{P}\}$ is closure-preserving for every subset $H(P) \subset P \in \mathcal{P}$. A space X is called a Σ^* -space (or, strong Σ^* -space) [3] if there is a covering \mathcal{K} of X by closed countable compact subsets (or, closed compact

*Partly supported by the National Natural Science Foundation of China.

subsets) and a σ -HCP collection \mathcal{P} of closed subsets of X such that whenever $K \subset U$ with $K \in \mathcal{K}$ and U open in X , then $K \subset P \subset U$ for some $P \in \mathcal{P}$. The \mathcal{P} is called a σ -HCP closed (mod \mathcal{K})-network for X .

Lemma (4, Lemma 1.1) *If \mathcal{P} is an HCP collection of subsets of X , then*

$$\{P_1 \cap P_2 \dots \cap P_n; P_i \in \mathcal{P}, i \leq n\}$$

is also HCP in X for each $N \in \mathbb{N}$.

Theorem Σ^* -spaces satisfy (*).

Proof. Suppose that f is a closed mapping from a Σ^* -space X onto Y . Let \mathcal{K} be a covering of X by closed countable compact subsets, and let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ -HCP closed (mod \mathcal{K})-network for X , where each \mathcal{P}_n is HCP in X . Here we can assume that \mathcal{P} is closed under finite intersections by Lemma, and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in \mathbb{N}$, put

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\},$$

then

(1) D_n is a σ -closed discrete subspace of X . In fact, for each $m \in \mathbb{N}$, put

$$E_m = \{x \in X : \cap\{P \in \mathcal{P}_m : x \in P\} = \{x\}\},$$

then E_m is a closed discrete subspace of X by [3, Lemma 2.5]. It is not difficult to check that $D_n \subset \cup\{E_m : m \in \mathbb{N}\}$ (cf.[2]), hence D_n is σ -closed discrete in X . Put

$$Q_n = \{P \setminus D_n : P \in \mathcal{P}_n\},$$

and

$$Q = \cup\{Q_n : n \in \mathbb{N}\},$$

then

(2) There are $m \geq n$, $F \in Q_m$, and $G \subset D_m$ with $\cap \mathcal{F} = F \cup G$ for each finite $\mathcal{F} \subset Q$ and $n \in \mathbb{N}$.

In fact; let $\mathcal{F} = \{F_i : i \leq k\} \subset Q$, we might as well grant $\cap \mathcal{F} \neq \emptyset$, and there are $n_i \in \mathbb{N}$, $P_i \in \mathcal{P}_{n_i}$ with $F_i = P_i \setminus D_{n_i}$ and $n_i \leq n_{i+1}$, then $\cap \mathcal{F} = \cap\{P_i : i \leq k\} \setminus D_{n_k}$, and $\cap\{P_i : i \leq$

$k\} = P$ for some $m \geq \max\{n_k, n\}$, $P \in \mathcal{P}_m$ because \mathcal{P} is closed under finite intersections. Put

$$F = P \setminus D_m, \text{ and}$$

$$G = P \cap (D_m \setminus D_{n_k}),$$

then $F \in \mathcal{Q}_m$, $G \subset D_m$ and $\cap \mathcal{F} = F \cup G$.

For each $n \in N$, put

$$Z_n = f(D_n) \cup \left(\bigcup \{f(Q) \cap f(Q') : Q, Q' \in \mathcal{Q}_n, \text{ and } F(Q) \cap f(Q') \text{ is finite}\} \right).$$

Since \mathcal{Q}_n is HCP in X , $\{f(Q) : Q \in \mathcal{Q}_n\}$ is HCP in Y . Thus Z_n is σ -closed discrete in Y by (1) and Lemma. Put

$$Z = \bigcup \{Z_n : n \in N\},$$

then Z is σ -closed discrete in Y . Take a $y \in Y \setminus Z$, then

(3) $\{Q \in \mathcal{Q}_n : Q \cap f^{-1}(y) \neq \emptyset\}$ is finite.

Assume the contrary, then there is an $m \in N$ and a sequence $\{Q_n\}$ of distinct members of \mathcal{Q}_m such that $Q_n \cap f^{-1}(y) \neq \emptyset$. Pick an $x \in f^{-1}(y)$, then $x \notin X \setminus D_n$ for each $n \in N$, put

$$R_n = \bigcap \{Q \in \mathcal{Q}_n : x \in Q\}.$$

Since \mathcal{Q}_n is point-finite on X there are a $k_n \in N$, $F_n \in \mathcal{Q}_{k_n}$ and $G_n \subset D_{k_n}$ with $m \leq k_n < k_{n+1}$ and $R_n = F_n \cup G_n$ by (2). Put

$$F'_n = Q_n \setminus D_{k_n},$$

$$G'_n = Q_n \cap D_{k_n},$$

then $F'_n \in \mathcal{Q}_{k_n}$, $G'_n \subset D_{k_n}$, and $Q_n = F'_n \cup G'_n$. Since $y \in f(R_n) \cap f(Q_n) \setminus Z = f(F_n) \cap f(F'_n) \setminus Z$, $f(F_n) \cap f(F'_n)$ is an infinite set. So we can choose a sequence $\{y_n\}$ of distinct points in Y such that $y_n \in f(F_n) \cap f(F'_n)$. Pick $p_n \in F_n \cap f^{-1}(y_n)$, and $q_n \in F'_n \cap f^{-1}(y_n)$.

Suppose that the sequence $\{p_n\}$ has not any cluster point in X . Take a $K \in \mathcal{K}$ with $x \in K$, then there is an $i \in N$ such that $K \cap \{p_n : n \geq i\} = \emptyset$, thus $x \in K \subset P \subset X \setminus \{p_n : n \geq i\}$ for some $j \geq i$, $P \in \mathcal{P}_j$. Since $y \in Y \setminus Z$, $x \notin D_j$, then $x \in P \setminus D_j \in \mathcal{Q}_j$, hence $R_j \subset P \setminus D_j$, so $p_j \in F_j \subset R_j \subset P \setminus D_j \subset X \setminus \{p_n : n \geq i\}$, a contradiction. Consequently, the sequence $\{p_n\}$ has a cluster point in X , and the sequence $\{y_n\}$ also has a cluster

point in Y . On the other hand, since $q_n \in F'_n \subset Q_n \in \mathcal{Q}_m$ for each $n \in N$, the sequence $\{q_n\}$ has not any cluster point in X so the sequence $\{y_n\}$ has not any cluster point in Y either, a contradiction. (3) holds. By

$$y \in Y \setminus Z, f^{-1}(y) \subset X \setminus \bigcup \{D_n : n \in N\},$$

thus

$$\{f^{-1}(y) \cap P : P \in \mathcal{P}\} = \{f^{-1}(y) \cap Q : Q \in \mathcal{Q}\},$$

therefore it is a countable closed (mod \mathcal{K}')-network from (3), where $\mathcal{K}' = \mathcal{K}|_{f^{-1}(y)}$. Hence $f^{-1}(y)$ is an ω_1 -compact subspace of X by [4, Lemma 1.2].

Corollary *If f is a closed mapping from a strong Σ^* -space X onto Y , then there is a σ -closed discrete subspace Z of Y such that $f^{-1}(y)$ is Lindelöf for each $y \in Y \setminus Z$.*

Proof. An ω_1 -compact strong Σ^* -space is a space with a countable (mod \mathcal{K})-network with respect to \mathcal{K} by compact subsets. A space with a countable (mod \mathcal{K})-network with respect to \mathcal{K} by compact subsets is Lindelöf.

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