

# The topological properties of $q$ -spaces in free topological groups



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## ABSTRACT

Given a Tychonoff space  $X$ , let  $F(X)$  and  $A(X)$  be respectively the free topological group and the free Abelian topological group over  $X$  in the sense of Markov. In this paper, we provide some topological properties of  $X$  whenever one of  $F(X)$ ,  $A(X)$ , some finite level of  $F(X)$  and some finite level of  $A(X)$  is  $q$ -space (in particular, locally  $\omega$ -bounded spaces and  $r$ -spaces), which give some partial answers to a problem posed in [10].

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## 1. Introduction

Let  $F(X)$  and  $A(X)$  be respectively the free topological group and the free Abelian topological group over a Tychonoff space  $X$  in the sense of Markov [9]. For every  $n \in \mathbb{N}$ , by  $F_n(X)$  we denote the subspace of  $F(X)$  that consists of all words of reduced length at most  $n$  with respect to the free basis  $X$ . The subspace  $A_n(X)$  is defined similarly. We always use  $G(X)$  to denote  $F(X)$  or  $A(X)$ , and  $G_n(X)$  to  $F_n(X)$  or  $A_n(X)$  for each  $n \in \mathbb{N}$ . Therefore, any statement about  $G(X)$  applies to  $F(X)$  and  $A(X)$ , and  $G_n(X)$  applies to  $F_n(X)$  and  $A_n(X)$ .

One of the techniques of studying the topological structure of free topological groups is to clarify the relations of subspaces  $X$ ,  $G(X)$ , and  $G_n(X)$ , where  $n \in \mathbb{N}$ . It is well known that only when  $X$  is discrete,  $G(X)$  can be first-countable. Therefore,  $G(X)$  is first-countable if and only if  $X$  is discrete [7]. Similarly, the

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group  $G(X)$  is locally compact if and only if  $X$  is discrete [4]. More generally, P. Nickolas and M. Tkachenko proved that if  $G(X)$  is almost metrizable, then  $X$  is discrete [10]. Further, K. Yamada gave a characterization for a metrizable space  $X$  such that some  $G_n(X)$  is first-countable [14]. Then P. Nickolas and M. Tkachenko showed that  $F_4(X)$  is of pointwise countable type if and only if  $X$  is either compact or discrete [10]. Since each space being of pointwise countable type is a  $q$ -space, P. Nickolas and M. Tkachenko posed the following problem in [10].

**Problem 1.1.** [10, Problem 5.2] Characterize the spaces  $X$  such that  $F_2(X)$  is a  $q$ -space. What about  $F_n(X)$  for all  $n \in \mathbb{N}$ ? The Abelian case?

In this article, we shall give some partial answers to Problem 1.1. This article is organized as follows. Section 2 introduces the notation and terminology used throughout the paper. In Section 3, we characterize the spaces  $X$  such that  $F(X)$  and  $A(X)$  are  $q$ -spaces. Moreover, the  $q$ -subspaces of the groups  $F(X)$  and  $A(X)$  are studied. In Section 4, we show that  $X$  must belong to some class of spaces if  $F_4(X)$  is a  $q$ -space. We also show that if  $F_2(X)$  is a  $q$ -space then the set  $X'$  of all non-isolated points of  $X$  is bounded in  $X$ . In Section 5, we mainly prove that  $F_2(X)$  is locally  $\omega$ -bounded if  $X$  is homeomorphic to the topological sum of an  $\omega$ -bounded space and a discrete space. In Section 6, some examples and questions are presented and posed respectively.

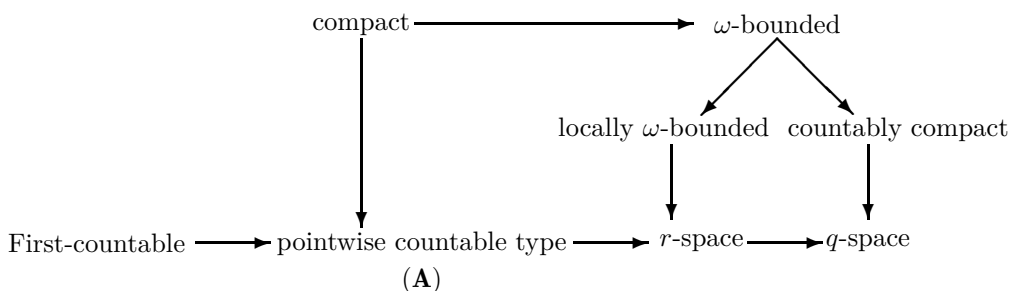
**2. Notation and terminologies**

In this section, we introduce the necessary notation and terminologies. Throughout this paper, all topological spaces are assumed to be Tychonoff, unless explicitly stated otherwise. For undefined notation and terminologies, refer to [3,6,8]. First of all, let  $\mathbb{N}$  and  $\omega_1$  denote the set of positive integers and the first uncountable ordinal, respectively.

Let  $X$  be a space and  $x \in X$ . If there exists a sequence  $\{U_n : n \in \mathbb{N}\}$  of open neighborhoods of  $x$  in  $X$  satisfying the following  $(\blacklozenge)$ , then we say that  $x$  is an  $r$ -point (resp. a  $q$ -point) in  $X$  and the sequence  $\{U_n : n \in \mathbb{N}\}$  is an  $r$ -sequence (resp. a  $q$ -sequence).

$(\blacklozenge)$  For an arbitrary sequence  $\{x_n \in U_n : n \in \mathbb{N}\}$ , there exists a compact subset (resp. countably compact subset)  $K$  of  $X$  such that  $\{x_n \in U_n : n \in \mathbb{N}\} \subset K$ .

Let  $X$  be a space. The space  $X$  is said to be an  $r$ -space (resp. a  $q$ -space) if each point of  $X$  is an  $r$ -point (resp.  $q$ -point) in  $X$ . Recall that  $X$  is the space of pointwise countable type [1] if every point of  $X$  is contained in a compact subspace  $K$  that is of countable character in  $X$ . Recall that a space  $X$  is locally  $\omega$ -bounded at point  $x \in X$  if there exists an open neighborhood  $U$  of the point  $x$  in  $X$  such that the closure of every countable subset of  $\bar{U}$  is compact in  $\bar{U}$ . In particular,  $X$  is said to be locally  $\omega$ -bounded if it is locally  $\omega$ -bounded at each point of  $X$ . A space  $X$  is  $\omega$ -bounded [2] if the closure of every countable subset of  $X$  is compact. The following implications follow directly from definitions:



Note that none of the above implications can be reversed.

Let  $X$  be a topological space  $X$  and  $A$  be a subset of  $X$ . The *closure* of  $A$  in  $X$  is denoted by  $\bar{A}$ . The set  $A$  is called *bounded* in  $X$  provided every continuous real-valued function  $f$  defined on  $X$  is bounded on  $A$ . If the closure of every bounded set in  $X$  is compact, then  $X$  is called a  $\mu$ -space. The *derived set* of  $X$  is denoted by  $X'$ . In addition,  $X$  is called a  $k$ -space provided that a subset  $C \subseteq X$  is closed in  $X$  if  $C \cap K$  is closed in  $K$  for each compact subset  $K$  of  $X$ . In particular,  $X$  is called a  $k_\omega$ -space if there exists a family of countably many compact subsets  $\{K_n : n \in \mathbb{N}\}$  in  $X$  such that each subset  $F$  of  $X$  is closed in  $X$  if  $F \cap K_n$  is closed in  $K_n$  for each  $n \in \mathbb{N}$ .

Let  $X$  be a non-empty space. Throughout this paper,  $X^{-1} := \{x^{-1} : x \in X\}$  and  $-X := \{-x : x \in X\}$ , which are just two copies of  $X$ . Let  $e$  be the neutral element of  $F(X)$  (i.e., the empty word) and  $0$  be that of  $A(X)$ . For every  $n \in \mathbb{N}$  and an element  $(x_1, x_2, \dots, x_n)$  of  $(X \oplus X^{-1} \oplus \{e\})^n$  we call  $g = x_1 x_2 \dots x_n$  a *form*. In the Abelian case,  $x_1 + x_2 \dots + x_n$  is also called a *form* for  $(x_1, x_2, \dots, x_n) \in (X \oplus -X \oplus \{0\})^n$ . This word  $g$  is called *reduced* if it does not contains  $e$  or any pair of consecutive symbols of the form  $xx^{-1}$  or  $x^{-1}x$ . It follows that if the word  $g$  is reduced and non-empty, then it is different from the neutral element  $e$  of  $F(X)$ . Similar assertions (with the obvious changes for commutativity) are valid for  $A(X)$ . For every  $n \in \mathbb{N}$ , let  $i_n : (X \oplus X^{-1} \oplus \{e\})^n \rightarrow F_n(X)$  be the natural mapping defined by

$$i_n(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n$$

for each  $(x_1, x_2, \dots, x_n) \in (X \oplus X^{-1} \oplus \{e\})^n$ . We also use the same symbol in the Abelian case, that is, in means the natural mapping from  $(X \oplus -X \oplus \{0\})^n$  onto  $A_n(X)$ . Clearly, each  $i_n$  is a continuous mapping.

### 3. The characterizations of $q$ -spaces in free topological groups

In this section, we shall give a characterization for a space  $X$  such that  $G(X)$  is a  $q$ -space. Then, we show that  $X$  contains a copy of  $P$  if the free topological group  $G(X)$  contains a copy of some  $q$ -space  $P$ .

We first show two lemmas and a proposition which play an important role in the proof of our main theorem in this section.

**Lemma 3.1.** *Let  $X$  be a non-discrete space. Then, for each open neighborhood  $U$  of  $e$  in  $A(X)$ , we have  $U \not\subseteq A_k(X)$  for each  $k \in \mathbb{N}$ .*

**Proof.** Assume on the contrary that there exist an open neighborhood  $U$  of  $e$  in  $A(X)$  and  $k_0 \in \mathbb{N}$  such that  $U \subseteq A_{k_0}(X)$ . Since  $A(X)$  is a topological group, there is an open neighborhood  $V$  of  $e$  in  $A(X)$  with  $V^{k_0+1} \subseteq U$ . Since  $X$  is not discrete,  $A(X)$  is also not discrete. Take an arbitrary point  $x \in V \setminus \{e\}$ . Then we have

$$x^{k_0+1} \in V^{k_0+1} \subseteq U \subseteq A_{k_0}(X).$$

Since  $A(X)$  is Abelian, the length of  $x^{k_0+1}$  is at least  $k_0 + 1$ . However, each element of  $A_{k_0}(X)$  is at most  $k_0$ . A contradiction occurs.  $\square$

It is well known that each  $A(X)$  is a quotient group of  $F(X)$  for each space  $X$ , hence  $A(X)$  is an open image of  $F(X)$ . Then it follows from Lemma 3.1 that we also have the following lemma.

**Lemma 3.2.** *Let  $X$  be a non-discrete space. Then, for each open neighborhood  $U$  of  $e$  in  $F(X)$ , we have  $U \not\subseteq F_k(X)$  for each  $k \in \mathbb{N}$ .*

**Proposition 3.3.** *Suppose that  $X$  is a space. If  $Y$  is a  $q$ -subspace in  $G(X)$ . Then the family  $\{(G_{n+1} \setminus G_n) \cap Y : n \in \mathbb{N}\}$  is discrete in  $Y$ .*

**Proof.** Assume on the contrary that the family  $\{(G_{n+1} \setminus G_n) \cap Y : n \in \mathbb{N}\}$  is non-discrete in  $Y$ . Then there exists a point  $g \in Y$  such that the family  $\{(G_{n+1} \setminus G_n) \cap Y : n \in \mathbb{N}\}$  is non-discrete at the point  $g$  in  $Y$ . Since  $Y$  is a  $q$ -space, there exists a sequence  $\{U_n : n \in \mathbb{N}\}$  of open neighborhoods at  $g$  in  $Y$  satisfying  $(\blacklozenge)$ . By the assumption, we can choose an increasing subsequence  $\{k_n : n \in \mathbb{N}\}$  of  $\mathbb{N}$  such that  $U_n \cap (G_{k_{n+1}} \setminus G_{k_n}) \neq \emptyset$  for each  $n \in \mathbb{N}$ , and then pick a point  $g_n \in U_n \cap (G_{k_{n+1}} \setminus G_{k_n})$  for each  $n \in \mathbb{N}$ . By  $(\blacklozenge)$ , the set  $\{g_n : n \in \mathbb{N}\}$  is contained in some countably compact subset  $K$  in  $Y$ . Obviously,  $K$  is also a countably compact subset in  $G(X)$ , then it follows from [3, Corollary 7.4.4] that  $K$  is contained in some  $G_m(X)$  for some  $m \in \mathbb{N}$ , which is a contradiction.  $\square$

By Lemmas 3.1, 3.2 and Proposition 3.3, we obtain one of the main theorems in this section, which generalizes a result in [10].

**Theorem 3.4.** *Let  $X$  be a space. If  $G(X)$  is a  $q$ -space, then  $X$  is discrete.*

Moreover, by Proposition 3.3, we have the following result.

**Proposition 3.5.** *Let  $X$  be a non-discrete space. Then, for every sequence  $\{n_i : i \in \mathbb{N}\}$  of natural numbers, the set  $\bigcup_{i \in \mathbb{N}} (G_{n_i}(X) \setminus G_{n_{i-1}}(X))$  is not a  $q$ -subspace.*

**Proof.** By the proof of [13, Corollary 3.4], it easily checks that  $\{G_{n_i}(X) \setminus G_{n_{i-1}}(X) : i \in \mathbb{N}\}$  is not discrete in  $\bigcup_{i \in \mathbb{N}} (G_{n_i}(X) \setminus G_{n_{i-1}}(X))$ . Therefore, it follows from Proposition 3.3 that  $\bigcup_{i \in \mathbb{N}} (G_{n_i}(X) \setminus G_{n_{i-1}}(X))$  is not a  $q$ -space.  $\square$

Next, we shall consider the question when  $X$  contains a copy of a  $q$ -space  $P$  if  $G(X)$  contains a copy of some  $q$ -space  $P$ . We first recall some concepts and show a technical lemma.

We say that a space  $X$  is *densely self-embeddable*, if each non-empty open set in  $X$  contains a copy of  $X$ . A space  $P$  is said to be *prime* if for any spaces  $X$  and  $Y$ , we have the following statement:

*If  $X \times Y$  contains a copy of  $P$ , then either  $X$  or  $Y$  contains a copy of  $P$ .*

**Lemma 3.6.** *Assume that  $Y$  is a densely self-embeddable,  $q$ -space and  $G(X)$  contains a copy of  $Y$ . Then, some  $G_n(X)$  contains a copy of  $Y$ .*

**Proof.** Assume that  $G(X)$  contains a copy of  $Y$ . Since  $Y$  is a densely self-embeddable, it suffices to show that there exists a non-empty open set  $U$  of  $Y$  such that  $U$  is contained in some  $G_n(X)$ . Pick an arbitrary point  $y \in Y$ . Then  $y$  is a  $q$ -point in  $Y$ , and hence there exists a sequence  $\{U_n : n \in \mathbb{N}\}$  of open neighborhoods at  $y$  in  $Y$  such that  $\{U_n : n \in \mathbb{N}\}$  satisfies the condition  $(\blacklozenge)$ . We claim that  $U_k \subseteq G_n(X)$  for some  $k, n \in \mathbb{N}$ . Assume on the contrary that  $U_k \not\subseteq G_n(X)$  for any  $k, n \in \mathbb{N}$ . Then it is easy to see that there exist a countably compact subset  $K$  and a sequence  $\{y_n : n \in \mathbb{N}\}$  in  $Y$  such that  $\{y_n : n \in \mathbb{N}\} \subseteq K$  and  $y_n \notin G_n(X)$  for each  $n \in \mathbb{N}$ . However, it follows from [3, Corollary 7.4.4] that  $K$  is contained in some  $G_n(X)$ , which is a contradiction. Therefore,  $U_k \subseteq G_n(X)$  for some  $k, n \in \mathbb{N}$ , hence  $G_n(X)$  contains a copy of  $Y$ .  $\square$

Now we can show the second main theorem in this section, which generalizes a result in [5]. Indeed, K. Eda, H. Ohta and K. Yamada showed that, for each densely self-embeddable and prime space which is either compact or first countable,  $X$  contains a copy of  $P$  iff  $F(X)$  contains a copy of  $P$  iff  $A(X)$  contains a copy of  $P$ . By the implications of diagram **(A)**, we see that each compact or first-countable space is a  $q$ -space.

**Theorem 3.7.** *Let  $P$  be a densely self-embeddable, prime  $q$ -space. For a space  $X$ , the following are equivalent:*

- (1)  $X$  contains a copy of  $P$ .
- (2)  $F(X)$  contains a copy of  $P$ .
- (3)  $A(X)$  contains a copy of  $P$ .

**Proof.** Clearly, it suffices to show that (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1).

(2)  $\Rightarrow$  (1). Assume that  $F(X)$  contains a copy of  $P$ . By Lemma 3.6, some  $F_n(X)$  contains a copy of  $P$ . Without loss of generality, we may assume that  $n$  is the least number such that  $F_n(X)$  contains a copy of  $P$ . Since  $P$  is densely self-embeddable,  $F_n(X) \setminus F_{n-1}(X)$  contains a copy of  $P$ . Further, we may assume that  $P$  is a subspace of  $F_n(X) \setminus F_{n-1}(X)$ . Then  $(X \oplus X^{-1})^n$  contains a copy of  $P$  since  $F_n(X) \setminus F_{n-1}(X)$  is homeomorphic to a subspace of  $(X \oplus X^{-1})^n$ . Since  $P$  is prime and densely self-embeddable,  $X$  contains a copy of  $P$ .

(3)  $\Rightarrow$  (1). Similar to (2)  $\Rightarrow$  (1), one see that  $A_n(X) \setminus A_{n-1}(X)$  contains a copy of  $P$  for some least number  $n \in \mathbb{N}$ . It is well known that  $A_n(X) \setminus A_{n-1}(X)$  is homeomorphic to a subspace of finite symmetric products  $(X \oplus X^{-1})^n/n!$  (see [5]). It follows from [5, Proposition 2.1] that  $X$  itself contains a copy of  $P$ .  $\square$

#### 4. The topological properties of $q$ -spaces in $F_n(X)$

In this section, we shall give some topological properties for a space  $X$  whenever  $G_n(X)$  is a  $q$ -space or an  $r$ -space. First, we recall a theorem which was proved in [14].

Let  $X$  be a space, and let  $\mathcal{U}_X$  be the finest uniformity on  $X$  compatible with the topology of  $X$ . For each  $U \in \mathcal{U}_X$ , put

$$O_2(U) := \{x^\varepsilon y^{-\varepsilon} : (x, y) \in U, \varepsilon = \pm 1\}.$$

In [14], K. Yamada obtained the following theorem.

**Theorem 4.1.** ([14]) *The family  $\{O_2(U) : U \in \mathcal{U}_X\}$  is a neighborhood base at  $e$  in  $F_2(X)$ .*

Now we can give a topological property for a space  $X$  when  $F_2(X)$  is a  $q$ -space, which is similar to that in [10, Lemma 2.1].

**Proposition 4.2.** *If  $F_2(X)$  is a  $q$ -space, then the derived set  $X'$  is bounded in  $X$ .*

**Proof.** Since  $F_2(X)$  is a  $q$ -space, there exists a sequence  $\{U_n : n \in \mathbb{N}\}$  of open neighborhoods at  $e$  in  $F_2(X)$  such that  $\{U_n : n \in \mathbb{N}\}$  satisfies  $(\blacklozenge)$ . Suppose that  $X'$  is not bounded in  $X$ . Then there exists in  $X$  a discrete family of open subsets  $\{V_n : n \in \mathbb{N}\}$  each of which intersects  $X'$ . Choose a point  $x_n \in V_n \cap X'$  for each  $n \in \mathbb{N}$ . Since each  $x_n \in X'$ , we can find  $y_n \in V_n$  such that  $x_n \neq y_n$  and  $x_n^{-1}y_n \in U_n$  for each  $n \in \mathbb{N}$ . It is easy to see that  $x_n^{-1}y_n \neq x_m^{-1}y_m$  for distinct  $m, n \in \mathbb{N}$ . Set  $P := \{x_n^{-1}y_n : n \in \mathbb{N}\}$ .

**Claim:** The set  $P$  is closed and discrete in  $F_2(X)$ .

Indeed, it suffices to show that each point in  $F_2(X)$  is not a cluster point of  $P$ .

**Subclaim 1:** Each point in  $X \cup X^{-1}$  is not a cluster point of  $P$ .

Obviously, all elements of  $P$  have length 2. Since the set  $X \cup X^{-1}$  is open in  $F_2(X)$ , the set  $P$  has no cluster points in  $X \cup X^{-1}$ .

**Subclaim 2:** The point  $e$  is not a cluster point of  $P$ .

Since the family  $\{V_n : n \in \mathbb{N}\}$  is discrete in  $X$  and  $x_n$  and  $y_n$  are distinct points in  $V_n$ , it follows from [6, 8.1.C] that the set

$$U := (X \times X) \setminus \{(x_n, y_n), (y_n, x_n) : n \in \mathbb{N}\}$$

belongs to the finest uniformity on  $X$ . Then it follows from [Theorem 4.1](#) that the set  $O_2(U)$  is an open neighborhood of the point  $e$  in  $F_2(X)$ . Obviously,  $O_2(U) \cap P = \emptyset$ . Therefore,  $e$  is not a cluster point of  $P$ .

**Subclaim 3:** Each point in  $F_2(X) \setminus F_1(X)$  is not a cluster point of  $P$ .

Clearly, the natural mapping  $i_2 : (X \oplus X^{-1} \oplus \{e\})^2 \rightarrow F_2(X)$  is a local homeomorphism from  $(X \oplus X^{-1} \oplus \{e\})^2$  to  $F_2(X)$  at each point of  $(X \oplus X^{-1} \oplus \{e\})^2 \setminus i_2^{-1}(e)$ . Moreover, the set  $L := \{(x_n^{-1}, y_n) : n \in \mathbb{N}\}$  is closed and discrete in  $(X \oplus X^{-1} \oplus \{e\})^2$ , which shows that the set  $P = j_2(L)$  has no cluster points in  $F_2(X) \setminus F_1(X)$ .

Therefore, the claim is verified.

However,  $F_2(X)$  is a  $q$ -space, then  $P$  has a cluster point in  $F_2(X)$  since each  $x_n^{-1}y_n \in U_n$ , which is a contradiction.

Therefore, the derived set  $X'$  is bounded in  $X$ .  $\square$

By [Proposition 4.2](#), the following two corollaries are obvious.

**Corollary 4.3.** *If  $X$  is a homogeneous space and  $F_2(X)$  is a  $q$ -space, then  $X$  is pseudocompact or discrete.*

**Corollary 4.4.** *If  $F_2(X)$  is a  $q$ -space and  $X$  is a  $\mu$ -space, then  $X'$  is compact.*

Next, we shall consider the question that  $X$  belongs to what kind of classes of spaces if  $F_4(X)$  is a  $q$ -space. We first recall two concepts.

A space is called a *cf-space* if each compact subset of it is finite. Recall that a subspace  $Y$  of a space  $X$  is said to be  *$P$ -embedded in  $X$*  if each continuous pseudometric on  $Y$  admits a continuous extensions over  $X$ .

**Theorem 4.5.** *Let  $X$  be a space. If  $F_4(X)$  is a  $q$ -space, then  $X$  is either pseudocompact or a  $cf$ -space.*

**Proof.** Suppose  $X$  is not a  $cf$ -space. Then there exists an infinite compact subset  $C$  in  $X$ . Next we shall show that  $X$  is pseudocompact.

Assume on the contrary that we can find a countably infinite discrete family of open sets  $\{U_n : n \in \mathbb{N}\}$  in  $X$ . It easily checks that the family of  $\{\overline{U_n} : n \in \mathbb{N}\}$  is also discrete in  $X$ , and hence  $\bigcup_{n \in \mathbb{N}} \overline{U_n}$  is closed in  $X$ . Since the set  $C$  is compact, it can intersect at most a finite number of the sets  $\overline{U_n}$ . Without loss of generality, we may assume that  $C \cap \bigcup_{n \in \mathbb{N}} \overline{U_n} = \emptyset$ . Now the family  $\{C\} \cup \{U_n : n \in \mathbb{N}\}$  is discrete in  $X$ .

Since  $C$  is an infinite compact subset in  $X$ , there exists a non-isolated point  $x$  in  $C$ . For each  $n \in \mathbb{N}$ , pick  $y_n \in U_n$ , and then put  $C_n := y_n^{-1}x^{-1}Cy_n$ . Let

$$Y := C \cup \{y_n : n \in \mathbb{N}\} \text{ and } Z := \bigcup_{n \in \mathbb{N}} C_n.$$

Obviously,  $Y$  is closed,  $\sigma$ -compact, non-discrete and  $P$ -embedded in  $X$ . By [\[12\]](#), the subgroup  $F(Y, X)$  of  $F(X)$  generated by  $Y$  is naturally topologically isomorphic to the free topological group  $F(Y)$ . Moreover, since  $Y$  is a  $k_\omega$ -space, it follows from [\[3\]](#) that the free group  $F(Y)$  is also a  $k_\omega$ -space, hence  $F_4(Y)$  is also a  $k_\omega$ -space. Next, we claim that  $Z$  is closed in  $F_4(Y)$ .

Indeed, for each  $n \in \mathbb{N}$ , let

$$K_n := C \cup C^{-1} \cup \{y_i, y_i^{-1} : i \leq n\} \cup \{e\}$$

and

$$X_n := \overbrace{K_n \cdot \dots \cdot K_n}^n$$

Then it follows from the proof of [3, Theorem 7.4.1] that the topology of  $F(Y)$  is determined by the family of compact subsets  $\{X_n : n \in \mathbb{N}\}$ . Hence the topology of  $F_4(Y)$  is determined by the family  $\{X_n \cap F_4(Y) : n \in \mathbb{N}\}$ . Then we have  $Z \cap X_n \cap F_4(Y) = \bigcup_{i=1}^n C_i$  for each  $n \in \mathbb{N}$ , which shows that  $Z$  is closed in  $F_4(Y)$ . Therefore,  $Z$  is a  $q$ -space. Moreover, one sees that the topology of  $Z$  is determined by the family  $\{C_n : n \in \mathbb{N}\}$ , which shows that  $Z$  is not a  $q$ -space since any compact subset of  $Z$  lies in a finite union  $\bigcup_{i \leq n} C_i$  for some  $n \in \mathbb{N}$ . Hence  $e$  is not a  $q$ -point in  $F_4(X)$ , which is a contradiction.  $\square$

**Remark 4.6.** Let  $Z = X \oplus Y$ , where  $X$  is an infinite compact space and  $Y$  is an infinite discrete space. Then  $F_4(Z)$  is not a  $q$ -space by Theorem 4.5. By [14], one sees that “ $F_4(X)$ ” in Theorem 4.5 cannot be replaced by “ $F_3(X)$ ”. Indeed, if  $X$  is a compact metrizable space, then  $F_3(Z)$  is metrizable [14]. However,  $Z$  is neither pseudocompact nor a  $cf$ -space.

It is natural to consider what conditions on a space  $X$  can guarantee  $X$  to be countably compact or discrete if  $F_4(X)$  is a  $q$ -space. We first recall two concepts.

A topological space is said to be *collectionwise Hausdorff* if given any closed discrete collection of points in the topological space, there are pairwise disjoint open sets containing the points. A topological space  $X$  is *weakly- $k$*  if and only if a subset  $F$  of  $X$  is closed in  $X$  if  $F \cap C$  is finite for every compact subset  $C$  in  $X$ . Obviously, we have the following two facts:

- (a) Each  $k$ -space is a weakly- $k$  space.
- (b) Each weakly- $k$   $cf$ -space is discrete.

However, there exists a weakly- $k$  space which is not a  $k$ -space, see [11].

Obviously, each collectionwise Hausdorff pseudocompact space is countably compact. Hence, we have the following corollary.

**Corollary 4.7.** *Let  $X$  be a collectionwise Hausdorff space. If  $F_4(X)$  is a  $q$ -space, then  $X$  is either countably compact or a  $cf$ -space.*

Further, we have the following Theorem 4.8, which is a more stronger form of Corollary 4.7.

**Theorem 4.8.** *Let  $X$  be a collectionwise Hausdorff, weakly- $k$  and non-discrete space. If  $F_4(X)$  is a  $q$ -space, then  $X^2$  is countably compact.*

**Proof.** Since  $X$  is a non-discrete weakly- $k$ -space, we see that  $X$  is not a  $cf$ -space, hence  $X$  is pseudocompact by Theorem 4.5. Then  $X$  is countably compact since it is collectionwise Hausdorff. Therefore, it follows from [11] that  $X^2$  is countably compact.  $\square$

If we replace “ $q$ -space” with “ $r$ -space” in Theorem 4.5, we can obtain a more stronger result.

**Corollary 4.9.** *Let  $X$  be a space. If  $F_4(X)$  is an  $r$ -space, then  $X$  is either pseudocompact or discrete.*

**Proof.** Suppose that  $X$  is a  $cf$ -space, then  $X$  is discrete since  $X$  is an  $r$ -space. Therefore, the corollary holds by Theorem 4.5.  $\square$

By Corollary 4.9, we have the following theorem.

**Theorem 4.10.** *The following conditions are equivalent for a  $\mu$ -space  $X$ :*

- (1)  $F_4(X)$  is an  $r$ -space.
- (2)  $F_4(X)$  is of pointwise countable type.
- (3)  $F_4(X)$  is locally compact.
- (4)  $F_n(X)$  is of pointwise countable type for each  $n \in \mathbb{N}$ .
- (5)  $F_n(X)$  is locally compact for each  $n \in \mathbb{N}$ .
- (6) The space  $X$  is either compact or discrete.

**Proof.** For any space  $X$ , the equivalence of items (2), (3), (4) and (5) of the theorem were established in [10]. Obviously, we have (5)  $\Rightarrow$  (1). By Corollary 4.9,  $X$  is pseudocompact or discrete. If  $X$  is pseudocompact, it follows that  $X$  is compact since it is a  $\mu$ -space. Hence (1)  $\Rightarrow$  (2).  $\square$

### 5. The topological properties of $\omega$ -bounded spaces in $F_2(X)$

In this section, we shall give some topological properties for a space  $X$  when  $G_2(X)$  is an  $\omega$ -bounded space. First, we give some topological properties of an  $\omega$ -bounded space.

Obviously, each compact space is  $\omega$ -bounded, and each  $\omega$ -bounded space is a countably compact  $r$ -space. Moreover, it is easy to see that  $[0, \omega_1)$  is an  $\omega$ -bounded and non-compact space. It is well known that there exist two countably compact spaces such that their product is not countably compact. However, it easily checked that the product of a family of  $\omega$ -bounded spaces is again  $\omega$ -bounded.

The following two propositions are obvious.

**Proposition 5.1.** *Let  $f : X \rightarrow Y$  be a continuous mapping. If  $X$  is  $\omega$ -bounded, then  $Y$  is also  $\omega$ -bounded.*

By [6, Propositions 3.72 and 3.7.6], it is easy to show the following proposition.

**Proposition 5.2.** *Let  $f : X \rightarrow Y$  be a continuous perfect mapping. If  $Y$  is  $\omega$ -bounded (resp. locally  $\omega$ -bounded), then  $X$  is also  $\omega$ -bounded (locally  $\omega$ -bounded).*

**Proposition 5.3.** *Let  $X = Y \oplus D$  be the topological sum of a space  $Y$  and a discrete space  $D$ . Then we have the following statements.*

- (a)  $F_2(X)$  is an  $r$ -space iff  $F_2(Y)$  is an  $r$ -space.
- (b)  $F_2(X)$  is a locally  $\omega$ -bounded space iff  $F_2(Y)$  is a locally  $\omega$ -bounded space.
- (c) For every  $n \in \mathbb{N}$ ,  $A_n(X)$  is a  $q$ -space (resp.  $r$ -space, locally  $\omega$ -bounded) iff  $A_n(Y)$  is a  $q$ -space (resp.  $r$ -space, locally  $\omega$ -bounded).

**Proof.** (a) Since  $Y$  is a retract of  $X$ , the closed subgroup  $F(Y, X)$  of  $F(X)$  is topologically isomorphic to the group  $F(Y)$ . Therefore,  $F_2(Y)$  can be identified with the closed subspace  $F_2(Y, X) := F(Y, X) \cap F_2(X)$  of  $F_2(X)$ . Thus  $F_2(Y)$  is an  $r$ -space if  $F_2(X)$  is an  $r$ -space.

Conversely, suppose that  $F_2(Y)$  is an  $r$ -space. Since  $Y$  is closed in  $X$ , it is easy to see that  $X$  is an  $r$ -space. Then  $(X \oplus X^{-1} \oplus \{e\})^2$  is an  $r$ -space. Further, since the natural mapping  $i_2 : (X \oplus X^{-1} \oplus \{e\})^2 \rightarrow F_2(X)$  is a local homeomorphism from  $(X \oplus X^{-1} \oplus \{e\})^2$  to  $F_2(X)$  at each point of  $(X \oplus X^{-1} \oplus \{e\})^2 \setminus i_2^{-1}(e)$ , each point of  $F_2(X) \setminus \{e\}$  is an  $r$ -point. It remains to show that  $e$  is an  $r$ -point in  $F_2(X)$ .

Since  $e$  is an  $r$ -point in  $F_2(Y)$ , there exists a sequence of open neighborhoods  $\{U_n : n \in \mathbb{N}\}$  of  $e$  in  $F_2(Y)$  such that  $\{U_n : n \in \mathbb{N}\}$  is an  $r$ -sequence. For every  $n \in \mathbb{N}$ , it follows from Theorem 4.1 that there exists a  $W_n \in \mathcal{U}_Y$  such that  $O_2(W_n) \subset U_n$ . Put  $W_n^* := W_n \cup \{(d, d) : d \in D\}$  for each  $n \in \mathbb{N}$ . Clearly, each  $W_n^*$  belongs to the finest uniformity of  $X$ , and  $O_2(W_n) = O_2(W_n^*)$ . Since  $F(Y)$  is topologically isomorphic to the closed subgroup  $F(Y, X)$  of  $F(X)$ , the sequence  $\{O_2(W_n^*) : n \in \mathbb{N}\}$  is an  $r$ -sequence at  $e$  in  $F_2(X)$ .

(b) The proof is quite similar to (a), so we omit it.



(c) It is well known that  $A(X)$  is topologically isomorphic in the natural way with  $A(Y) \times A(D)$ . If we identify  $A(X)$  and  $A(Y) \times A(D)$  under this isomorphism, then we have  $A_n(X) \subseteq A_n(Y) \times A_n(D)$  and  $A_n(X)$  is closed in  $A_n(Y) \times A_n(D)$  for each  $n \in \mathbb{N}$ . Since  $A(D)$  is discrete, each  $A_n(D)$  is discrete, and hence it is easy to see that (c) holds.  $\square$

**Proposition 5.4.** *If  $A_2(X)$  is locally  $\omega$ -bounded, then  $F_2(X)$  is also locally  $\omega$ -bounded*

**Proof.** Let  $\varphi : F(X) \rightarrow A(X)$  be the canonical homomorphism from  $F(X)$  onto  $A(X)$ . Then the restriction  $\varphi_2 := \varphi \upharpoonright F_2(X)$  is a perfect mapping from  $F_2(X)$  onto  $A_2(X)$  by [10, Lemma 2.10]. Hence  $F_2(X)$  is locally  $\omega$ -bounded by Proposition 5.2.  $\square$

By Proposition 5.3, we obtain the main result in this section.

**Theorem 5.5.** *If  $X$  is homeomorphic to the topological sum of an  $\omega$ -bounded space and a discrete space, then  $F_2(X)$  is locally  $\omega$ -bounded.*

## 6. Examples and questions

First, some examples are presented to show the applications of our results.

**Example 6.1.** There exists a first-countable space  $X$  such that  $F_4(X)$  is an  $\omega$ -bounded space (in particular, it is an  $r$ -space) and  $F_2(X)$  is not of pointwise countable type.

**Proof.** Let  $X := [0, \omega_1)$  be endowed with the order topology. Then  $X$  is an  $\omega$ -bounded and non-compact space. By Proposition 5.1, the subspace  $F_4(X)$  is an  $\omega$ -bounded space. Suppose that  $F_2(X)$  is the space of pointwise countable type. Then the derived set  $X'$  is compact in  $X$  by [10, Lemma 2.1], and hence  $X$  is paracompact, which is a contradiction.  $\square$

**Remark 6.2.** (1) By Example 6.1, we can not omit the condition “ $\mu$ -space” in Theorem 4.10.

(2) By Example 6.1, we know that  $X$  may not be compact in Theorem 4.5.

**Example 6.3.** There exists a space  $X$  such that  $F_2(X)$  is an  $r$ -space and  $F_4(X)$  is not an  $r$ -space.

**Proof.** Let  $X := [0, \omega_1) \oplus D$ , where  $[0, \omega_1)$  is endowed with the order topology and  $D$  is an uncountable set with a discrete topology. By Theorem 5.5,  $F_2([0, \omega_1) \oplus D)$  is an  $r$ -space. However, it follows from Corollary 4.9 that  $F_4([0, \omega_1) \oplus D)$  is not an  $r$ -space since  $[0, \omega_1) \oplus D$  is not pseudocompact.  $\square$

In [10], P. Nickolas and M. Tkachenko proved that  $F_2(X)$  is locally compact iff  $A_2(X)$  is locally compact iff  $X$  is homeomorphic to the topological sum of a compact space and a discrete space. From Theorem 5.5, it is natural to pose the following question.

**Question 6.4.** *Let  $F_2(X)$  be locally  $\omega$ -bounded. Is  $X$  homeomorphic to the topological sum of an  $\omega$ -bounded space and a discrete space?*

Further, we have the following question.

**Question 6.5.** *If  $F_2(X)$  is a non-discrete  $r$ -space, is  $X$  homeomorphic to the topological sum of an  $\omega$ -bounded space and a discrete space?*

In [10], P. Nickolas and M. Tkachenko proved that  $F_2(X)$  is locally compact iff  $F_3(X)$  is locally compact for any space. Therefore, it is natural to pose the following question.

**Question 6.6.** *If  $F_2(X)$  is an  $r$ -space (resp.  $q$ -space), is  $F_3(X)$  also an  $r$ -space ( $q$ -space)?*

By Theorem 4.5, we do not know the answer to the following question.

**Question 6.7.** *Let  $X$  be a space. If  $F_4(X)$  is a  $q$ -space, is  $X$  either countably compact or a  $cf$ -space?*

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