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G-methods, G-sequential spaces and G-continuity in topological spaces ☆

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ABSTRACT

G-methods and G-continuity for real functions are induced by changing the definition of the convergence of sequences on the set of real numbers. In this paper we introduce the concepts of G-methods, G-submethods and G-topologies on arbitrary sets and the related notion of G-continuity. We investigate operations on subsets that deal with G-hulls, G-closures, G-kernels and G-interiors, and we study topological spaces that are G-sequential spaces, G-Fréchet spaces or G-topologizable spaces. The G-methods on first-countable topological groups and several convergence methods on topological spaces are extended and studied in a unified way. In particular, several results for G-methods on first-countable topological groups are improved.

1. G-methods in arbitrary sets

Sequential convergence is an important research object in topology and analysis. On the one hand, convergence is closely related to continuity, compactness and other related properties. On the other hand, it has played a fundamental role in mathematics and its applications. Besides the ordinary convergence of sequences, there exists a wide variety of convergence types that are very important not only in pure mathe-

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matics but also in other branches of science involving mathematics especially in information theory, biological science and dynamical systems. We mention, for example, A-convergence of a matrix method in summability theory, almost convergence in functional analysis, Cesàro convergence and statistical convergence in real analysis [7]. Based on several kinds of convergence properties in real analysis, Connor and Grosse-Erdmann [7] introduced G-methods defined on a linear subspace of the vector space of all real sequences, G-convergence on real spaces and G-continuity for real functions, studied the relationship among G-continuous functions, linear functions and continuous functions, established the dichotomy theorem of G-continuity and extended several known results in the literature. Çakallı [4–6] extended the concepts to topological groups satisfying the first axiom of countability, defined G-sequential compactness and G-sequential connectedness. At the same time, he also discussed G-sequential continuity by means of G-sequential closures and G-sequentially closed sets. Recently, Mucuk and Şahan [16] have introduced the notions of G-sequentially open sets and G-sequential neighborhoods of first-countable topological groups, studied the operations of G-sequentially closed sets and G-sequentially open sets, and investigated G-sequential continuity in topological groups.

There have been a number of similar investigations that replace the usual definition of sequential convergence with one of a variety of other definitions that is typically related to statistical convergence. As a generalization of convergence, statistical convergence is a special case of G-methods. The notion of statistical convergence for real sequences was introduced by Fast [11] in 1951. Çakallı [3] discussed statistical convergence in first-countable topological groups. Di Maio and Kočinac [9] defined statistical convergence in topological spaces, introduced statistically sequential spaces and statistically Fréchet spaces, and considered their applications in selection principles theory, function spaces and hyperspaces. Liu, Tang and Lin [15] further studied statistically sequentially continuous mappings in topological spaces. Topological spaces are not only the abstract version of the space of all real numbers but also the basis of topological groups. The concepts of convergence, sequential continuity and sequential compactness essentially only relate to open or closed sets in a topological space. Motivated by the investigations in statistical convergence described above we try to introduce G-convergence and G-methods in arbitrary sets, study the operations of G-hulls, G-closures, G-kernels and G-interiors, discuss the G-generalized topology induced by G-methods, and establish some conditions implying G-continuity in topological spaces, or even in arbitrary sets.

We obtain conditions for the coincidence of closures and G-closures, interiors and G-interiors in topological spaces, discuss characterizations, hereditary properties and mapping properties of G-sequential spaces and G-Fréchet spaces, compare the topology of a topological space and the family of all G-open sets on the space, and give the mutual relationship between continuity and G-continuity. These show that G-methods on topological groups and sequential convergence methods on topological spaces can be treated in a unified way. Some results for G-methods on first-countable topological groups are extended and improved. Thus G-methods really become a method to study convergence and continuity in general topology. Although some proofs of the results in this paper are not difficult, and the questions considered in topological spaces lose the properties of algebraic operations, they are more general. We believe that this work, especially in G-continuity, will have a positive impact on future research and applications.

In the sequel of this section, we introduce G-methods and G-convergence on arbitrary sets. Let X be a set, $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x} \in s(X)$ if and only if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If $f : X \to Y$ is a mapping, then $f(\boldsymbol{x}) = \{f(x_n)\}_{n\in\mathbb{N}}$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X)$. If X is a topological space, the set of all X-valued convergent sequences is denoted by $c(X)$, and we put $\lim \boldsymbol{x} = \lim\limits_{n\to\infty} x_n$ for any $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c(X)$. All topological spaces are assumed to satisfy the T_2 separation property.

Definition 1.1. Let X be a set.

(1) A *method* on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$. A sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be *G-convergent* to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$.

(2) Let X be a topological space.
  (2.1) A method $G : c_G(X) \to X$ is called *regular* if $c(X) \subset c_G(X)$ and $G(\boldsymbol{x}) = \lim \boldsymbol{x}$ for each $\boldsymbol{x} \in c(X)$.
  (2.2) A method $G : c_G(X) \to X$ is called *subsequential* if, whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{x}' \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}' = l$.

Firstly, Connor and Grosse-Erdmann [7] described G-methods and related concepts in the space of all real numbers. Subsequently, Çakallı [4] extended these concepts to first-countable topological groups. Inspired by their work, we discuss these methods in arbitrary sets.

We allow that $c_G(X) = \emptyset$ for a method G in a set X. The definition of G-methods and G-convergence does not involve a topology of a set X. It is only by the notions of regular and subsequential methods that a link is established between G-convergence and convergence in the topology of a topological space X. On the other hand, because X is only a set or a topological space, $c_G(X)$ and G are not required to be a group or a homomorphic mapping in Definition 1.1, respectively.

Readers may refer to [10] for some terminology unstated here.

2. G-hulls and G-closures

As a generalization of the concept of closures in topological spaces, G-hulls and G-closures are essential concepts in G-methods.

A subset A of a topological space X is called a *sequentially closed set* of X if, whenever $\boldsymbol{x} \in s(A) \cap c(X)$, then $\lim \boldsymbol{x} \in A$ [13].

Definition 2.1. Let X be a set, and G be a method on X. A subset A of X is called a *G-closed set* of X if, whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$.

Remark 2.2. The appropriate choice of terminology is important. The notion in Definition 2.1 was called a *G-closed set* for the real line in [7], and a *G-sequentially closed set* for first-countable topological groups in [4]. Since the definition of G already contains the fact that only sequences are involved, "*sequentially*" in *G-sequentially closed sets* seems redundant, so we choose the terminology of *G-closed sets.* In the sequel of this paper, the terminology related to G-methods does not usually say "G-sequentially" or "G-sequential" except for Definition 5.1(1).

The following proposition is obvious.

Proposition 2.3. *If G is a method on a set X, the intersection of any family of G-closed sets in X is G-closed.*

Suppose X is a topological space and $A \subset X$. The *sequential closure* of A is defined as the set $\{\lim \boldsymbol{x} : \boldsymbol{x} \in s(A) \cap c(X)\}$, and it is denoted by $[A]_{seq}$ [1, p. 13].

Definition 2.4. Let X be a set, G be a method on X and $A \subset X$.

(1) The *G-hull* of A is defined as the set $\{G(\boldsymbol{x}) : \boldsymbol{x} \in s(A) \cap c_G(X)\}$, and the G-hull of A is denoted by $\mathrm{hu}_G(A)$ or $[A]_G$.
(2) The *G-closure* of A is defined as the intersection of all G-closed sets containing A, and the G-closure of A is denoted by $\mathrm{cl}_G(A)$ or $\overline{A}^G$.

Remark 2.5. (1) The G-hull of a set A was used in [7, p. 102] and was denoted by $\overline{A}^G$, and it was called the G-sequential closure of A in [4, p. 596].

(2) It follows from Proposition 2.3 that the G-closure of a set A is the smallest G-closed set containing A. This is quite similar to the property of the closure of a set in a topological space.

(3) The most important properties of the closure operator in topological spaces are given by the Kuratowski axioms. It is easy to see that $[A]_{seq} \cup [B]_{seq} = [A \cup B]_{seq}$ for each $A, B \subset X$. Unfortunately, $[A]_G \cup [B]_G = [A \cup B]_G$ or $\overline{A}^G \cup \overline{B}^G = \overline{A \cup B}^G$ is not always true, see Example 2.14(1). In fact, J.L. Kelley had noticed this, and pointed out [14, p. 109]: "However, the really damaging fact is that the sequential closure of a set may fail to be sequentially closed; sequential closure is not a Kuratowski closure operator."

Obviously, $G(s(A) \cap c_G(X)) = [A]_G$, $A \subset [A]_{seq} \subset \overline{A}$, and A is sequentially closed if and only if $[A]_{seq} \subset A$ in a topological space X.

Proposition 2.6. *Let G be a method on a set X and $A \subset X$. The following are equivalent.*

(1) *A is G-closed.*
(2) *$[A]_G \subset A$.*
(3) *$\overline{A}^G \subset A$, i.e., $\overline{A}^G = A$.*

Proof. (1) ⇔ (2) by Definition 2.1, and (1) ⇔ (3) by Definition 2.4(2) and Proposition 2.3. □

Thus $\overline{\overline{A}^G}^G = \overline{A}^G$ for each $A \subset X$. But, it is possible that $[[A]_G]_G \neq [A]_G$, see Example 2.13(3).

Corollary 2.7. *Let G be a method on a set X. Then $[A]_G \subset \overline{A}^G$ for each $A \subset X$.*

Proof. $\overline{A}^G$ is G-closed by Proposition 2.3. It follows from $A \subset \overline{A}^G$ and Proposition 2.6 that $[A]_G \subset [\overline{A}^G]_G \subset \overline{A}^G$. □

Generally speaking, $A \subset [A]_G$ or $[A]_G = \overline{A}^G$ is not always true. We may not even have $X = [X]_G$, see Example 2.13(1).

Next, we discuss the relations between the closed sets and the G-closed sets in topological spaces.

For a topological space X and a sequence $\boldsymbol{x} = \{x_n\}_{n \in \mathbb{N}} \in s(X)$, let

$$[\boldsymbol{x}] = \{x_n : n \in \mathbb{N}\} \cup \{l \in X : l \text{ is an accumulation point of } \boldsymbol{x}\}.$$

Lemma 2.8. *Let G be a method on a topological space X. The following are equivalent.*

(1) *$[A]_G \subset \overline{A}$ for each $A \subset X$.*
(2) *$\overline{A}^G \subset \overline{A}$ for each $A \subset X$.*
(3) *Every closed set in X is G-closed.*
(4) *If $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in [\boldsymbol{x}]$ for each $A \subset X$.*

Proof. (1) ⇒ (4). For each $A \subset X$, let $\boldsymbol{x} = \{x_n\}_{n \in \mathbb{N}} \in s(A) \cap c_G(X)$ and $l = G(\boldsymbol{x})$. We can assume that $l \neq x_n$ for each $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, by condition (1), we have that

$$l \in [\{x_n : n \in \mathbb{N}\}]_G \subset \overline{\{x_n : n \in \mathbb{N}\}} = \{x_n : n \leq m\} \cup \overline{\{x_n : n > m\}},$$

so $l \in \overline{\{x_n : n > m\}}$. Thus $l \in \bigcap_{m \in \mathbb{N}} \overline{\{x_n : n > m\}}$, and l is an accumulation point of $\boldsymbol{x}$. Thus $G(\boldsymbol{x}) \in [\boldsymbol{x}]$.

(4) ⇒ (3). Suppose that A is a closed set in X. If $l \in [A]_G$, then there exists an $\boldsymbol{x} \in s(A) \cap c_G(X)$ with $G(\boldsymbol{x}) = l$. By condition (4), we know that $G(\boldsymbol{x}) \in [\boldsymbol{x}] \subset \overline{A} = A$. So $[A]_G \subset A$, i.e., A is G-closed in X by Proposition 2.6.

(3) ⇒ (2). Let $A \subset X$. Because $\overline{A}$ is G-closed by condition (3), $\overline{\overline{A}}^G \subset \overline{A}$ by Definition 2.4(2).

(2) ⇒ (1) by Corollary 2.7. □

The following are the main results in this section.

Theorem 2.9. *Let G be a method on a topological space X. The following are equivalent.*

(1) *For each $A \subset X$, $\overline{A} = \overline{A}^G$.*
(2) *For each $A \subset X$, the set A is closed if and only if it is G-closed.*

Proof. (1) ⇒ (2). For each $A \subset X$, suppose $\overline{A} = \overline{A}^G$. It follows from Lemma 2.8 that each closed set in X is G-closed. On the other hand, if A is G-closed in X, then $A = \overline{A}^G = \overline{A}$, thus A is closed in X.

(2) ⇒ (1). Suppose that a subset A of X is closed if and only if it is G-closed. For any $A \subset X$, we have that $\overline{A}^G \subset \overline{A}$ by Lemma 2.8. Since $\overline{A}^G$ is always G-closed, $\overline{A}^G$ is closed, $\overline{A} \subset \overline{A}^G$. This shows that $\overline{A} = \overline{A}^G$. □

Theorem 2.10. *Let G be a method on a topological space X. The following are equivalent.*

(1) *For each $A \subset X$, $\overline{A} = [A]_G$.*
(2) *For each $A \subset X$, $\overline{A} = \overline{A}^G = [A]_G$.*
(3) *For each $A \subset X$, $\overline{A} \subset [A]_G$, and A is closed if and only if it is G-closed.*
(4) *For each $A \subset X$, $\overline{A} \subset [A]_G$, and if A is closed, then it is G-closed.*
(5) *For each $A \subset X$, $\overline{A} \subset [A]_G$, and if $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in [\boldsymbol{x}]$.*

Proof. (1) ⇒ (3). For each $A \subset X$, suppose $\overline{A} = [A]_G$. If A is a closed set in X, then $[A]_G = \overline{A} = A$, so A is G-closed by Proposition 2.6. On the other hand, if A is a G-closed set in X, then $[A]_G \subset A \subset \overline{A} = [A]_G$, thus $A = \overline{A}$, i.e., A is closed in X.

It can easily be obtained that (3) ⇒ (2) ⇒ (4) ⇒ (5) ⇒ (1) by Theorem 2.9, Corollary 2.7 and Lemma 2.8. □

Examples 2.13(2) and 2.13(3) show that the condition "$\overline{A} = \overline{A}^G$ for each $A \subset X$" and the condition "$\overline{A}^G = [A]_G$ for each $A \subset X$" in a topological space X do not imply each other.

The following lemma is easy to check.

Lemma 2.11. *Let X be a topological space.*

(1) *If G is a regular method on X, then $[A]_{seq} \subset [A]_G$ for each $A \subset X$. Thus, every G-closed set of X is sequentially closed.*
(2) *If G is a subsequential method on X, then $[A]_G \subset [A]_{seq}$ for each $A \subset X$. Thus, every sequentially closed set of X is G-closed.*

As a result, if G is a regular subsequential method on a topological space X, then $[A]_G = [A]_{seq}$ for each $A \subset X$. It shows that sequentially closed sets coincide with G-closed sets in the topological space X.

Let G be a method on a set X. G is said to satisfy a *cofinal condition* if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(X)$, $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in s(X)$ and there exists an $m \in \mathbb{N}$ such that $x_n = y_n$ for each $n > m$, then $\boldsymbol{y} \in c_G(X)$ and $G(\boldsymbol{x}) = G(\boldsymbol{y})$. If G is a regular method on a topological group X in the sense of Çakallı [4], the method G satisfies the cofinal condition on X. Indeed, let $c_G(X)$ be a subgroup of the group $s(X)$, and

$G : c_G(X) \to X$ be a homomorphism. Suppose that $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(X)$, $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in s(X)$ and there exists an $m \in \mathbb{N}$ such that $x_n = y_n$ if $n > m$. Then $\lim(\boldsymbol{x} \cdot \boldsymbol{y}^{-1}) = e$ (the unit of the topological group X). Since G is a regular method, it follows that $\boldsymbol{x} \cdot \boldsymbol{y}^{-1} \in c_G(X)$ and $G(\boldsymbol{x} \cdot \boldsymbol{y}^{-1}) = e$. Because $c_G(X)$ is a group, $\boldsymbol{y} = (\boldsymbol{x}^{-1} \cdot (\boldsymbol{x} \cdot \boldsymbol{y}^{-1}))^{-1} \in c_G(X)$, by the homomorphism of G, then $G(\boldsymbol{x}) = G(\boldsymbol{y})$.

Proposition 2.12. *Let X be a first-countable topological space, and G be a method satisfying the cofinal condition on X. Then G is a subsequential method on X if and only if $[A]_G \subset [A]_{seq}$ for each $A \subset X$.*

Proof. It follows from Lemma 2.11(2) that we need only show the sufficiency. Suppose that $[A]_G \subset [A]_{seq}$ for each $A \subset X$. Put $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(X)$ and $G(\boldsymbol{x}) = l \in X$. If U is an open neighborhood of l in X, we claim that U must contain infinitely many terms of $\boldsymbol{x}$. If it is not the case, there exists an $m \in \mathbb{N}$ with $U \cap \{x_n : n > m\} = \emptyset$. Put $A = \{x_n : n > m\}$. Then $U \cap \overline{A} = \emptyset$, hence $l \notin \overline{A}$. On the other hand, put $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}}$ such that $y_n = x_{m+1}$ if $n \le m$; and $y_n = x_n$ if $n > m$. Since the method G satisfies the cofinal condition, $\boldsymbol{y} \in c_G(X)$ and $G(\boldsymbol{x}) = G(\boldsymbol{y}) \in [A]_G \subset [A]_{seq} \subset \overline{A}$, and $l \in \overline{A}$, which is a contradiction.

Because X is a first-countable space, let $\{U_i\}_{i\in\mathbb{N}}$ be a decreasing base of open neighborhoods of l in X. Then each U_i contains infinitely many terms of $\boldsymbol{x}$, thus there exists a subsequence $\boldsymbol{x}' = \{x_{n_i}\}_{i\in\mathbb{N}}$ of $\boldsymbol{x}$ such that each $x_{n_i} \in U_i$, hence $\boldsymbol{x}'$ converges to l in X. Therefore, G is a subsequential method on X. □

Example 2.13(2) shows that the cofinal condition of the method G in Proposition 2.12 cannot be omitted.

Example 2.13. *G-hulls and G-closures.*

(1) *There exists a non-regular method G on a topological space X such that every G-closed set in X is closed.*

Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology. Put $c_G(X) = s(X)$, and $G : c_G(X) \to X$ is defined by $G(\boldsymbol{x}) = 0$ for each $\boldsymbol{x} \in c_G(X)$. Then G is a method on X. It is obvious that each G-closed set in X is closed, because X is a discrete space. Although $c(X) \subset c_G(X)$, G is not regular, even $X = [X]_G$ is not satisfied. Indeed, $[X]_G = \{0\}$.

For any non-empty subset A of X, we have that $[A]_G = \{0\}$, thus A is G-closed if and only if $0 \in A$, and $\overline{A}^G = \{0\} \cup A$.

(2) *There exists a non-subsequential method G on a topological space X such that $[A]_G \subset [A]_{seq}$ for each $A \subset X$.*

Let X be a non-discrete first-countable space. A method $G : s(X) \to X$ on X is defined by $G(\boldsymbol{x}) = x_1$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X)$. Obviously, the method G does not satisfy the cofinal condition on X. It is easy to see that, for each $A \subset X$, $[A]_G = A = \overline{A}^G \subset [A]_{seq}$, thus A is G-closed. The method G is not subsequential, so the converse of Lemma 2.11(2) does not hold. Since X is non-discrete, there exists a subset A of X such that $\overline{A} \neq A = \overline{A}^G$.

(3) *There exists a topological space X so that for the usual convergence method G on X one has that $\overline{A} = \overline{A}^G$ for each $A \subset X$, but $\overline{A}^G \neq [A]_G$ for some $A \subset X$.*

Let $X = \{0\} \cup \bigcup_{i\in\mathbb{N}} X_i$, where $X_i = \{1/i\} \cup \{1/i + 1/k : k \in \mathbb{N}, k \ge i^2\}$ for each $i \in \mathbb{N}$; suppose that X is endowed with the following topology.

(3.1) Each point of the form $1/i + 1/j$ is isolated.

(3.2) Each neighborhood of each point of the form $1/i$ contains a set of the form $\{1/i\} \cup \{1/i + 1/k : k \ge j\}$, where $j \ge i^2$.

(3.3) Each neighborhood of the point 0 contains a set obtained from X by removing a finite number of X_i's and a finite number of points of the form $1/i + 1/j$ in all the remaining X_i's.

The topological space X is called Arens' space and is denoted by S_2 [10, Example 1.6.19]. Let G be the ordinary convergence method on X. Then a subset A of X is closed if and only if it is sequentially closed, i.e., it is G-closed, thus $\overline{A} = \overline{A}^G$ by Theorem 2.9. Let $A = \{1/i + 1/k : i, k \in \mathbb{N}, k \geq i^2\}$. Then $[A]_G = X \setminus \{0\} \neq X = [[A]_G]_G = \overline{A}^G$. □

Example 2.14. *G-closed sets, closed sets and sequentially closed sets.*

(1) *The union of two G-closed sets is not always G-closed.*

Connor and Grosse-Erdmann [7] gave an example as follows. Let $\mathbb{R}$ be the set of all real numbers endowed with the usual topology. Put

$$c_G(\mathbb{R}) = \{\{x_n\}_{n\in\mathbb{N}} \in s(\mathbb{R}) : \{x_n + x_{n+1}\}_{n\in\mathbb{N}} \in c(\mathbb{R})\}.$$

Define $G : c_G(\mathbb{R}) \to \mathbb{R}$ by $G(\boldsymbol{x}) = \lim_{n\to\infty} \frac{x_n+x_{n+1}}{2}$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(\mathbb{R})$. Then G is a regular method on $\mathbb{R}$.

Let $A = \{0\}$ and $B = \{1\}$. Then $[A]_G = \overline{A}^G = \{0\}$, $[B]_G = \overline{B}^G = \{1\}$, and $[A \cup B]_G = \{0, \frac{1}{2}, 1\}$, so A and B are G-closed in $\mathbb{R}$, but $A \cup B = \{0, 1\}$ is not G-closed, and $[A]_G \cup [B]_G \neq [A \cup B]_G$. Since $[[A \cup B]_G]_G = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, the set $[A \cup B]_G$ is not G-closed in $\mathbb{R}$. It can be checked that $\overline{A \cup B}^G = [0, 1]$, and $\overline{A}^G \cup \overline{B}^G \neq \overline{A \cup B}^G$.

(2) *Closed sets are not always G-closed.*

In the above (1), the set $A \cup B$ is a closed set in $\mathbb{R}$, but $A \cup B$ is not G-closed. This shows that G is not a subsequential method on $\mathbb{R}$ by Lemma 2.11(2), and $[A \cup B]_G \not\subset \overline{A \cup B}$.

(3) *G-closed sets are not always closed.*

The topological space X and the method on X are defined in Example 2.13(2). Then, for each $A \subset X$, $[A]_G = A$, thus A is G-closed, but it is not always closed in X.

(4) *G-closedness and sequential closedness do not imply each other.*

Since closed sets coincide with sequentially closed sets in a first-countable space, examples (2) and (3) above also show that G-closedness and ordinary closedness do not imply each other in general. □

3. *G*-kernels and *G*-interiors

In this section we discuss the dual concepts of G-hulls and G-closures: G-kernels and G-interiors.

A subset A of a topological space X is called *sequentially open* [13] if $X \setminus A$ is sequentially closed in X.

Definition 3.1. Let X be a set and G be a method on X.

(1) A subset A of X is called a *G-neighborhood* of a point $x \in X$ if there exists a G-open set U with $x \in U \subset A$.
(2) A subset A of X is called *G-open* if $X \setminus A$ is G-closed in X.

Mucuk, Şahan [16] proved the following proposition on first-countable topological groups. In fact, it is still true on arbitrary sets.

Proposition 3.2. *Let G be a method on a set X.*

(1) *The union of any family of G-open sets of X is G-open.*
(2) *For each $A \subset X$, the set A is G-open in X if and only if A is a G-neighborhood of each point in A.*

We know that the interior of a set in a topological space coincides with the complement of the closure of the complement of the set, and the interior of a set in a topological space is also the union of all open sets contained in the set. Based on the above knowledge, the following notions for G-methods are introduced.

Definition 3.3. Let G be a method on a set X and $A \subset X$.

(1) The *G-kernel* of A is defined as the set

$$\{l \in X : \text{ there is no } \boldsymbol{x} \in s(X \setminus A) \cap c_G(X) \text{ with } l = G(\boldsymbol{x})\},$$

and the G-kernel of A is denoted by $\ker_G(A)$ or $(A)_G$.

(2) The *G-interior* of A is defined as the union of all G-open sets contained in A, and the G-interior of A is denoted by $\mathrm{int}_G(A)$ or $A^{\circ G}$.

Remark 3.4. (1) The G-kernel of a set seems to be unnatural. But, the definition ensures that the formula $(A)_G = X \setminus [X \setminus A]_G$ holds, see Theorem 3.5(1). A geometric meaning of G-kernels is given in Proposition 3.12 and Remark 3.13.

(2) In Example 2.13(1), $(\emptyset)_G = X \setminus \{0\} \not\subset \emptyset = \emptyset^{\circ G}$.

(3) It follows from Proposition 3.2(1) that the G-interior $A^{\circ G}$ of a set A is the largest G-open set contained in A.

(4) The G-interior of a set A of a first-countable topological group was defined and denoted by $A^{\circ G}$ in [16, Definition 2.7], where it was called *G-sequential interior*.

The relationship between the G-kernels (resp. G-interiors) and the G-hulls (resp. G-closures) of spaces is established in the following theorem.

Theorem 3.5. *Let G be a method on a set X and $A \subset X$. Then*

(1) $(A)_G = X \setminus [X \setminus A]_G$.
(2) $A^{\circ G} = X \setminus \overline{X \setminus A}^G$.

Proof. (1) is true by Definition 3.3(1). Since $A^{\circ G} \subset A$, we have that $X \setminus A \subset X \setminus A^{\circ G}$, thus $\overline{X \setminus A}^G \subset \overline{X \setminus A^{\circ G}}^G = X \setminus A^{\circ G}$, and $A^{\circ G} \subset X \setminus \overline{X \setminus A}^G$. On the other hand, since $X \setminus A \subset \overline{X \setminus A}^G$, $X \setminus \overline{X \setminus A}^G \subset A$, thus $X \setminus \overline{X \setminus A}^G \subset A^{\circ G}$. This shows that $A^{\circ G} = X \setminus \overline{X \setminus A}^G$, i.e., (2) is true. □

By Proposition 2.6, Corollary 2.7, Lemma 2.8, Theorems 2.9, 2.10 and 3.5 we have the following corollaries.

Corollary 3.6. *Let G be a method on a set X and $A \subset X$. The following are equivalent.*

(1) *A is G-open.*
(2) $A \subset (A)_G$.
(3) $A \subset A^{\circ G}$, *i.e.*, $A^{\circ G} = A$.

Corollary 3.7. *Let G be a method on a set X. Then $A^{\circ G} \subset (A)_G$ for each $A \subset X$.*

Corollary 3.8. *Let G be a method on a topological space X. The following are equivalent.*

(1) $A^{\circ} \subset (A)_G$ *for each* $A \subset X$.
(2) $A^{\circ} \subset A^{\circ G}$ *for each* $A \subset X$.
(3) *Every open set in X is G-open.*
(4) *For any $x \in X$, every neighborhood of x is a G-neighborhood of x in X.*

Proof. By Lemma 2.8 and Theorem 3.5, (1) ⇔ (2). By Corollary 3.6, (2) ⇒ (3). By Definitions 3.1(1) and 3.3(2), (3) ⇒ (4) ⇒ (2). □

Corollary 3.9. *Let G be a method on a topological space X. The following are equivalent.*

(1) *For each $A \subset X$, $A^\circ = A^{\circ G}$.*
(2) *For each $A \subset X$, the set A is open if and only if it is G-open.*

Corollary 3.10. *Let G be a method on a topological space X. The following are equivalent.*

(1) *For each $A \subset X$, $(A)_G = A^\circ$.*
(2) *For each $A \subset X$, $(A)_G = A^\circ = A^{\circ G}$.*
(3) *For each $A \subset X$, $(A)_G \subset A^\circ$, and A is open if and only if it is G-open.*
(4) *For each $A \subset X$, $(A)_G \subset A^\circ$, and if A is open, then it is G-open.*
(5) *For each $A \subset X$, $(A)_G \subset A^\circ$, and every neighborhood of x is a G-neighborhood of x in X for any $x \in X$.*

Proof. By Theorems 2.10 and 3.5, (1) ⇔ (2) ⇔ (3) ⇔ (4). It follows from (3) ⇔ (4) in Corollary 3.8 that (4) ⇔ (5). □

Proposition 3.11. *Let G be a method on a set X and $A \subset X$. Then*

(1) *$x \in [A]_G$ if and only if every subset U of X with $x \in (U)_G$ intersects A.*
(2) *$x \in \overline{A}^G$ if and only if every subset U of X with $x \in U^{\circ G}$ intersects A.*

Proof. (1) Suppose $x \in X \setminus [A]_G$. Let $U = X \setminus A$. Then $x \in (U)_G$ by Theorem 3.5, and $U \cap A = \emptyset$. On the other hand, suppose $x \in [A]_G \cap (U)_G$. Then $x \in [A]_G \setminus [X \setminus U]_G$, thus $A \not\subset X \setminus U$, i.e., $U \cap A \neq \emptyset$.

(2) If $x \in X \setminus \overline{A}^G$, the set $X \setminus \overline{A}^G$ is a G-open set containing x that does not intersect A, as desired. Conversely, suppose $x \in \overline{A}^G \cap U^{\circ G}$. Then $x \in \overline{A}^G \setminus \overline{X \setminus U}^G$, thus $A \not\subset X \setminus U$, i.e., $U \cap A \neq \emptyset$. □

Let G be a method on a set X. For any subset A of X, because the set $A^{\circ G}$ is always G-open in X, a point $x \in A^{\circ G}$ if and only if the set A is a G-neighborhood of x in X. We do not know how to characterize the property $x \in (A)_G$ for a subset A of a set X. A partial answer to the question is the following.

Proposition 3.12. *Let G be a regular subsequential method on a topological space X. For any subset A of X, a point $p \in (A)_G$ if and only if whenever $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = p$ then there exists a subsequence $\boldsymbol{x}' \in c(X) \cap s(A)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}' = p$.*

Proof. Suppose $p \in (A)_G$. Since G is regular and subsequential, it follows from Theorem 3.5 and Lemma 2.11 that $p \in X \setminus [X \setminus A]_G = X \setminus [X \setminus A]_{seq}$. Let $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = p$. Since G is subsequential, there exists a subsequence $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{y} = p$. Now suppose that $y_n \in X \setminus A$ for all large $n \in \mathbb{N}$. Then $p = \lim \boldsymbol{y} \in [X \setminus A]_{seq}$, which is not the case. Hence there exists a subsequence $\boldsymbol{x}' \in c(X) \cap s(A)$ of $\boldsymbol{y}$ (and $\boldsymbol{x}$) with $\lim \boldsymbol{x}' = p$.

Conversely, suppose $p \notin (A)_G$. By Definition 3.3(1), there exists a $\boldsymbol{y} \in c_G(X) \cap s(X \setminus A)$ with $G(\boldsymbol{y}) = p$. Since G is subsequential, there exists a subsequence $\boldsymbol{x} \in c(X)$ of $\boldsymbol{y}$ with $\lim \boldsymbol{x} = p$. Then, by regularity, $\boldsymbol{x} \in c_G(X)$ with $G(\boldsymbol{x}) = p$. But since $\boldsymbol{y} \in s(X \setminus A)$, $\boldsymbol{x} \in s(X \setminus A)$, and no subsequence of $\boldsymbol{x}$ can belong to $s(A)$, which is a contradiction. □

Remark 3.13. A subset A of a topological space X is called a *sequential neighborhood* of a point $x \in X$ [12] if, whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to x, then there exists an $m \in \mathbb{N}$ with $\{x_n : n > m\} \subset A$, i.e., $\{x_n\}_{n\in\mathbb{N}}$ is eventually in A.

(1) By Proposition 3.12, if G is the ordinary convergence method on a topological space X, then, for each $A \subset X$, $x \in (A)_G$ if and only if the set A is a sequential neighborhood of x in X. There may not exist a sequentially open set (i.e., a G-open set) U in X with $x \in U \subset A$ in this case. Indeed, consider Arens' space S_2 in Example 2.13(3). Let $A = \{0\} \cup \{1/i : i \in \mathbb{N}\}$. Then A is a sequential neighborhood of the point 0 in X, i.e., $0 \in (A)_G$, but there does not exist a G-open set U of X with $0 \in U \subset A$.

(2) Generally speaking, suppose that G is a method on a set X and $G(\boldsymbol{x}) \in (A)_G$, then the sequence $\boldsymbol{x}$ need not even have a subsequence in A. Consider Example 2.13(2). It is easy to see that $(A)_G = A$ for each $A \subset X$. Take two distinct points $a, b \in X$. Put $A = \{a\}$ and $\boldsymbol{x} = \{a, b, b, b, \cdots\} \in c_G(X)$, then $G(\boldsymbol{x}) = a \in (A)_G$, but $\boldsymbol{x}$ does not have a subsequence in A.

4. Subspace methods

Let $G : c_G(X) \to X$ be a method on a set X, and $Y \subset X$. How can one induce naturally a method on the subset Y?

Put

$$c_{G|Y}(Y) = \{\boldsymbol{x} \in s(Y) \cap c_G(X) : G(\boldsymbol{x}) \in Y\},$$

and define a function $G|_Y : c_{G|Y}(Y) \to Y$ by

$$G|_Y(\boldsymbol{x}) = G(\boldsymbol{x}), \quad \boldsymbol{x} \in c_{G|Y}(Y).$$

Then $G|_Y$ is a method on the subset Y of X.

Definition 4.1. Let G be a method on a set X, and $Y \subset X$. The function $G|_Y : c_{G|Y}(Y) \to Y$ is called the *submethod* of G on the subset Y of X, or the method on the subset Y induced by G.

If Y is a subset of a set X, we shall assume that Y is given the submethod on the subset Y of X unless we specially state otherwise. Obviously,

$$G|_Y(c_{G|Y}(Y)) = [Y]_G \cap Y.$$

Thus, $c_{G|Y}(Y) \neq \emptyset$ if and only if $[Y]_G \cap Y \neq \emptyset$.

Let $G : c_G(X) \to X$ be the method on the discrete space X in Example 2.13(1). Put $Y = X \setminus \{0\}$. Then $[Y]_G = \{0\}$, and $c_{G|Y}(Y) = \emptyset$.

It is well-known that if X is a topological space and $A \subset Y \subset X$ then

$$\mathrm{cl}_Y(A) = \overline{A} \cap Y \text{ and } A^\circ = Y^\circ \cap \mathrm{int}_Y(A).$$

We have the following results on G-methods and its submethods.

Proposition 4.2. *Let G be a method on a set X, and $A \subset Y \subset X$. Then*

(1) $[A]_{G|Y} = [A]_G \cap Y$.
(2) $\overline{A}^{G|Y} \subset \overline{A}^G \cap Y$.

(3) $Y \cap (A)_G \subset (Y)_G \cap (A)_{G|Y}$.
(4) $A^{\circ G} \subset Y^{\circ G} \cap A^{\circ G|Y}$.

Proof. (1) If $l \in [A]_{G|Y}$, there exists an $\boldsymbol{x} \in c_{G|Y}(Y) \cap s(A)$ with $G(\boldsymbol{x}) = l$, thus $l \in [A]_G \cap Y$. On the other hand, if $l \in [A]_G \cap Y$, there exists an $\boldsymbol{x} \in s(A) \cap c_G(X)$ with $G(\boldsymbol{x}) = l \in Y$, hence $\boldsymbol{x} \in c_{G|Y}(Y)$, and $l = G|_Y(\boldsymbol{x}) \in [A]_{G|Y}$. Therefore, $[A]_{G|Y} = [A]_G \cap Y$.

(2) Since $\overline{A}^G$ is G-closed in X, $[\overline{A}^G]_G \subset \overline{A}^G$, then $[\overline{A}^G \cap Y]_{G|Y} = [\overline{A}^G \cap Y]_G \cap Y \subset [\overline{A}^G]_G \cap Y \subset \overline{A}^G \cap Y$, thus $\overline{A}^G \cap Y$ is $G|_Y$-closed in Y and $A \subset \overline{A}^G \cap Y$, hence $\overline{A}^{G|Y} \subset \overline{A}^G \cap Y$.

(3) $(A)_{G|Y} = Y \setminus [Y \setminus A]_{G|Y} = Y \setminus ([Y \setminus A]_G \cap Y) = Y \cap (X \setminus [Y \setminus A]_G) = Y \cap (X \setminus (Y \setminus A))_G = Y \cap ((X \setminus Y) \cup A)_G \supset Y \cap (A)_G$. Since $(A)_G \subset (Y)_G$, $Y \cap (A)_G \subset (Y)_G \cap (A)_{G|Y}$.

(4) $A^{\circ G|Y} = Y \setminus \overline{Y \setminus A}^{G|Y} \supset Y \setminus (\overline{Y \setminus A}^G \cap Y) = Y \cap (X \setminus \overline{Y \setminus A}^G) = Y \cap (X \setminus (Y \setminus A))^{\circ G} = Y \cap ((X \setminus Y) \cup A)^{\circ G} \supset Y \cap A^{\circ G}$. Since $A^{\circ G} \subset Y^{\circ G} \cap Y$, $A^{\circ G} = Y^{\circ G} \cap Y \cap A^{\circ G} \subset Y^{\circ G} \cap A^{\circ G|Y}$. □

Example 4.8(2) shows that it is possible that $(A)_G \not\subset (Y)_G \cap (A)_{G|Y}$. In special cases we have that $(A)_G \subset (Y)_G \cap (A)_{G|Y}$, see Proposition 4.5.

Corollary 4.3. *Let G be a method on a set X, and $Y \subset X$. If F is G-closed (resp. G-open) in X, then the set $F \cap Y$ is $G|_Y$-closed (resp. $G|_Y$-open) in Y.*

Proof. If F is G-closed in X, then $[F]_G \subset F$. It follows from Proposition 4.2(1) that $[F \cap Y]_{G|Y} = [F \cap Y]_G \cap Y \subset [F]_G \cap Y \subset F \cap Y$, i.e., $F \cap Y$ is $G|_Y$-closed in Y.

If F is G-open in X, then $X \setminus F$ is G-closed in X, thus $(X \setminus F) \cap Y$ is $G|_Y$-closed in Y, hence $Y \setminus ((X \setminus F) \cap Y) = Y \cap F$ is $G|_Y$-open in Y. □

Corollary 4.4. *Let G be a method on a set X, and $A \subset Y \subset X$. If A is $G|_Y$-closed in Y and Y is G-closed in X, then A is G-closed in X.*

Proof. Since Y is G-closed in X, $[A]_G \subset [Y]_G \subset Y$. And since A is $G|_Y$-closed in Y, $[A]_{G|Y} \subset A$. It follows from Proposition 4.2(1) that $[A]_G = [A]_G \cap Y = [A]_{G|Y} \subset A$, thus A is G-closed in X. □

Example 4.7(3) shows that Corollary 4.4 is not always true if "closed" is replaced by "open" throughout.

Proposition 4.5. *Let G be a method on a set X, and $A \subset Y \subset X$. If $X \setminus Y \subset [X \setminus Y]_G$, then $(A)_G \subset X \setminus ([X \setminus Y]_G \cup [Y \setminus A]_G) = (Y)_G \cap (A)_{G|Y}$.*

Proof. By Theorem 3.5 and Proposition 4.2(1), we have

$$\begin{aligned}(A)_{G|Y} &= Y \setminus [Y \setminus A]_{G|Y} \\ &= X \setminus ((X \setminus Y) \cup ([Y \setminus A]_G \cap Y)) \\ &= X \setminus ((X \setminus Y) \cup [Y \setminus A]_G).\end{aligned}$$

Thus

$$\begin{aligned}(Y)_G \cap (A)_{G|Y} &= (X \setminus [X \setminus Y]_G) \cap (X \setminus ((X \setminus Y) \cup [Y \setminus A]_G)) \\ &= X \setminus ([X \setminus Y]_G \cup (X \setminus Y) \cup [Y \setminus A]_G) \\ &= X \setminus ([X \setminus Y]_G \cup [Y \setminus A]_G) \\ &\supset X \setminus [X \setminus A]_G = (A)_G. \quad \square\end{aligned}$$

Corollary 4.6. *Let G be a regular subsequential method on a topological space X, and $A \subset Y \subset X$. Then $(A)_G = (Y)_G \cap (A)_{G|Y}$.*

Proof. By Lemma 2.11, $[A]_G = [A]_{seq}$ for each $A \subset X$. Thus, $X \setminus Y \subset [X \setminus Y]_G$ and $[X \setminus Y]_G \cup [Y \setminus A]_G = [X \setminus A]_G$. By Proposition 4.5 we have $(A)_G = (Y)_G \cap (A)_{G|Y}$. □

Example 4.7. *$G|_Y$-closed sets and $G|_Y$-open sets.*

Let X be Arens' space S_2 in Example 2.13(3), and G be the ordinary convergence method on X. Take $A = \{1/i + 1/k : i, k \in \mathbb{N}, k \geq i^2\}$, $B = \{0\}$ and $Y = A \cup B$. Then $[A]_{G|Y} = A$ and $(B)_{G|Y} = B$. Thus, A is $G|_Y$-closed and B is $G|_Y$-open in Y.

(1) *A is $G|_Y$-closed but $A \neq Y \cap F$ for any G-closed set F in X.*

Suppose that F is G-closed in X and $A = Y \cap F$. Since $A \subset F$ and F is G-closed in X, $1/i \in F$ for each $i \in \mathbb{N}$, and $0 \in F$, thus $F = X$, and $Y \cap F = Y \neq A$.

(2) *B is $G|_Y$-open but $B \neq Y \cap U$ for any G-open set U in X.*

Suppose that U is G-open in X and $B = Y \cap U$. Since U is G-open in X, $0 \in U \subset (U)_G$. It follows from Proposition 3.12 that $1/i \in U$ for some $i \in \mathbb{N}$, and $1/i + 1/k \in U$ for some $k \in \mathbb{N}$ and $k \geq i^2$, thus $1/i + 1/k \in (Y \cap U) \setminus B$, hence $B \neq Y \cap U$.

(3) *There exist a method G on a topological space X and $A \subset Y \subset X$ such that Y is G-open in X and A is $G|_Y$-open in Y, but A is not G-open in X.*

Let G be the method on the topological space $\mathbb{R}$ in Example 2.14(1). Take $Y = (0, +\infty)$, $A = (0, 1)$. It is easy to check that $[\mathbb{R} \setminus Y]_G = (-\infty, 0]$, $[Y \setminus A]_{G|Y} = [1, +\infty)$ and $[\mathbb{R} \setminus A]_G = \mathbb{R}$, thus $(Y)_G = (0, +\infty)$, $(A)_{G|Y} = (0, 1)$ and $(A)_G = \emptyset$. Therefore, Y is G-open in X and A is $G|_Y$-open in Y, but A is not G-open in X. □

Example 4.8. *$G|_Y$-closures, $G|_Y$-kernels and $G|_Y$-interiors.*

(1) $\overline{A}^G \cap Y \not\subset \overline{A}^{G|Y}$ *for $A \subset Y \subset X$.*

Let G be the method on the topological space X in Example 4.7. Take $A = \{1/i + 1/k : i, k \in \mathbb{N}, k \geq i^2\}$, and $Y = A \cup \{0\}$. It is easy to see that $\overline{A}^G = X$, and $\overline{A}^{G|Y} = A$. Then $\overline{A}^G \cap Y \not\subset \overline{A}^{G|Y}$. This shows that the converse inclusion in Proposition 4.2(2) is false.

(2) $(A)_G \not\subset (Y)_G \cap (A)_{G|Y}$ *for $A \subset Y \subset X$.*

Let G be the method on the topological space X in Example 2.13(1), which is not a regular method. Then $(A)_G = X \setminus \{0\}$ for any $A \subset X$. Take $Y = \{2k : k \in \mathbb{Z}\}$, $A = \{4k : k \in \mathbb{Z}\}$. Then $A \subset Y \subset X$, and $(A)_{G|Y} = Y \setminus \{0\}$. Hence, $(A)_G \not\subset (Y)_G \cap (A)_{G|Y}$. This shows that one cannot drop "$Y\cap$" on the left of Proposition 4.2(3).

(3) $(Y)_G \cap (A)_{G|Y} \not\subset (A)_G$, *and* $Y^{\circ G} \cap A^{\circ G|Y} \not\subset A^{\circ G}$ *for $A \subset Y \subset X$.*

Let G be the method on the topological space $\mathbb{R}$ in Example 2.14(1), which is not a subsequential method. Take $Y = [0, +\infty)$, $A = [0, 1]$. It is easy to check that $[\mathbb{R} \setminus A]_G = \overline{\mathbb{R} \setminus A}^G = \mathbb{R}$, $[\mathbb{R} \setminus Y]_G = \overline{\mathbb{R} \setminus Y}^G = (-\infty, 0]$ and $[Y \setminus A]_{G|Y} = \overline{Y \setminus A}^{G|Y} = [1, +\infty)$, thus $(A)_G = A^{\circ G} = \emptyset$, $(Y)_G = Y^{\circ G} = (0, +\infty)$ and $(A)_{G|Y} = A^{\circ G|Y} = [0, 1)$. Therefore, $(Y)_G \cap (A)_{G|Y} \not\subset (A)_G$, and $Y^{\circ G} \cap A^{\circ G|Y} \not\subset A^{\circ G}$. This shows that the converse inclusions in Propositions 4.2(3) and 4.2(4) are not true. □

5. *G*-sequential spaces and *G*-Fréchet spaces

Sequential spaces and Fréchet spaces are two kinds of spaces which are determined by convergent sequences of topological spaces. A topological space X is said to be *sequential* [13] if any subset A of X with $[A]_{seq} \subset A$ is closed in X, i.e., every sequentially closed set in X is closed. A topological space X is said to be *Fréchet* [13] if $\overline{A} \subset [A]_{seq}$ for each $A \subset X$. Statistically sequential spaces and statistically Fréchet spaces were defined by Di Maio and Kočinac [9].

Definition 5.1. Let G be a method on a topological space X.

(1) X is said to be a *G-sequential space* if any subset A of X with $[A]_G \subset A$ is closed in X, i.e., every G-closed set in X is closed.
(2) X is said to be a *G-Fréchet space* if $\overline{A} \subset [A]_G$ for each $A \subset X$.

Remark 5.2. If we want to keep the consistency with the terminology for G-methods, *G-sequential spaces* in Definition 5.1(1) should be called *G-spaces*. But, this sounds strange, so we prefer the terminology of *G-sequential space*.

The topological space X in Example 2.13(2) is not a G-sequential space. The topological space X in Example 2.13(1) is a G-sequential space, but it is not a G-Fréchet space. In Theorem 2.10 and Corollary 3.10 we have used the property of being a G-Fréchet space. In this section, we discuss a characterization and hereditary properties of G-sequential spaces and G-Fréchet spaces. The concepts of G-open sets and G-neighborhoods will help us understand G-sequential spaces and G-Fréchet spaces.

Proposition 5.3. *Let G be a method on a topological space X. The following are equivalent.*

(1) *X is a G-sequential space.*
(2) *$\overline{A} \subset \overline{A}^G$ for each $A \subset X$.*
(3) *$A^{\circ G} \subset A^\circ$ for each $A \subset X$.*
(4) *Every G-open set of X is open.*
(5) *Every G-neighborhood of a point in X is a neighborhood of the point.*

Proof. (1) $\Rightarrow$ (2). Suppose each G-closed set in the topological space X is closed. For each $A \subset X$, the G-closed set $\overline{A}^G$ of X is closed and $A \subset \overline{A}^G$, thus $\overline{A} \subset \overline{A}^G$.

Obviously, (2) $\Leftrightarrow$ (3) by Theorem 3.5.

(3) $\Rightarrow$ (4). Let A be G-open in X. Then $A^{\circ G} = A$, and $A^{\circ G} \subset A^\circ$ by condition (3). Thus, $A \subset A^\circ$, i.e., A is open in X.

(4) $\Rightarrow$ (5). If a subset A of X is a G-neighborhood of a point $x \in X$, there exists a G-open set U with $x \in U \subset A$. Then U is open in X by condition (4), so A is a neighborhood of x in X.

(5) $\Rightarrow$ (1). Let A be G-closed in X. Then $X \setminus A$ is G-open in X. For each $x \in X \setminus A$, it follows from Definition 3.1(1) that $X \setminus A$ is a G-neighborhood of x in X, thus $X \setminus A$ is a neighborhood of x in X by condition (5). Hence, $X \setminus A$ is open in X, i.e., A is closed in X. □

By Corollary 2.7 we have the following corollary.

Corollary 5.4. *Every G-Fréchet space is a G-sequential space.*

The following corollary is easily obtained by Lemma 2.11(1) and Definition 5.1.

Corollary 5.5. *Let G be a regular method on a topological space X.*

(1) *If X is a sequential space, then X is a G-sequential space.*
(2) *If X is a Fréchet space, then X is a G-Fréchet space.*

Let X be the set of all real numbers endowed with the usual topology in Example 2.13(2). Then X is a first-countable space, but X is not a G-sequential space. This shows that the regularity of the method G in Corollary 5.5 cannot be omitted.

By Lemma 2.11(2), Theorems 2.10 and 3.5, the following corollary can easily be proved.

Proposition 5.6. *Let G be a subsequential method on a topological space X. The following are equivalent.*

(1) *X is a G-Fréchet space.*
(2) *$\overline{A} = [A]_G$ for each $A \subset X$.*
(3) *$A^\circ = (A)_G$ for each $A \subset X$.*

Remark 5.7. (1) It follows from Corollary 5.5 that the topological space $\mathbb{R}$ in Example 2.14(1) is G-Fréchet, but $\mathbb{R}$ does not satisfy Proposition 5.6(2), because $\overline{A \cup B} \neq [A \cup B]_G$. This shows that the subsequentiality of the method G in Proposition 5.6 cannot be omitted.

(2) Proposition 5.6 improves the following result. Let X be a statistically Fréchet space. If $A \subset X$, then a point $x \in A^\circ$ if and only if A is a sequential neighborhood of x in X [18].

In the second part of this section, we discuss hereditary properties of G-sequential spaces and a characterization of G-Fréchet spaces. Let G be a method on a topological space X, and $Y \subset X$. The subspace Y is called a G-sequential space (resp. a G-Fréchet space) for short if Y is a $G|_Y$-sequential space (resp. a $G|_Y$-Fréchet space). Since a subspace of a sequential space is not necessarily sequential [13], a subspace of a G-sequential space is not always G-sequential.

Lemma 5.8. *Every G-closed subset of a G-sequential space is G-sequential.*

Proof. Suppose that G is a method on a G-sequential space X, and Y is a G-closed set of X. We will show that the subspace Y is a $G|_Y$-sequential space. Let A be a subset of Y with $[A]_{G|Y} \subset A$. Since $[A]_G \subset [Y]_G \subset Y$, $[A]_G = [A]_G \cap Y = [A]_{G|Y} \subset A$, then A is closed in X, thus A is closed in Y. Hence, Y is a G-sequential space. □

Theorem 5.9. *Let G be a method on a topological space X. The following are equivalent.*

(1) *X is a G-Fréchet space.*
(2) *Every subspace of X is a G-Fréchet space.*
(3) *Every subspace of X is a G-sequential space, and $A \subset [A]_G$ for each $A \subset X$.*

Proof. (1) ⇒ (2). Suppose that Y is a subspace of X, and $A \subset Y$. Since X is a G-Fréchet space, by Definition 5.1(2) and Proposition 4.2(1), we have that $\mathrm{cl}_Y(A) = \overline{A} \cap Y \subset [A]_G \cap Y = [A]_{G|Y}$. Thus, the subspace Y is a $G|_Y$-Fréchet space, i.e., Y is a G-Fréchet space.

(2) ⇒ (3). By Corollary 5.4, it suffices to show that $A \subset [A]_G$ for each $A \subset X$. Since the subspace A is a $G|_A$-Fréchet space, by Proposition 4.2(1), $A = \mathrm{cl}_A(A) \subset [A]_{G|A} = [A]_G \cap A \subset [A]_G$.

(3) ⇒ (1). Suppose that $A \subset X$ and $l \in \overline{A}$. If $l \in A$, then $l \in [A]_G$ because $A \subset [A]_G$. If $l \notin A$, put $Y = \{l\} \cup A$, then A is not closed in the subspace Y. Since Y is a $G|_Y$-sequential space, it follows that there exists a sequence $\boldsymbol{x} \in s(A) \cap c_{G|Y}(Y)$ such that $G|_Y(\boldsymbol{x}) \in Y \setminus A = \{l\}$, hence $l \in [A]_{G|Y} = [A]_G \cap Y \subset [A]_G$. This shows that $\overline{A} \subset [A]_G$. Thus X is a G-Fréchet space. □

Remark 5.10. Hereditary properties.

(1) If G is a regular method on a topological space X, then $A \subset [A]_G$ for each $A \subset X$, thus the topological space X is a G-Fréchet space if and only if each subspace of X is a G-sequential space, i.e., X is a hereditary G-sequential space. Therefore, Theorem 5.9 unifies the hereditary properties of Fréchet spaces [1, p. 14] and statistically Fréchet spaces [18].

(2) A hereditary G-sequential space is not always a G-Fréchet space. Let G be the method on the discrete space X defined in Example 2.13(1). Then X is a hereditary G-sequential space, but it is not a G-Fréchet space because $X \not\subset [X]_G$.

The following question was posed by Professor Salvador García-Ferreira in the first Pan Pacific International Conference on Topology and Applications (Zhangzhou, November, 2015).

Question 5.11. *Does there exist a topological space X that is not G-sequential for any G-method on X?*

6. G-generalized topology

In this section we will discuss some relations between the family of all G-open sets on a topological space $(X, \mathcal{T})$ and the topology $\mathcal{T}$ on the set X.

A *generalized topology* [8] on a set X is a collection μ of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. Let G be a method on a set X, and put

$$\mathcal{T}_G = \{A \subset X : A \text{ is } G\text{-open in } X\}.$$

Obviously, $\mathcal{T}_G$ is a generalized topology on the set X by Proposition 3.2(1). Generally speaking, the family $\mathcal{T}_G$ is not a topology on the set X, see Example 2.14(1).

Definition 6.1. Let G be a method on a set X. The family $\mathcal{T}_G$ is called the *G-generalized topology* on the set X.

(1) $\mathcal{T}_G$ is called the *G-topology* on the set X if it is a topology on X.
(2) If X carries a topology $\mathcal{T}$ then $(X, \mathcal{T})$ is called *G-topologizable* if $\mathcal{T} = \mathcal{T}_G$.

Clearly, $\emptyset$, $X \in \mathcal{T}_G$, thus $\mathcal{T}_G$ is a topology on the set X if and only if the intersection of any two G-open sets in X is G-open, if and only if the union of any two G-closed sets in X is G-closed. A sufficient condition for this to happen is the following by Proposition 2.6: for any $A, B \subset X$, $[A \cup B]_G \subset [A]_G \cup [B]_G$. It is easy to see that a topological space X is a G-topologizable space if and only if $A^\circ = A^{\circ G}$ for each $A \subset X$ by Corollary 3.9, if and only if $\overline{A} = \overline{A}^G$ for each $A \subset X$ by Theorem 2.9.

Proposition 6.2. *If G is a regular subsequential method on a topological space X, then $\mathcal{T}_G$ is a G-topology on the set X.*

Proof. It follows from Lemma 2.11 that $[A]_G = [A]_{seq}$ for each $A \subset X$. It is easy to see that $[A]_G \cup [B]_G = [A]_{seq} \cup [B]_{seq} = [A \cup B]_{seq} = [A \cup B]_G$ for every $A, B \subset X$. Thus, $\mathcal{T}_G$ is a G-topology on the set X. $\square$

Lemma 6.3. *Let $(X, \mathcal{T})$ be a topological space and G be a method on X. Then*

(1) *X is a G-sequential space if and only if $\mathcal{T}_G \subset \mathcal{T}$.*
(2) *If G is a subsequential method on X, then $\mathcal{T} \subset \mathcal{T}_G$.*

Proof. (1) By Proposition 5.3, the space X is a G-sequential space if and only if every G-open set of X is open in X, if and only if $\mathcal{T}_G \subset \mathcal{T}$.

(2) Suppose G is a subsequential method on the topological space X. By Lemma 2.11(2), every closed set of X is G-closed, thus $\mathcal{T} \subset \mathcal{T}_G$. $\square$

Theorem 6.4. *If G is a method on a topological space X, then X is a G-topologizable sequential space if and only if it is a G-sequential space such that each sequentially closed set of X is G-closed.*

Proof. The topology of the topological space X is denoted by $\mathcal{T}$. Suppose X is a G-topologizable sequential space. It follows from $\mathcal{T}_G = \mathcal{T}$ that a subset of X is G-open if and only if it is open. Thus, X is a G-sequential space by Proposition 5.3. Let A be a sequentially closed set of X. Since A is a sequential space, A is closed, and A is G-closed in X.

Conversely, suppose X is a G-sequential space such that each sequentially closed set of X is G-closed. Since each closed set is sequentially closed, $\mathcal{T} \subset \mathcal{T}_G$, and $\mathcal{T}_G \subset \mathcal{T}$ by Lemma 6.3(1), thus $\mathcal{T}_G = \mathcal{T}$, i.e., X is a G-topologizable space. Let A be a sequentially closed set in X. Then A is G-closed, thus A is closed in X. Hence, X is a sequential space. □

The following corollary is obtained by Theorem 6.4 and Lemma 2.11(2).

Corollary 6.5. *If G is a subsequential method on a topological space X, then X is a G-sequential space if and only if it is a G-topologizable sequential space.*

Corollary 6.6. *Suppose G is a subsequential method on a topological space X. If X is a G-Fréchet space, then X is a G-topologizable Fréchet space.*

Proof. The topology of the topological space X is denoted by $\mathcal{T}$. By Corollary 5.4 and Corollary 6.5, X is a G-topologizable space. Let $A \subset X$ and $x \in \overline{A}$. By Proposition 5.6 and Lemma 2.11(2), $\overline{A} \subset [A]_{seq}$, thus $x \in [A]_{seq}$, and there is an $\boldsymbol{x} \in s(A) \cap c(X)$ such that $\lim \boldsymbol{x} = x$. Hence, X is a Fréchet space. □

By Corollaries 5.5(2) and 6.6 the following result is obtained.

Corollary 6.7. *If G is a regular subsequential method on a topological space X, then X is a G-Fréchet space if and only if it is a G-topologizable Fréchet space.*

Recall the notion of statistical convergence in topological spaces. For each subset A of $\mathbb{N}$ the *asymptotic density* of A [17], denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \leq n\}|,$$

if this limit exists, where $|B|$ denotes the cardinality of the set B. Let X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to *converge statistically* to a point $x \in X$ [9], if $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$, i.e., $\delta(\{n \in \mathbb{N} : x_n \in U\}) = 1$ for every neighborhood U of x in X. Obviously, every convergent sequence in a topological space is statistically convergent to the same limit, but the converse is not true in general [9]. Statistical convergence on a topological space is a regular method on the space.

Example 6.8. *G-topology.*

(1) *There is a method G on a topological space X such that $\mathcal{T}_G$ is a G-topology, but the method G is neither a regular nor a subsequential method on X.*

Let G be the method on the topological space X in Example 2.13(1), which is neither a regular nor a subsequential method. It is easy to see that $\mathcal{T}_G = \{X\} \cup \{U \subset X : 0 \notin U\}$. Hence, $\mathcal{T}_G$ is a topology on X. This shows that the converse of Proposition 6.2 is false.

(2) *There is a regular method G on a topological space X such that X is a G-sequential and G-topologizable space, but X is not a sequential space.*

Let $S = \{1/n : n \in \mathbb{N}\}$ and $X = \{0\} \cup S$. The topology $\mathcal{T}$ on X is defined in the following way: i) each point $1/n$ is isolated; ii) each open neighborhood of the point 0 is a set U of the form $U = \{0\} \cup M$, where $M \subset S$ and $\delta(\{n \in \mathbb{N} : 1/n \in M\}) = 1$. Then X is not a sequential space [18]. Let G be the statistical convergence method on X. Then G is a regular method. It was proved that X is a G-sequential space [18]. By Corollary 6.5, G is not a subsequential method on X. Since each open set of X is G-open (i.e., statistically open [18]), $\mathcal{T} \subset \mathcal{T}_G$. This shows that the converse of Lemma 6.3(2) is false.

By Lemma 6.3(1), $\mathcal{T}_G \subset \mathcal{T}$. Thus, $\mathcal{T}_G = \mathcal{T}$, and X is a G-topologizable space. This shows that the subsequentiality of the method G in Corollary 6.5 cannot be omitted.

(3) *There is a regular method G on a topological space X such that X is a G-Fréchet space, but $\mathcal{T}_G$ is not a topology on X.*

Let G be the method on the topological space X in Example 2.14(1), which is a regular method and $\mathcal{T}_G$ is not a topology on X. Since X is first-countable, by Corollary 5.5(2), X is a G-Fréchet space. This shows that the subsequentiality of the method G in Corollary 6.6 cannot be omitted. □

Question 6.9. *Suppose G is a subsequential method on a topological space X. Is X a G-Fréchet space if it is a G-topologizable Fréchet space?*

7. *G*-continuous mappings

Let X and Y be topological spaces. A mapping $f : X \to Y$ is called *sequentially continuous* [1] if, whenever a sequence $\boldsymbol{x}$ of X converges to a point $l \in X$, then the sequence $f(\boldsymbol{x})$ of Y converges to $f(l) \in Y$. Çakallı [4] defined G-sequential continuity as follows. Let G be a method on a first-countable topological group X. A mapping $f : X \to X$ is called *G-sequentially continuous* if $f(\boldsymbol{x}) \in c_G(X)$ and $G(f(\boldsymbol{x})) = f(G(\boldsymbol{x}))$ for each $\boldsymbol{x} \in c_G(X)$. The mapping f is a self-mapping on the topological group. Since G-methods and G-convergence are defined on sets, we introduce the following notion of G-continuity in sets.

Definition 7.1. Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is called *(G_1, G_2)-continuous* if $f(\boldsymbol{x}) \in c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x}))$ for each $\boldsymbol{x} \in c_{G_1}(X)$. (G_1, G_2)-continuity is called the *G-continuity* if G_1 and G_2 are the same method G.

We can define (G_1, G_2)-continuity at a point of a set, and then define (G_1, G_2)-continuity on some subsets or a set. Connor, Grosse-Erdmann [7] and Mucuk, Şahan [16] considered G-continuity at a point in the topological space of all real numbers or first-countable topological groups, respectively.

Lemma 7.2. *Let G_1, G_2 be methods on sets X and Y, respectively. The following are equivalent for a mapping $f : X \to Y$.*

1. $f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2}$ *for each $A \subset X$.*
2. $f^{-1}(F)$ *is a G_1-closed set of X for each G_2-closed set F of Y.*
3. $f^{-1}(W)$ *is a G_1-open set of X for each G_2-open set W of Y.*
4. *For each $x \in X$ if U is a G_2-neighborhood of $f(x)$, then there exists a G_1-neighborhood V of x with $f(V) \subset U$.*

Proof. Obviously, (2) ⇔ (3). To complete the proof, we will prove that (3) ⇒ (4) ⇒ (1) ⇒ (2).

(3) ⇒ (4). For each $x \in X$, if U is a G_2-neighborhood of $f(x)$, then there exists a G_2-open set W with $f(x) \in W \subset U$, i.e., $x \in f^{-1}(W) \subset f^{-1}(U)$. By condition (3), $f^{-1}(W)$ is a G_1-open set of X, hence $f^{-1}(W)$ is a G_1-neighborhood of x, and $f(f^{-1}(W)) \subset U$.

(4) ⇒ (1). For any $A \subset X$, if $x \in \overline{A}^{G_1}$ and U is a G_2-open set of Y containing $f(x)$, then there exists a G_1-open set V of X containing x with $f(V) \subset U$ by condition (4). By Proposition 3.11(2), $V \cap A \neq \emptyset$, then $f(V) \cap f(A) \neq \emptyset$, and $U \cap f(A) \neq \emptyset$, hence $f(x) \in \overline{f(A)}^{G_2}$. This shows that $f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2}$.

(1) ⇒ (2). Let F be a G_2-closed set of Y and let $A = f^{-1}(F)$. Then $\overline{F}^{G_2} = F$, $f(A) \subset F$, and $f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2} \subset \overline{F}^{G_2} = F$, thus $\overline{A}^{G_1} \subset f^{-1}(F) = A$, hence $f^{-1}(F)$ is G_1-closed in X. □

Theorem 7.3. *Let $f : X \to Y$ be a mapping, where G_1, G_2 are methods on sets X and Y, respectively. Then (1) ⇒ (2) ⇒ (3) in the following conditions.*

(1) *f is a (G_1, G_2)-continuous mapping.*
(2) *$f([A]_{G_1}) \subset [f(A)]_{G_2}$ for each $A \subset X$.*
(3) *$f^{-1}(F)$ is a G_1-closed set of X for each G_2-closed set F of Y.*

Proof. (1) ⇒ (2). Let $A \subset X$ and $l \in [A]_{G_1}$. There exists an $\boldsymbol{x} \in s(A) \cap c_{G_1}(X)$ with $G_1(\boldsymbol{x}) = l$. By condition (1), we have $f(\boldsymbol{x}) \in s(f(A)) \cap c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x}))$, then $f(l) \in [f(A)]_{G_2}$. Thus, $f([A]_{G_1}) \subset [f(A)]_{G_2}$.

(2) ⇒ (3). Suppose that F is a G_2-closed set of Y. Put $A = f^{-1}(F)$, then $f(A) \subset F$. By condition (2), we have $f([A]_{G_1}) \subset [f(A)]_{G_2} \subset [F]_{G_2} \subset F$, hence $[A]_{G_1} \subset f^{-1}(F) = A$, i.e., $f^{-1}(F)$ is a G_1-closed set of X. □

It is worth noting that Lemma 7.2 and Theorem 7.3 have nothing to do with the topology of the sets. Thus one can always introduce an appropriate topology on the sets to make the map f continuous.

Example 7.4. *Continuity and G-continuity.*

(1) *There exists a mapping which satisfies Theorem 7.3(3), but does not satisfy Theorem 7.3(2).*

Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology. Put

$$c_{G_1}(X) = \{\{x_n\}_{n\in\mathbb{N}} \in s(X) : \text{there exists an } m \in \mathbb{N} \text{ such that } \{x_n - x_{n-1}\}_{n>m} \text{ is a constant sequence}\}.$$

Define $G_1 : c_{G_1}(X) \to X$ by $G_1(\boldsymbol{x}) = \lim_{n\to\infty}(x_{n+1} - x_n)$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$. Then G_1 is a method on X.

Let $Y = \{0, 1\}$. Define a mapping $f : X \to Y$ as follows: $f(x) = 0$ if and only if $x = 2k,\ k \in \mathbb{Z}$.

Let $G_2 = G_1|_Y$. Then G_2 is a method on the set Y. If F is a G_2-closed set of Y, then F is equal to $\{0\}$ or Y, thus $f^{-1}(F)$ is a G_1-closed set of X. Therefore, f satisfies Theorem 7.3(3). On the other hand, $\mathbb{N} \subset X$, $1 \in [\mathbb{N}]_{G_1}$ and $[f(\mathbb{N})]_{G_2} = \{0\}$, hence $f([\mathbb{N}]_{G_1}) \not\subset [f(\mathbb{N})]_{G_2}$, and f does satisfy Theorem 7.3(2).

(2) *There exists a bijective mapping which satisfies Theorem 7.3(2), but does not satisfy Theorem 7.3(1).*

Let G_1 be the method G on the topological space X defined in Example 2.13(1). Let G_2 be the method G_1 on the topological space X defined in Example 7.4(1). Let $f : X \to X$ be the identity mapping. Then f is a bijective mapping. For each non-empty set $A \subset \mathbb{Z}$, $[A]_{G_1} = \{0\} \subset [A]_{G_2}$, i.e., $f([A]_{G_1}) \subset [f(A)]_{G_2}$. Let $\boldsymbol{x} = \{1, 1, 2, 3, 5, 8, \cdots\}$ be the Fibonacci sequence. Then $\boldsymbol{x} \in c_{G_1}(X)$, but $f(\boldsymbol{x}) \notin c_{G_2}(X)$, so f is not (G_1, G_2)-continuous. □

Connor and Grosse-Erdmann [7, p. 109] posed the following question. Are there a regular method G on the topological space $\mathbb{R}$ and a G-continuous mapping $f : \mathbb{R} \to \mathbb{R}$ such that f is not continuous?

Lemma 7.5. [2] *Let X, Y be topological spaces and $f : X \to Y$ be a mapping. f is a sequentially continuous mapping if and only if $f^{-1}(F)$ is a sequentially closed set of X whenever F is a sequentially closed set of Y.*

By Lemma 2.11, Theorem 7.3 and Lemma 7.5, we have the following corollaries.

Corollary 7.6. *Let G_1, G_2 be regular subsequential methods on topological spaces X and Y, respectively. If $f : X \to Y$ is a (G_1, G_2)-continuous mapping, then f is sequentially continuous.*

Corollary 7.7. *Let G_1, G_2 be regular subsequential methods on topological spaces X and Y, respectively. If X is a sequential space, and $f : X \to Y$ is a (G_1, G_2)-continuous mapping, then f is continuous.*

Theorem 7.8. *Let X be a Tychonoff space satisfying the first axiom of countability, and G be a method satisfying the cofinal condition on X. If each continuous function $f : X \to \mathbb{R}$ is $(G, G_{\mathbb{R}})$-continuous, where $G_{\mathbb{R}}$ is a regular method on $\mathbb{R}$, then G is a subsequential method on X.*

Proof. If G is not a subsequential method on X, there exists a subset A of X with $[A]_G \not\subset [A]_{seq} = \overline{A}$ by Proposition 2.12 and first-countability of X. Pick a point $l \in [A]_G \setminus \overline{A}$. Because X is a Tychonoff space, it follows that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(l) = 0$ and $f(\overline{A}) = \{1\}$. There is an $\boldsymbol{x} \in s(A) \cap c_G(X)$ such that $G(\boldsymbol{x}) = l$ by $l \in [A]_G$. We know that f is $(G, G_{\mathbb{R}})$-continuous, hence $G(f(\boldsymbol{x})) = G_{\mathbb{R}}(f(\boldsymbol{x})) = 1 \neq 0 = f(l)$, which is a contradiction. Therefore, G is a subsequential method on X. □

Theorem 7.8 improves the following result on $\mathbb{R}$ with the ordinary topology (see [7, Theorem 2]): Let G be a regular method. If every continuous function is G-continuous, then G is a subsequential method.

At the end of this section, we discuss some mappings which preserve G-sequential spaces or G-Fréchet spaces.

Let X and Y be topological spaces. A surjective mapping $f : X \to Y$ is called *quotient* if, whenever $f^{-1}(U)$ is an open set of X, then U is an open set of Y. Continuity is not required for a quotient mapping. f is called *pseudo-open* [13] if for each $y \in Y$ and each open subset U in X with $f^{-1}(\{y\}) \subset U$, then we have that $f(U)$ is a neighborhood of y in Y. It is easy to verify that every pseudo-open mapping is a quotient mapping.

Theorem 7.9. *Let G_1, G_2 be methods on topological spaces X and Y, respectively. If X is a G_1-sequential space and $f : X \to Y$ is a (G_1, G_2)-continuous quotient mapping, then Y is a G_2-sequential space.*

Proof. Let U be a G_2-open set of Y. Since $f : X \to Y$ is a (G_1, G_2)-continuous mapping, and X is a G_1-sequential space, it is clear that $f^{-1}(U)$ is open in X by Theorem 7.3, Lemma 7.2 and Proposition 5.3. It follows that U is an open set of Y because f is quotient. By Proposition 5.3, Y is a G_2-sequential space. □

Theorem 7.10. *Let G_1, G_2 be methods on topological spaces X and Y, respectively. If X is a G_1-Fréchet space and $f : X \to Y$ is a (G_1, G_2)-continuous pseudo-open mapping, then Y is a G_2-Fréchet space.*

Proof. Let $A \subset Y$, and $y \in \overline{A}$. Then $f^{-1}(\{y\}) \cap \overline{f^{-1}(A)} \neq \emptyset$. In fact, if $f^{-1}(\{y\}) \cap \overline{f^{-1}(A)} = \emptyset$, that is $f^{-1}(\{y\}) \subset X \setminus \overline{f^{-1}(A)}$, since $f : X \to Y$ is a pseudo-open mapping, it follows that

$$y \in [f(X \setminus \overline{f^{-1}(A)})]^\circ \subset [f(X \setminus f^{-1}(A))]^\circ = (Y \setminus A)^\circ = Y \setminus \overline{A},$$

which is a contradiction. Hence, there exists a point $l \in f^{-1}(\{y\}) \cap \overline{f^{-1}(A)}$. Because X is a G_1-Fréchet space, it follows that $\overline{f^{-1}(A)} \subset [f^{-1}(A)]_{G_1}$, so there exists an $\boldsymbol{x} \in s(f^{-1}(A)) \cap c_{G_1}(X)$ with $G_1(\boldsymbol{x}) = l$. Since $f : X \to Y$ is a (G_1, G_2)-continuous mapping, $f(\boldsymbol{x}) \in s(A) \cap c_{G_2}(Y)$, $G_2(f(\boldsymbol{x})) = f(G_1(\boldsymbol{x})) = f(l) = y$, and $y \in [A]_{G_2}$. Therefore, $\overline{A} \subset [A]_{G_2}$. Hence Y is a G_2-Fréchet space. □

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References

[1] A.V. Arhangel'skiĭ, L.S. Pontryagin, General Topology I, Encyclopaedia of Mathematical Sciences, vol. 17, Springer-Verlag, Berlin, 1990.
[2] J.R. Boone, F. Siwiec, Sequentially quotient mappings, Czechoslov. Math. J. 26 (1976) 174–182.
[3] H. Çakallı, Lacunary statistical convergence in topological groups, Indian J. Pure Appl. Math. 26 (2) (1995) 113–119.
[4] H. Çakallı, Sequential definitions of compactness, Appl. Math. Lett. 21 (6) (2008) 594–598.
[5] H. Çakallı, On G-continuity, Comput. Math. Appl. 61 (2011) 313–318.
[6] H. Çakallı, Sequential definitions of connectedness, Appl. Math. Lett. 25 (2012) 461–465.
[7] J. Connor, K. Grosse-Erdmann, Sequential definitions of continuity for real functions, Rocky Mt. J. Math. 33 (1) (2003) 93–121.
[8] Á. Császár, Generalized topology, generalized continuity, Acta Math. Hung. 96 (2002) 351–357.
[9] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, Topol. Appl. 156 (2008) 28–45.
[10] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[11] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[12] L. Foged, Sequential coreflections of stratifiable spaces, Proc. Am. Math. Soc. 92 (1984) 470–472.
[13] S.P. Franklin, Spaces in which sequences suffice, Fundam. Math. 57 (1) (1965) 107–115.
[14] J.L. Kelley, General Topology, van Nostrand, New York, 1955 (Graduate Texts Math., vol. 27, Springer-Verlag, Berlin, 1975).
[15] L. Liu, Z.B. Tang, S. Lin, Statistically sequential spaces and statistically sequentially quotient mappings (in Chinese), Appl. Math. J. Chin. Univ. 30 (4) (2015) 485–493.
[16] O. Mucuk, T. Şahan, On G-sequential continuity, Filomat 28 (6) (2014) 1181–1189.
[17] I. Niven, H.S. Zuckerman, H. Montgomery, An Introduction to the Theory of Numbers, fifth ed., Wiley, New York, 1991.
[18] Z.B. Tang, F.C. Lin, Statistical versions of sequential and Fréchet–Urysohn spaces, Adv. Math. (PRC) 44 (6) (2015) 945–954.