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CLOSED MAPPINGS, BOUNDARY-COMPACT MAPPINGS AND SEQUENCE-COVERING MAPPINGS

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Abstract. Yoshio Tanaka and Chuan Liu posed the following question in 1999: Let $f : X \to Y$ be a closed mapping. Under what conditions on X or Y, does $\partial f^{-1}(y)$ have some nice properties for every $y \in Y$?

In this paper, the following two related questions are discussed.

(1) When is a closed mapping to be a boundary-compact mapping or boundary-Lindelöf mapping?

(2) When is a sequence-covering boundary-compact mapping or boundary-Lindelöf mapping to be a 1-sequence-covering mapping?

The following results on generalized metric spaces are obtained, which answers a few questions in literature.

(a) Suppose that $f : X \to Y$ is a closed mapping, where X is a regular k-space with a point-countable k-network or a regular sequential space with a point-countable w-system. If Y contains no closed copy of S_ω, then f is a boundary-compact mapping.

(b) Suppose that $f : X \to Y$ is a closed mapping, where X is a k^*-metrizable k-space. If Y contains no closed copy of S_{ω_1}, then f is a boundary-s-mapping.

(c) Suppose that $f : X \to Y$ is a sequence-covering boundary-compact mapping. If X is first-countable, then f is a 1-sequence-covering mapping.

(d) Suppose that $f : X \to Y$ is a sequence-covering boundary-Lindelöf mapping, where X is first-countable. Then Y is snf-countable if and only if f is a 1-sequence-covering mapping.

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1. Introduction

Mappings are an important tool of investigating topological spaces. Recently, mutual relations among closed mappings, boundary-compact mappings and sequence-covering mappings have been one of the topics focused by topologists [3, 16, 21, 22, 29, 40, 41, 42, 49, 50]. Research concerning this topic mainly originated from the deepening to some mappings theorems on metrizable spaces.

The classic mapping theorem on metrizable spaces is as follows.

Theorem 1.1. [35, 45] *Suppose that X is a metrizable space. If $f : X \to Y$ is a closed mapping, then the following are equivalent.*

(1) Y is a metrizable space.

(2) Y is a first-countable space.

(3) f is a boundary-compact mapping.

This shows that, under certain conditions, a closed mapping is boundary-compact and so inductively perfect and compact-covering. Tanaka and Liu [47] posed the following Question 1.2.

Question 1.2. [47] *Let $f : X \to Y$ be a closed mapping. Under what conditions on X or Y, does $\partial f^{-1}(y)$ have some nice properties for every $y \in Y$?*

This guides us to discuss following Question 1.3 in this paper.

Question 1.3. *When is a closed mapping to be a boundary-compact mapping or boundary-Lindelöf mapping?*

Theorem 1.1 implies that open and closed mappings preserve metrizability, where openness of mappings can not omitted, but can be weakened from different stand point. Siwiec [44] introduced sequence-covering mappings and proved that every open mapping on first-countable spaces is sequence-covering. This shows that sequence-covering mappings on metrizable spaces are a generalization of open mappings. Yan, Jiang and Lin [49] further proved that sequence-covering closed mappings preserve metrizability, where the key is to show that every sequence-covering compact mapping on metrizable spaces is 1-sequence-covering [25]. Every open mapping on first-countable spaces is also 1-sequence-covering [17].

These lead to following Question 1.4.

Question 1.4. *When is a sequence-covering boundary-compact mapping or boundary-Lindelöf mapping to be a 1-sequence-covering mapping?*

In this paper, we mainly discuss Questions 1.3 and 1.4 on generalized metric spaces, in particular, affirmatively answer following Questions 1.5.

Question 1.5. [16, Question 3.1] *Is every sequence-covering boundary-compact mapping on the spaces in which every compact subset has a countable neighbourhood base 1-sequence-covering? More generally, is every sequence-covering boundary-compact mapping on first-countable spaces 1-sequence-covering?*

The paper is organized as follows.

In Sections 2 and 3, around Questions 1.2 and 1.3, we discuss the properties of closed mappings on the spaces with a point-countable k-network or k^*-metrizable spaces.

In Sections 4 and 5, around Questions 1.4 and 1.5, we mainly discuss the properties of sequentially quotient or sequence-covering boundary-compact mappings on the spaces in which every compact subset has a countable sn-network, and investigate the properties of sequence-covering boundary-compact mappings or boundary-Lindelöf mappings on first-countable spaces.

In addition, a few questions are also posed.

Recall some concepts relevant to mappings or k-networks.

Definition 1.6. Let $f : X \to Y$ be a mapping.

(1) f is called a *boundary-compact mapping* (resp. *boundary-Lindelöf mapping*, *boundary-s-mapping*), if $\partial f^{-1}(y)$ is compact (resp. Lindelöf, separable) for every $y \in Y$.

(2) f is a *compact-covering mapping* [30] if for every compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

(3) f is a *sequence-covering mapping* [44] if $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}$ in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.

(4) f is a *pseudo-sequence-covering mapping* [13, 14] if for every convergent sequence L including its limit point in Y, there exists a compact subset K of X such that $f(K) = L$.

(5) f is a *subsequence-covering mapping* [23] if $\{y_n\}$ is a convergent sequence including its limit point in Y, there exists a compact subset K of X such that $f(K)$ is a subsequence of $\{y_n\}$.

(6) f is a *sequentially quotient mapping* [4] if $\{y_n\}$ is a convergent sequence in Y, there exists a convergent sequence $\{x_k\}$ in X such that the sequence $\{f(x_k)\}$ is a subsequence of the sequence $\{y_n\}$.

(7) f is a *1-sequence-covering mapping* [17] if for every $y \in Y$, there exists a point $x \in f^{-1}(y)$ such that whenever a sequence $\{y_n\}$ converges to y in Y, there is a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.

The following implications about mappings are evident.

$$
\begin{tikzcd}
\text{1-sequence-covering} \arrow[r] & \text{sequence-covering} \arrow[r] \arrow[d] & \text{sequentially quotient} \arrow[d] \\
\text{compact-covering} \arrow[r] & \text{pseudo-sequence-covering} \arrow[r] & \text{subsequence-covering}
\end{tikzcd}
$$

Definition 1.7. Let $\mathcal{P}$ be a family of subsets of a topological space X and $A \subset X$. The symbol $\mathrm{cl}_1(A)$ denotes the set consisting of the limits of convergent sequences of points of A in X.

(1) $\mathcal{P}$ is a *network* [2] for X if for every point $x \in X$ and any neighbourhood U of x, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is a *k-network* [36] for X if whenever K is a compact subset of an open set U, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset U$.

(3) $\mathcal{P}$ is a wcs^*-network [24] for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is a *cl_1-osed k-network* [3] (briefly, *k_1-network*) for X if whenever K is a compact subset of an open set U, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset \mathrm{cl}_1(\cup\mathcal{F}) \subset U$.

(5) $\mathcal{P}$ is a *cl_1-osed cs^*-network* [3] (briefly, *cs_1^*-network*) for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset \mathrm{cl}_1(P) \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

Obviously, every k-network (resp. wcs^*-network) in regular spaces is a k_1-network (resp. cs_1^*-network). The following implications are also evident.

$$
\begin{tikzcd}
 & cs_1^*\text{-network} \arrow[dr] & & \\
k_1\text{-network} \arrow[ur] \arrow[dr] & & wcs^*\text{-network} \arrow[r] & \text{network} \\
 & k\text{-network} \arrow[ur] & &
\end{tikzcd}
$$

For an infinite cardinal number κ, the *fan space* S_κ [46] is the quotient space obtained by identifying all limit points of the topological sum of κ many convergent sequences. We only use S_ω and S_{ω_1} in the paper.

In what follows, all topological spaces are assumed to be Hausdorff and all mappings are continuous and surjective. For some terminology unstated here, readers may refer to [11].

2. BOUNDARY-COMPACT MAPPINGS

The following theorem complements Theorem 1.1.

Theorem 2.1. [46] *Let $f : X \to Y$ be a closed mapping, where X is a metrizable space. If Y contains no closed copy of S_ω, then f is a boundary-compact mapping.*

In this section, around Question 1.3, we shall discuss which generalized metric spaces Theorem 2.1 can be generalized to.

A topological space X is said to be *sequentially separable* [7], if there exists a countable subset $D \subset X$ such that $\mathrm{cl}_1(D) = X$, where D is called a *sequentially dense* subset of X.

Lemma 2.2. *Suppose that X is a sequentially separable space. If X has a point-countable cs_1^*-network, then it has a countable network.*

Proof. Let D be a sequentially dense subset of X and $\mathcal{P}$ a point-countable cs_1^*-network. Then $\mathcal{F} = \{\mathrm{cl}_1(P) : P \in \mathcal{P}, P \cap D \neq \emptyset\}$ is countable. Let $x \in U$ with U open in X. There exists a sequence $\{x_n\}$ of points of D such that $\{x_n\}$ converges to x in X, and $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset \mathrm{cl}_1(P) \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$. Thus $\mathrm{cl}_1(P) \in \mathcal{F}$ and $x \in \mathrm{cl}_1(P) \subset U$. Hence $\mathcal{F}$ is a countable network for X. □

A topological space X is called a *sequential space* [9] if a set $A \subset X$ is closed if and only if together with any sequence it contains its limit. A topological space X is of *countable tightness* [32] if whenever $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subset A$. Every sequential space is of countable tightness [32].

Lemma 2.3. *Let $f : X \to Y$ be a closed mapping, where X is a sequential space and Y contains no closed copy of S_ω. If every sequentially separable subspace of X is normal, then $\partial f^{-1}(y)$ is a countably compact subset of X for every $y \in Y$.*

Proof. For every $y \in Y$, put $A = f^{-1}(y) \cap \mathrm{cl}_1(X \setminus f^{-1}(y))$.

(3.1) $\overline{A} = \partial f^{-1}(y)$.

Obviously, $\overline{A} \subset \partial f^{-1}(y)$. It is easy to see the set $\overline{A} \cup (X \setminus f^{-1}(y))$ is sequentially closed. Since X is sequential, the set $U = f^{-1}(y) \cap (X \setminus \overline{A})$ is open and $\partial f^{-1}(y) \setminus \overline{A} \subset U$, thus $\overline{A} = \partial f^{-1}(y)$.

Now, we shall show that $\partial f^{-1}(y)$ is a countably compact subset of X for every $y \in Y$.

Otherwise, $\partial f^{-1}(y)$ contains an infinite closed discrete subset $\{x_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, by (3.1), $x_n \in \overline{A}$ and there exists a countable subset $\{x_{n,m} : m \in \mathbb{N}\} \subset A$ such that $x_n \in \overline{\{x_{n,m} : m \in \mathbb{N}\}}$, since X is of countable tightness. For every $n, m \in \mathbb{N}$, there exists a sequence $L_{n,m}$ of points of $X \setminus f^{-1}(y)$ converging to $x_{n,m}$ in X. Let

$$D = \mathrm{cl}_1(\{x_n : n \in \mathbb{N}\} \cup \bigcup\{L_{n,m} : n, m \in \mathbb{N}\}).$$

Obviously, $\{x_n : n \in \mathbb{N}\} \subset D$ and the subspace D is sequentially separable. By the hypothesis, D is a normal space. There exists a discrete family $\{U_n : n \in \mathbb{N}\}$ of open subsets in D such that $x_n \in U_n$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose

$m_n \in \mathbb{N}$ such that $x_{n,m_n} \in U_n$. Since L_{n,m_n} converges to x_{n,m_n} in D, without loss of generality, we may assume that $L_{n,m_n} \subset U_n$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, write $P_n = L_{n,m_n}$, thus $y \notin f(P_n)$ and the sequence $f(P_n)$ converges to y in Y.

(3.2) $\bigcup_{n\in\mathbb{N}} Q_n$ is a closed discrete subset of X, where Q_n is a finite subset of P_n for every $n \in \mathbb{N}$.

Otherwise, since X is a sequential space, there exists a sequence $\{b_k\}$ converging to some point $b \notin \bigcup_{n\in\mathbb{N}} Q_n$ in X such that $b_k \in Q_{n_k} \subset U_{n_k}$ for every $k \in \mathbb{N}$ with $n_1 < n_2 < \cdots$. Thus $b \in D$ and $\{b_k : k \in \mathbb{N}\}$ is not closed in D, which contradicts the fact that $\{U_{n_k} : k \in \mathbb{N}\}$ is discrete in D. Hence, $\bigcup_{n\in\mathbb{N}} Q_n$ is closed, further, is closed discrete in X.

For every $n \in \mathbb{N}$, there exist a finite subset F_n of $f(P_n)$ and $n_k \in \mathbb{N}$ such that

$$f(P_n) \cap \bigcup_{m\geq n_k} f(P_m) \subset F_n.$$

Otherwise, there exist a non-trivial sequence $\{y_i\}$ of points of $f(P_n)$ and a sequence $\{f(P_{k_i})\}$ such that $y_i \in f(P_{k_i})$ for every $i \in \mathbb{N}$. According to (3.2) and the closedness of f, the set $\{y_i : i \in \mathbb{N}\}$ is a closed discrete subset of Y. This is a contradiction. Now, we can assume that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let $K_k = f(P_{n_k}) \setminus F_{n_k}$. Then the sequence K_k converges to y in Y for every $k \in \mathbb{N}$ and $\{K_k : k \in \mathbb{N}\}$ is a pairwise disjoint family.

Put $S = \{y\} \cup \bigcup_{k\in\mathbb{N}} K_k$. Since X is a sequential space and f is closed, Y is also a sequential space. Using again (3.2) and the closedness of f, we conclude that S is closed in Y and S is homeomorphic to S_ω. This contradicts the hypothesis that Y contains no closed copy of S_ω. In a word, $\partial f^{-1}(y)$ is a countably compact subset of X for every $y \in Y$. □

Lemma 2.4. [24] *Suppose that X is a countably compact sequential space with a point-countable wcs*-network. Then X is a compact metrizable space.*

Lemma 2.5. [13] *Every k-space with a point-countable k-network is a sequential space.*

Theorem 2.6. *Suppose that $f : X \to Y$ is a closed mapping, where X is a regular k-space with a point-countable k-network. If Y contains no closed copy of S_ω, then f is a boundary-compact mapping.*

Proof. Since X is a regular space with a point-countable k-network, X has a point-countable cs_1^*-network. By Lemma 2.2, every sequentially separable subspace of X has a countable network, and so is normal. Lemmas 2.3, 2.4 and 2.5 imply together that f is a boundary-compact mapping. □

Let X be a topological space and $x \in X$. A subset G of X is called a *sequential neighborhood* of x in X if any sequence $\{x_n\}$ converging to x is eventually in G, i.e., $\{x_n : n \geq k_0\} \cup \{x\} \subset G$ for some $k_0 \in \mathbb{N}$. Let $\mathcal{W}$ be a family of subsets of a topological space X. $\mathcal{W}$ is called a *w-system* [6] for X, if for every $x \in X$ and every open neighbourhood U of x in X, there exists a subfamily $\mathcal{V} \subset \mathcal{W}$ such that $x \in \cap\mathcal{V}$, $\cup\mathcal{V} \subset U$ and $\cup\mathcal{V}$ is a sequential neighbourhood of x in X.

Burke [6, Proposition 4.2, Theorem 4.4] obtained the following results.

(1) Quotient s-images of metric spaces are a sequential space with a point-countable w-system.

(2) Sequential spaces with a point-countable w-system are a D-space[1].

Next, we shall discuss Questions 1.2 and 1.3 on sequential spaces with a point-countable w-system. In order to make better use of w-systems, we introduce a new notion. Let $\mathcal{P}$ be a family of subsets of a topological space X. $\mathcal{P}$ is called a *cs'-network* for X, if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x, x_m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.

Lemma 2.7. *Let $\mathcal{P}$ be a family of subsets of a topological space X. Then $\mathcal{P}$ is a cs'-network for X if and only if $\mathcal{P}$ is a w-system for X.*

Proof. Necessity. Let $\mathcal{P}$ be a cs'-network for X and U be an open neighbourhood of a point x in X. Put $\mathcal{V} = \{P \in \mathcal{P} : x \in P \subset U\}$. Then $x \in \cap\mathcal{V}$ and $\cup\mathcal{V} \subset U$. Assume that $\cup\mathcal{V}$ is not a sequential neighbourhood of x in X. Then there exists a sequence $\{x_n\}$ converging to x such that $x_n \notin \cup\mathcal{V}$ for every $n \in \mathbb{N}$. Since $\mathcal{P}$ is a cs'-network for X, then $\{x, x_m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$. Thus $x_m \in P \subset \cup\mathcal{V}$. This is a contradiction. Hence $\mathcal{P}$ is a w-system for X.

Sufficiency. Let $\mathcal{P}$ be a w-system for X and $\{x_n\}$ be a sequence converging to a point $x \in U$ with U open in X. There exists a subfamily $\mathcal{V} \subset \mathcal{P}$ such that $x \in \cap\mathcal{V}$, $\cup\mathcal{V} \subset U$ and $\cup\mathcal{V}$ is a sequential neighbourhood of x in X. We may choose $m \in \mathbb{N}$ and $P \in \mathcal{V}$ such that $x_m \in P$. Hence $\{x, x_m\} \subset P \subset U$ and $\mathcal{P}$ is a cs'-network for X. □

Theorem 2.8. *Suppose that $f : X \to Y$ is a closed mapping, where X is a regular sequential space with a point-countable w-system. If Y contains no closed copy of S_ω, then f is a boundary-compact mapping.*

[1] A *neighbourhood assignment* of a topological space (X, τ) is a function $\phi : X \to \tau$ satisfying $x \in \phi(x)$ for every $x \in X$. A topological space X is called a *D-space* [8] if for every neighbourhood assignment ϕ of X, there exists a closed discrete subset D of X such that $\{\phi(x) : x \in D\}$ covers X.

Proof. Since every sequential space with a point-countable w-system is a D-space [6, Theorem 4.4] and every countably compact D-space is compact, by Lemma 2.3, it suffices to show that every sequentially separable subspace of X has a countable network. Let A be a sequentially separable subspace of X and D be a sequentially dense subset of A. By Lemma 2.7, X has a point-countable cs'-network $\mathcal{P}$. Let $\mathcal{F} = \{P \cap A : P \in \mathcal{P}, P \cap D \neq \emptyset\}$. Obviously, $\mathcal{F}$ is countable. Let $x \in U \cap A$ with U open in X. Then there exists a sequence $\{d_n\}$ of points of D converging to x in A. So $\{x, d_m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$. Then $P \cap A \in \mathcal{F}$, $x \in P \cap A \subset U \cap A$ and $\mathcal{F}$ is a countable network for A. $\square$

Corollary 2.9. *Let $f : X \to Y$ be a closed mapping, where X is regular. If one of following conditions is satisfied, then f is a compact-covering mappings:*

(1) X is a sequential space with a point-countable w-system.

(2) X is a k-space with a point-countable k-network [22].

Proof. Let K be a compact subset of Y. Put $L = f^{-1}(K)$ and $g = f|_L : L \to K$. Since K contains no closed copy of S_ω, by Theorems 2.8 and 2.6, g is a boundary-compact mapping. Therefore, there is a closed subspace Z of L such that $g|_Z : Z \to K$ is a perfect mapping. Then $(g|_Z)^{-1}(K)$ is a compact subset of Z and $f((g|_Z)^{-1}(K)) = K$. So f is a compact-covering mapping. $\square$

Remark 2.10. (1) [43] There exists a closed mapping $f : X \to Y$ such that f is not compact-covering, where X has a countable base, and $Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual topology.

(2) [39] There exists a closed mapping $f : X \to Y$ such that f is not compact-covering, where X is a regular space with a point-countable k-network and a point-countable w-system, and every compact subset of X is finite.

(3) There exists a paracompact sequential space with a point-countable k-network, but without a point-countable w-system.

In fact, the fan space S_{ω_1} is such a space. It is easy to see that S_{ω_1}, as a closed image of a metric space, is a paracompact sequential space with a point-countable k-network. We shall show that S_{ω_1} does not have a point-countable w-system or point-countable cs'-network. Indeed, write $S_{\omega_1} = \{s\} \cup \bigcup_{\alpha<\omega_1} X_\alpha$, where X_α converges to s for every $\alpha < \omega_1$. Assume that S_{ω_1} has a point-countable cs'-network $\mathcal{P}$. Write

$$\{P \in \mathcal{P} : s \in P, |\{\alpha < \omega_1 : X_\alpha \cap P \neq \emptyset\} \geq \omega|\} = \{P_n : n \in \mathbb{N}\}.$$

By induction, we pick a subset $\{x_n : n \in \mathbb{N}\}$ of S_{ω_1} such that $x_n \in P_n \cap X_{\alpha_n}$, where $X_{\alpha_i} \cap X_{\alpha_j} \neq \emptyset$ for any two distinct $i, j \in \mathbb{N}$. Then the set $\{x_n : n \in \mathbb{N}\}$ is closed in S_{ω_1}. Let

$$V = S_{\omega_1} \setminus \{x_n : n \in \mathbb{N}\},$$
$$\mathcal{F} = \{P \in \mathcal{P} : P \subset V\},$$
$$H = \cup\{F \in \mathcal{F} : s \in F\}.$$

If $s \in P \in \mathcal{F}$, then $P \notin \{P_n : n \in \mathbb{N}\}$ and so P intersects only finitely many X_α's. Thus H intersects only countably many X_α's. We choose $\beta < \omega_1$ such that $X_\beta \cap H = \emptyset$. Let $V \cap X_\beta = \{v_n : n \in \mathbb{N}\}$. Then the sequence $\{v_n\}$ converges to $s \in V$ and so $\{s, v_m\} \subset P$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{F}$. Hence $v_m \in X_\beta \cap H$. This is a contradiction.

(4) There exists a paracompact sequential space with a point-countable w-system, but without a point-countable k-network.

Indeed, there exists a non-first-countable, strongly Fréchet, countable and regular space S [38, Example 2.3]. Obviously, S is a paracompact sequential space. Since a regular strongly Fréchet space with a point-countable k-network is first-countable [13, Corollary 3.6], the space S does not have a point-countable k-network. Write $S = \{s_n : n \in \mathbb{N}\}$. Let $\mathcal{P} = \{\{s_n, s_m\} : n, m \in \mathbb{N}\}$. Then $\mathcal{P}$ is a countable cs'-network for S. Namely, $\mathcal{P}$ is a countable w-system for S by Lemma 2.7.

3. boundary-Lindelöf mappings

Tanaka [46] established the following mapping theorem on metrizable spaces.

Theorem 3.1. [46] *Let $f : X \to Y$ be a closed mapping, where X is a metrizable space. If Y contains no closed copy of S_{ω_1}, then f is a boundary-Lindelöf mapping.*

Much research shows that Theorem 3.1 can be generalized to some generalized metric spaces [13, 21, 28, 29]. In this section, we mainly discuss the mapping theorems on k^*-metrizable spaces.

Definition 3.2. [3] A topological space is called a *k^*-metrizable space* if there exist a metric space M and a mapping $f : M \to X$ satisfying the following $(*)$.

$(*)$ There exists a subspace $Z \subset M$ such that $f(Z) = X$, and if K is precompact[2] in X, then $Z \cap f^{-1}(K)$ is precompact in M.

The new class of k^*-metrizable spaces, which was also collected in [12] by Gruenhage, has involved various applications in topological algebra, functional analysis and measure theory.

[2]A subset A of a topological space X is *precompact* in X, if $\overline{A}$ is a compact subset of X.

A family $\mathcal{P}$ of subsets of a topological space X is said to be *compact-finite* (resp. *compact-countable*), if $(\mathcal{P})_K = \{P \in \mathcal{P} : P \cap K \neq \emptyset\}$ is finite (resp. countable) for every compact subset K of X.

Lemma 3.3. [3] *A topological space X is k^*-metrizable if and only if X has a σ-compact-finite k_1-network.*

A topological X is ω_1-*compact* if every uncountable subset of X has an accumulation point.

Lemma 3.4. *Let $\mathcal{P}$ be a compact-finite family of a k-space X. If A is an ω_1-compact subset of X, then $(\mathcal{P})_A$ is countable.*

Proof. Suppose $(\mathcal{P})_A$ is not countable. Since $\mathcal{P}$ is point-finite, there exist a subfamily $\mathcal{F} = \{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$ and a subset $B = \{x_\alpha : \alpha < \omega_1\} \subset A$ satisfying

(1) $x_\alpha \in P_\alpha$ for every $\alpha < \omega_1$; and

(2) $x_\alpha \neq x_\beta$ and $P_\alpha \neq P_\beta$ for any two distinct $\alpha, \beta < \omega_1$.

The space A being ω_1-compact, let x be an accumulation point of B in A, whence the set $B \setminus \{x\}$ is not closed in X. Since X is a k-space, there is a compact subset K of X such that $(B \setminus \{x\}) \cap K$ is not closed in K. Then $(B \setminus \{x\}) \cap K$ is infinite, which contradicts the hypothesis that $\mathcal{P}$ is compact-finite for X. □

Theorem 3.5. *Let $f : X \to Y$ be a closed mapping, where X is a k^*-metrizable k-space. If Y contains no closed copy of S_{ω_1}, then f is a boundary-s-mapping.*

Proof. By Lemma 3.3, X has a σ-compact-finite k_1-network $\mathcal{P}$. By Lemma 2.5, X is a sequential space. For every $y \in Y$, put $A = f^{-1}(y) \cap \mathrm{cl}_1(X \setminus f^{-1}(y))$.

We shall prove that A is an ω_1-compact subset of X.

Otherwise, there exists an uncountable closed discrete subset $D = \{x_\alpha : \alpha < \omega_1\}$ in A. For every $\alpha < \omega_1$, choose an open neighbourhood V_α of x_α in X such that $V_\alpha \cap \overline{D \setminus \{x_\alpha\}} = \emptyset$. Since $x_\alpha \in A \cap V_\alpha$, there exists a sequence L_α of points of $(X \setminus f^{-1}(y)) \cap V_\alpha$ converging to x_α in X. Since $\mathcal{P}$ is a k_1-network for X, without loss of generality, we choose $P_\alpha \in \mathcal{P}$ such that $L_\alpha \subset P_\alpha \subset \mathrm{cl}_1(P_\alpha) \subset V_\alpha$, which implies that $x_\alpha \in \mathrm{cl}_1(P_\alpha) \subset X \setminus \overline{D \setminus \{x_\alpha\}}$. Thus $P_\alpha \neq P_\beta$ for any two distinct $\alpha, \beta < \omega_1$. Since $\mathcal{P}$ is σ-compact-finite, we may assume that the family $\mathcal{F} = \{P_\alpha : \alpha < \omega_1\}$ is compact-finite. For every $\alpha < \omega_1$, the compact set $\overline{L_\alpha}$ intersects only finitely many elements of $\mathcal{F}$, and it can be considered that $\overline{L_\alpha} \cap \overline{L_\beta} = \emptyset$ for any two distinct $\alpha, \beta < \omega_1$.

(5.1) $\bigcup_{\alpha<\omega_1} Q_\alpha$ is a closed discrete subset of X, where Q_α is a finite subset of L_α for every $\alpha < \omega_1$.

In fact, for every compact subset K of X, since $\mathcal{F}$ is compact-finite, we conclude that $K \cap \bigcup_{\alpha<\omega_1} Q_\alpha$ is finite and so is closed in X. The space X being a k-space, $\bigcup_{\alpha<\omega_1} Q_\alpha$ is a closed discrete subset of X.

For every $\alpha < \omega_1$, we have that $y \notin f(L_\alpha)$ and the sequence $f(L_\alpha)$ converges to y in Y. According to (5.1), in a way similar to Lemma 2.3, without loss of generality, we can prove that for every $\alpha < \omega_1$, there exists a finite subset F_α of $f(L_\alpha)$ such that Y contains a closed copy $\{y\} \cup \bigcup_{\alpha<\omega_1}(f(L_\alpha) \setminus F_\alpha)$ of S_{ω_1}. This is a contradiction. Hence, A is an ω_1-compact subset of X.

Now, by Lemma 3.4, the space A has a countable k-network, and so is separable. From the proof of (3.1) in Lemma 2.3, we conclude that $\overline{A} = \partial f^{-1}(y)$ is separable. □

Banakh, Bogachev and Kolesnikov [3] proved that, under **CH**, every separable k^*-metrizable regular space has a countable k-network.

Corollary 3.6. *Under* **CH**, *let $f : X \to Y$ be a closed mapping, where X is a regular k^*-metrizable k-space. If Y contains no closed copy of S_{ω_1}, then f is a boundary-Lindelöf-mapping.*

Question 3.7. *Let $f : X \to Y$ be a closed mapping, where X is a k-space with a compact-countable k-network. Is f a boundary-s-mapping if Y contains no closed copy of S_{ω_1}?*

In Question 3.7, if X is replaced by a Moore space or a regular space with a σ-locally finite k-network, then the answer is negative [20, Example 3.4.19].

4. PSEUDO-SEQUENCE-COVERING MAPPINGS

Michael [33] proved that there exists a pseudo-sequence-covering quotient and compact mapping $f : X \to Y$ such that f is not compact-covering, where X is a separable metrizable space and Y is a compact metrizable space. This shows that there is a vast difference between pseudo-sequence-covering mappings and compact-covering mappings. However, every sequentially quotient and boundary-compact mapping on metrizable spaces is pseudo-sequence-covering [15]. In the other hand, every subsequence-covering mapping on a space in which every compact subset is metrizable is sequentially quotient. In this section, we shall further prove that every sequentially quotient and boundary-compact mapping on some generalized metric spaces is pseudo-sequence-covering.

If a topological space X is the open compact image of a metric space, then every sequentially quotient and boundary-compact mapping on X is pseudo-sequence-covering [16, Theorem 3.11]. Using outer bases, Michael and Nagami [34] proved that a topological space X is the open compact image of a metric space if and only if every compact subset K of X is metrizable has a countable neighbourhood base in X. Let us recall the definitions of sn-networks and outer sn-networks.

Let X be a topological space and $A \subset X$. A subset G of X is called a *sequential neighborhood* of A in X if any sequence converging to some point of A is eventually in G.

Definition 4.1. Let X be a topological space and $x \in X$.

(1) A family $\mathcal{P}_x$ of subsets of X is called a *network* of x in X, if $x \in \cap \mathcal{P}_x$ and for $x \in U$ with U open in X, $P \subset U$ for some $P \in \mathcal{P}_x$.

(2) A family $\mathcal{P}_x$ of subsets of X is called a *sequential neighbourhood network* (briefly, *sn-network*) of x in X [17], if the following conditions (a)–(c) hold.

(a) $\mathcal{P}_x$ is a network of x in X;

(b) every element of $\mathcal{P}_x$ is a sequential neighborhood of x in X; and

(c) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

Michael [31] made use of outer networks to investigate images of metric spaces. Outer bases and outer sn-networks are relevant to outer networks.

Definition 4.2. Let A be a non-empty subset of a topological space X.

(1) $\bigcup_{x \in A} \mathcal{P}_x$ is called an *outer sn-network* [27] (resp. an *outer base* [34]) of A in X, where $\mathcal{P}_x$ is an sn-network (resp. a neighbourhood base) of x in X for every $x \in A$.

(2) A family $\mathcal{P}$ of subsets of X is called an *sn-network* [27] (resp. a *neighbourhood base* [34]) of A in X, if every element of $\mathcal{P}$ is a sequential neighbourhood (resp. a neighbourhood) of every point of A in X and for every open subset V of X containing A, there exists a $P \in \mathcal{P}$ such that $P \subset V$.

We have the following lemma about the relation between outer sn-networks and sn-networks.

Lemma 4.3. [27] *Let K be a compact subset of a topological space X. Then K is metrizable and has a countable sn-network in X if and only if K has a countable outer sn-network $\mathcal{H}$ in X satisfying that every element of $\mathcal{H}$ is of the form $U \cap V$, where U is open in X and V is a sequential neighbourhood of K in X.*

Lemma 4.4. *Let $f : X \to Y$ be a sequentially quotient mapping. If $y \in Y$ and $\partial f^{-1}(y)$ has a countable sn-network in X, then the point y has a countable sn-network in Y.*

PROOF. Let $\{V_n : n \in \mathbb{N}\}$ be a countable sn-network of $\partial f^{-1}(y)$ in X, where $V_{n+1} \subset V_n$ for every $n \in \mathbb{N}$. We shall show that $\{f(V_n) : n \in \mathbb{N}\}$ is an sn-network of y in Y.

(4.1) $\{f(V_n) : n \in \mathbb{N}\}$ is a network of y in Y.

Let U be an open neighbourhood of y in Y. Then $f^{-1}(U) \supset \partial f^{-1}(y)$ and so there exists an $n \in \mathbb{N}$ such that $V_n \subset f^{-1}(U)$. Thus $y \in f(V_n) \subset U$.

(4.2) For every $n \in \mathbb{N}$, $f(V_n)$ is a sequential neighbourhood of y in Y.

Assume that $f(V_n)$ is not a sequential neighbourhood of y in Y for some $n \in \mathbb{N}$. Then there exists a sequence $\{y_i\}$ converging to y in Y such that $y_i \notin f(V_n)$ for every $i \in \mathbb{N}$. Since f is a sequentially quotient mapping, there exists a sequence $\{x_k\}$ converging to some point $x \in X$ in X such that $x_k \in f^{-1}(y_{i_k})$ for every $k \in \mathbb{N}$. Then $x \in \partial f^{-1}(y) \subset V_n$. Because V_n is a sequential neighbourhood of x in X, $\{x_k\}$ is eventually in V_n and $\{y_{i_k}\}$ is eventually in $f(V_n)$. This is a contradiction. □

Theorem 4.5. *Suppose that $f : X \to Y$ is a sequentially quotient and boundary-compact mapping. If every compact subset of X is has a countable sn-network in X, then f is a pseudo-sequence-covering mapping.*

PROOF. Suppose that $\{y_n\}$ converges to y_0 in Y. Let

$$S_1 = \{y_0\} \cup \{y_n : n \in \mathbb{N}\}, X_1 = f^{-1}(S_1) \text{ and } g = f|_{X_1} : X_1 \to S_1.$$

Then g is also a sequentially quotient and boundary-compact mapping. Let $\{V_n : n \in \mathbb{N}\}$ be a countable sn-network of the compact set $\partial g^{-1}(y_0)$ in X_1, where $V_{n+1} \subset V_n$ for every $n \in \mathbb{N}$. By Lemma 4.4, $\{g(V_n) : n \in \mathbb{N}\}$ is an sn-network of y_0 in S_1. For every $n \in \mathbb{N}$, there exists an $i_n \in \mathbb{N}$ such that $g^{-1}(y_i) \cap V_n \neq \emptyset$ when $i \geq i_n$. Without loss of generality, we assume that $1 < i_n < i_{n+1}$ for every $n \in \mathbb{N}$. For every $j \in \mathbb{N}$, pick

$$x_j \in \begin{cases} f^{-1}(y_j), & j < i_1; \\ f^{-1}(y_j) \cap V_n, & i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = \partial g^{-1}(y_0) \cup \{x_j : j \in \mathbb{N}\}$. Since $\{V_n : n \in \mathbb{N}\}$ is an sn-network of $\partial g^{-1}(y_0)$ in X_1, the sequence $\{x_j\}$ is eventually in every neighbourhood of $\partial g^{-1}(y_0)$ in X_1. Since $\partial g^{-1}(y_0)$ is a compact subset of X_1, K is also compact of X_1 by [30, Lemma 8.1]. Obviously, $f(K) = S_1$. So f is a pseudo-sequence-covering mapping. □

Corollary 4.6. [16, Corollary 3.12] *Suppose that $f : X \to Y$ is a sequentially quotient and boundary-compact mapping. If X satisfies one of the following conditions, then f is a pseudo-sequence-covering mapping.*

(1) X has a point-countable base.

(2) X is a developable space.

Proof. According to Theorem 4.5, it suffices to show that every compact subset of a space X with a point-countable base or a development has a countable neighbourhood base in X.

Since X is a space with a point-countable base or a developable space, it follows from [34, Theorem 1.3] or [48, Theorem 1] that X is a compact-covering open image of a metric space. Thus, by [34, Theorem 1.2], every compact subset of X has a countable neighborhood base in X. □

Question 4.7. *Is every sequentially quotient and boundary-compact mapping on first-countable spaces pseudo-sequence-covering?*

5. 1-sequence-covering mappings

In this section, we discuss Question 1.4 and give an affirmative answer to Question 1.5.

Lemma 5.1. *Let $f : X \to Y$ be a sequence-covering mapping and $y \in Y$. If $\{F_i : i \in \mathbb{N}\}$ is a decreasing network of y in Y and $\partial f^{-1}(y)$ is a non-empty Lindelöf subset of X. Then there exists a point $x_y \in \partial f^{-1}(y)$ such that if U is a neighbourhood of x_y in X, $F_i \subset f(U)$ for some $i \in \mathbb{N}$.*

Proof. Suppose the conclusion is not true. Then, for every $x \in \partial f^{-1}(y)$, there exists an open neighbourhood U_x of x in X such that $F_i \not\subset f(U_x)$ for every $i \in \mathbb{N}$. Since $\{U_x : x \in \partial f^{-1}(y)\}$ covers $\partial f^{-1}(y)$, there exists a countable subfamily $\mathcal{U} = \{U_{x_j} : j \in \mathbb{N}\}$ of $\{U_x : x \in \partial f^{-1}(y)\}$ covers $\partial f^{-1}(y)$. For every $i, j \in \mathbb{N}$, there exists a point $z_{i,j} \in F_i \setminus f(U_{x_j})$. For every $i, j \in \mathbb{N}$ and $i \geq j$, let $y_k = z_{i,j}$, where $k = j + \frac{i(i-1)}{2}$. Since $\{F_i : i \in \mathbb{N}\}$ is a decreasing network of y in Y, the sequence $\{y_k\}$ converges to y in Y. The mapping f being sequence-covering, there exists a sequence $\{u_k\}$ converging to some point $u \in \partial f^{-1}(y)$ in X such that $f(u_k) = y_k$ for every $k \in \mathbb{N}$. Pick $j_0 \in \mathbb{N}$ such that $u \in U_{x_{j_0}}$. Thus there exists a $k_0 \in \mathbb{N}$ such that $\{u_k : k > k_0\} \subset U_{x_{j_0}}$. Pick $i_0 \geq j_0$ such that $k = j_0 + \frac{i_0(i_0-1)}{2} > k_0$. Then $z_{i_0,j_0} = y_k = f(u_k) \in f(U_{x_{j_0}})$. This is a contradiction. □

Lemma 5.2. [17] *Let $f : X \to Y$ be a mapping, $\{B_n : n \in \mathbb{N}\}$ be a decreasing network of some point x in X and $\{y_i\}$ be a sequence converging to $f(x)$ in Y. If $\{y_i\}$ is eventually in $f(B_n)$ for every $n \in \mathbb{N}$, then there is a sequence $\{x_n\}$ converging to x in X such that $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.*

A topological space X is said to be *snf-countable* [18], if every point of X has a countable sn-network. Obviously, every first-countable space is snf-countable.

Theorem 5.3. *Suppose that $f : X \to Y$ is a sequence-covering boundary-Lindelöf mapping, where X is first-countable. Then Y is snf-countable if and only if f is a 1-sequence-covering mapping.*

PROOF. Necessity. Let $y \in Y$ and $\{F_i : i \in \mathbb{N}\}$ be a decreasing sn-network of y in Y. Without loss of generality, we assume that there exists a non-trivial sequence converging to y in Y. Thus $f^{-1}(y)$ is not open in X and so $\partial f^{-1}(y) \neq \emptyset$. By Lemma 5.1, there exists a point $x \in \partial f^{-1}(y)$ such that if U is a neighbourhood of x in X, then $F_i \subset f(U)$ for some $i \in \mathbb{N}$ and so $f(U)$ is a sequential neighbourhood of y in Y. Let $\{B_n : n \in \mathbb{N}\}$ be a decreasing neighbourhood base of x in X. Then $f(B_n)$ is a sequential neighbourhood of y in Y for every $n \in \mathbb{N}$. By Lemma 5.2, if $\{y_n\}$ is a sequence converging to y in Y, then there exists a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$. So f is 1-sequence-covering.

Sufficiency. Every 1-sequence-covering image on first-countable spaces is snf-countable [26]. □

Remark 5.4. (1) Theorem 5.3 improves [16, Theorems 3.3 and 3.9].

(2) Let (M, d) be a metric space and (Y, τ) be a topological space. A mapping $f : (M, d) \to (Y, \tau)$ is called a *π-mapping* [37], if $y \in U \in \tau$, then $d(f^{-1}(y), M \setminus f^{-1}(U)) > 0$. Every sequentially quotient and π-image of metric spaces is snf-countable [10]. Theorem 5.3 implies that every sequence-covering s-π-mapping on metric spaces is 1-sequence-covering, which was established in [1].

(3) There exists a sequence-covering quotient s-mapping f on a metrizable space such that f is not 1-sequence-covering [17, Example 3.7]. There exists a sequence-covering quotient π-mapping f on a metrizable space such that f is not 1-sequence-covering [19, Example 2].

Theorem 5.5. *Suppose that $f : X \to Y$ is a sequence-covering boundary-compact mapping, where X is first-countable. Then f is a 1-sequence-covering mapping.*

PROOF. By Theorem 5.3, it suffices to show that Y is snf-countable. Let $y \in Y$. Without loss of generality, we assume that there exists a non-trivial sequence converging to y in Y. Thus $\partial f^{-1}(y) \neq \emptyset$. Then there exists a point $x_y \in \partial f^{-1}(y)$ such that if U is a neighbourhood of x_y in X, $f(U)$ is a sequential neighbourhood of y in Y.

Otherwise, for every $x \in \partial f^{-1}(y)$, there exists an open neighbourhood U_x of x in X such that $f(U_x)$ is not a sequential neighbourhood of y in Y. Since $\{U_x : x \in \partial f^{-1}(y)\}$ covers $\partial f^{-1}(y)$, there exists a finite subfamily $\mathcal{U} = \{U_{x_j} : j \leq n\}$ of $\{U_x : x \in \partial f^{-1}(y)\}$ covers $\partial f^{-1}(y)$. For every $j \leq n$, since $f(U_{x_j})$ is not a sequential neighbourhood of y in Y, there exists a sequence $\{z_{i,j}\}_{i\in\mathbb{N}}$ converging to y in Y such that $z_{i,j} \notin f(U_{x_j})$ for every $i \in \mathbb{N}$. Now, we construct a sequence $\{y_k\}$ in Y satisfying $y_k = z_{i,j}$, where $k = j + (i-1)n, 1 \leq j \leq n$ and $i \in \mathbb{N}$. Obviously, the sequence $\{y_k\}$ converges to y in Y. The mapping f being sequence-covering, there exists a sequence $\{u_k\}$ converging to some point $u \in \partial f^{-1}(y)$ in X such that $f(u_k) = y_k$ for every $k \in \mathbb{N}$. Pick $j_0 \leq n$ such that $u \in U_{x_{j_0}}$. Thus there exists a $k_0 \in \mathbb{N}$ such that $\{u_k : k > k_0\} \subset U_{x_{j_0}}$. Pick $i_0 \in \mathbb{N}$ such that $k = j_0 + (i_0 - 1)n > k_0$. Then $z_{i_0,j_0} = y_k = f(u_k) \in f(U_{x_{j_0}})$. This is a contradiction.

Let $\{B_n : n \in \mathbb{N}\}$ be a decreasing neighbourhood base of x in X. Then $\{f(B_n) : n \in \mathbb{N}\}$ is an sn-network of y in Y. Hence, Y is snf-countable. □

Lemma 5.6. *Let $f : X \to Y$ be a sequence-covering mapping, $y \in Y$ and $K = \partial f^{-1}(y) \neq \emptyset$. If y has a countable sn-network in Y and K has a countable outer sn-network in X, then there exist a point $x \in f^{-1}(y)$ and a countable decreasing network $\mathcal{P}_x$ of x in X such that $f(\mathcal{P}_x)$ is an sn-network of y in Y.*

Proof. Let $\mathcal{P} = \bigcup_{x\in K} \mathcal{P}_x$ be a countable outer sn-network of K in X, where $\mathcal{P}_x$ is a decreasing sn-network of x in X for every $x \in K$. Suppose $\{F_i : i \in \mathbb{N}\}$ is a decreasing sn-network of y in Y. Assume the conclusion is not true. Then, for every $x \in \partial f^{-1}(y)$, there exists $P_x \in \mathcal{P}_x$ such that $F_i \not\subset f(P_x)$ for every $i \in \mathbb{N}$. Then $\mathcal{P}' = \{P_x : x \in K\}$ is a countable cover of K. Write $\mathcal{P}' = \{U_j : j \in \mathbb{N}\}$. For every $i, j \in \mathbb{N}$, there exists a point $z_{i,j} \in F_i \setminus f(U_j)$. For every $i, j \in \mathbb{N}$ and $i \geq j$, let $y_k = z_{i,j}$, where $k = j + \frac{i(i-1)}{2}$. Since $\{F_i : i \in \mathbb{N}\}$ is a decreasing network of y in Y, the sequence $\{y_k\}$ converges to y in Y. The mapping f being sequence-covering, there exists a sequence $\{u_k\}$ converging to some point $u \in \partial f^{-1}(y)$ in X such that $f(u_k) = y_k$ for every $k \in \mathbb{N}$. Pick $j_0 \in \mathbb{N}$ such that $P_u = U_{j_0}$. Thus U_{j_0} is a sequential neighbourhood of u in X and there exists a $k_0 \in \mathbb{N}$ such that $\{u_k : k > k_0\} \subset U_{j_0}$. Pick $i_0 \geq j_0$ such that $k = j_0 + \frac{i_0(i_0-1)}{2} > k_0$. Then $z_{i_0,j_0} = y_k = f(u_k) \in f(U_{j_0})$. This is a contradiction. □

Theorem 5.7. *Suppose that $f : X \to Y$ is a sequence-covering boundary-compact mapping, where every compact subset of X is metrizable and has a countable sn-network in X. Then f is a 1-sequence-covering mapping.*

Proof. Let $y \in Y$. Without loss of generality, we assume that there exists a non-trivial sequence converging to y in Y. Thus $\partial f^{-1}(y) \neq \emptyset$. By Lemma 4.4, y has a countable sn-network in Y. By Lemma 4.3, every compact subset of X has a countable outer sn-network. Lemma 5.6 implies that there exist a point $x \in f^{-1}(y)$ and a countable decreasing network $\mathcal{P}_x$ of x in X such that $f(\mathcal{P}_x)$ is an sn-network of y in Y. By Lemma 5.2, if $\{y_n\}$ is a sequence converging to y in Y, then there exists a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$. So f is 1-sequence-covering. □

Definition 5.8. [27] Let $f : X \to Y$ be a mapping.

(1) f is a *1-scc-mapping* if for every compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$, and for every $y \in K$, there exists a point $x \in L$ such that whenever $\{y_n\}$ converges to y in Y, there is a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.

(2) f is an *scc-mapping* if for every compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$, and whenever $\{y_n\}$ converges to some point of K in Y, there is a sequence $\{x_n\}$ converging to some point of L in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.

Obviously, every 1-*scc*-mapping is 1-sequence-covering and compact-covering, and every *scc*-mapping is sequence-covering and compact-covering. Every compact-covering open mapping on first-countable spaces is a 1-*scc*-mapping [27].

Theorem 5.9. *Suppose that $f : X \to Y$ is an scc-mapping, where X is first-countable. Then f is a 1-sequence-covering mapping.*

Proof. Let K be a compact subset of Y. There exists a compact subset L of X such that $f(L) = K$, and whenever $\{t_n\}$ converges to some point of K in Y, there is a sequence $\{s_n\}$ converging to some point of L in X with $s_n \in f^{-1}(t_n)$ for every $n \in \mathbb{N}$. Let $y \in K$. Then there exists a point $x_y \in \partial f^{-1}(y) \cap L$ such that if U is a neighbourhood of x_y in X, $f(U)$ is a sequential neighbourhood of y in Y.

Otherwise, for every $x \in \partial f^{-1}(y) \cap L$, there exists an open neighbourhood U_x of x in X such that $f(U_x)$ is not a sequential neighbourhood of y in Y. Since $\{U_x : x \in \partial f^{-1}(y) \cap L\}$ covers $\partial f^{-1}(y) \cap L$, there exists a finite subfamily $\mathcal{U} = \{U_{x_j} : j \leq n\}$ of $\{U_x : x \in \partial f^{-1}(y) \cap L\}$ covers $\partial f^{-1}(y) \cap L$. For every $j \leq n$, since $f(U_{x_j})$ is not a sequential neighbourhood of y in Y, there exists a sequence $z_{1,j}, z_{2,j}, \cdots$ converging to y in Y such that $z_{i,j} \notin f(U_{x_j})$ for every $i \in \mathbb{N}$. Now,we construct a sequence $\{y_k\}$ in Y satisfying $y_k = z_{i,j}$, where $k = j + (i-1)n, 1 \leq j \leq n$ and $i \in \mathbb{N}$. Obviously, $\{y_k\}$ converges to $y \in K$ in Y. The mapping f being sequence-covering, there exists a sequence $\{u_k\}$ converging to some point $u \in \partial f^{-1}(y) \cap L$ in X such that $f(u_k) = y_k$ for every $k \in \mathbb{N}$. Pick $j_0 \leq n$ such that $u \in U_{x_{j_0}}$. Thus there exists a $k_0 \in \mathbb{N}$ such that $\{u_k : k > k_0\} \subset U_{x_{j_0}}$. Pick $i_0 \in \mathbb{N}$ such that $k = j_0 + (i_0 - 1)n > k_0$. Then $z_{i_0,j_0} = y_k = f(u_k) \in f(U_{x_{j_0}})$. This is a contradiction.

Let $\{U_{y,n} : n \in \mathbb{N}\}$ be a decreasing neighbourhood base of x_y in X. Then $\{f(U_{y,n}) : n \in \mathbb{N}\}$ is an sn-network of y in Y. By Lemma 5.2, if $\{y_n\}$ is a sequence converging to y in Y, then there exists a sequence $\{x_n\}$ converging to x_y in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$. So f is 1-sequence-covering. □

We still do not know whether every *scc*-mapping on compact spaces is a 1-sequence-covering mapping or not [27].

Question 5.10. *Suppose that $f : X \to Y$ is a sequence-covering boundary-compact mapping. Is f a 1-sequence-covering mapping if X satisfies one of the following conditions?*

(1) Every compact subset of X has a countable sn-network in X.

(2) Every compact subset of X has a countable outer sn-network in X.

(3) X has a compact-countable sn-network.

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