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A few generalized metric properties on free paratopological groups ☆

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ABSTRACT

In this paper, some generalized metric properties on free paratopological groups are obtained. At first, a general stability theorem of free paratopological groups on submetrizable spaces is established. Secondly, countable tightness of free paratopological groups on k^*-metrizable k-spaces or k-semistratifiable k-spaces is investigated. Finally, first-countability of free paratopological groups is discussed and strongly Fréchetness of free paratopological groups is characterized. These complement and improve some known conclusions in literature. In addition, some questions are posed.

1. Introduction

A topological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous and the inversion mapping of G on itself is also continuous. A paratopological group is

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a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. The absence of continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups [2].

In 1941, free topological groups in the sense of Markov were introduced [22]. As in free topological groups, in 2002, S. Romaguera, M. Sanchis, and M. Tkachenko [29] introduced free paratopological groups on arbitrary topological spaces and discussed some of their topological properties. Recently, F. Lin, N. Pyrch, et al. have explored some generalized metric properties of free paratopological groups [16,28]. For example, the free (Abelian) paratopological group on a submetrizable space is submetrizable [28]. If the free Abelian paratopological group $AP(X)$ on a metrizable space X has countable tightness, then the subspace X' consisting of all non-isolated points in X is separable [16]. For a regular space X, $AP_2(X)$ is first-countable if and only if $AP_2(X)$ is metrizable if and only if $AP_n(X)$ is first-countable for every $n \in \mathbb{N}$ if and only if X is metrizable with only finitely many non-isolated points [16].

In this paper, we shall complement and improve the above results obtained by F. Lin, N. Pyrch, et al.

Firstly, in Section 3, we establish a general stability theorem of free paratopological groups on submetrizable spaces. Namely, if X is a submetrizable space, then X is a Ξ-space if and only if the free paratopological group $FP(X)$ on X is a Ξ-space (Theorem 3.6). As an application, we prove that if X is a submetrizable space, then X is a σ-space (semi-stratifiable space) if and only if $FP(X)$ is a σ-space (semi-stratifiable space).

Secondly, in Section 4, we investigate countable tightness of free (Abelian) paratopological groups on k^*-metrizable k-spaces or k-semistratifiable k-spaces. It is shown that if the free paratopological group $FP(X)$ or the free Abelian paratopological group $AP(X)$ on a Hausdorff k^*-metrizable k-space X has countable tightness, then X is an $\aleph_0'$-space (Theorem 4.9). If the free paratopological group $FP(X)$ or the free Abelian paratopological group $AP(X)$ on a Hausdorff k-semistratifiable k-space X has countable tightness, then the subspace X' consisting of all non-isolated points in X is ω_1-compact (Theorem 4.18).

Finally, in Section 5, we discuss first-countability of free paratopological groups and characterize strongly Fréchetness of free (Abelian) paratopological groups. We show that for a regular space X, $FP_2(X)$ is first-countable if and only if $FP_2(X)$ is metrizable if and only if X is metrizable with only finitely many non-isolated points (Theorem 5.5). For a completely regular space X, the free paratopological group $FP(X)$ if and only if the free Abelian paratopological group $AP(X)$ on X is strongly Fréchet if and only if the space X is discrete (Theorem 5.8).

2. Preparations about free paratopological groups

Definition 2.1. ([28]) Let X be a subspace of a paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X \rangle = G$; and
(2) every continuous mapping $f : X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group* (briefly, *free paratopological group*) on X and is denoted by $FP(X)$.

If all groups in the above definition are Abelian, we obtain the definition of *Markov free Abelian paratopological group* on X, which is denoted by $AP(X)$.

In the paper, $F_a(X)$ ($A_a(X)$) algebraically denotes the free group (free Abelian group) on a non-empty set X and e (0) is the identity of $F_a(X)$ ($A_a(X)$). The set X is called the free basis of $F_a(X)$ ($A_a(X)$). Here are some details, for instance, see [2].

Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say in this case that the length $l(g)$ of g equals to n. Every element $g \in F_a(X)$ distinct from the identity e can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$ and $x_i \neq x_{i+1}$ for every $i = 1, \ldots, n-1$. Similar assertions (with the obvious changes for commutativity) are valid for $A_a(X)$.

Remark 2.2. It is well known that the topology of $FP(X)$ ($AP(X)$) is the finest paratopological group topology on the group $F_a(X)$ ($A_a(X)$) which induces the original topology on X [28].

For every non-negative integer n, denote by $FP_n(X)$ ($AP_n(X)$) the subspace of the free paratopological group $FP(X)$ ($AP(X)$) that consists of all words of reduced length $\leq n$ with respect to the free basis X. Obviously, $FP_0(X) = \{e\}$. Denote by $\widetilde{X}$ the free topological sum $X \oplus \{e\} \oplus X^{-1}$. In the non-Abelian case, put $C_n(X) = FP_n(X) \setminus FP_{n-1}(X)$ for every $n \geq 1$. For every $n \geq 1$, denote by i_n the natural mapping of $\widetilde{X}^n$ onto $FP_n(X)$, i.e., $i_n(y_1, \ldots, y_n) = y_1 \cdots y_n$ for every point $(y_1, \ldots, y_n) \in \widetilde{X}^n$. It follows that the mapping i_n is continuous. We denote by $C_n^*(X) = i_n^{\leftarrow}(C_n(X))$ the inverse image of $C_n(X)$ under the mapping $i_n : \widetilde{X}^n \to FP_n(X)$. Analogously, in the Abelian case, put $C_n(X) = AP_n(X) \setminus AP_{n-1}(X)$ and $C_n^*(X)$ denotes the inverse image of $C_n(X)$ under the continuous multiplication mapping $i_n : \widetilde{X}^n \to AP_n(X)$.

Remark 2.3. If X is a T_1-space, then $FP(X)$ is also T_1, X^{-1} is closed and discrete, and the subspaces X and $FP_n(X)$ of $FP(X)$ are all closed in $FP(X)$ for every non-negative integer n [6]. The same is true for $AP(X)$ [28].

In what follows, the subspace X of $FP(X)$ and $AP(X)$ is assumed to be T_1 in the paper. All closed mappings are assumed to be continuous and surjective.

3. A general stability theorem of free paratopological groups on submetrizable spaces

A topological space (X, τ) is *submetrizable* if there exists a topology τ' on X such that $\tau' \subset \tau$ and (X, τ') is metrizable [10]. Submetrizability is stable with respect to taking free (Abelian) paratopological groups [28]. We shall establish a general stability theorem of free paratopological groups on submetrizable spaces.

In this paper, *the topological property Ξ is meant to satisfy the following*:

- *Ξ is hereditary with respect to closed subspaces*;
- *Ξ is finitely productive*;
- *Every discrete space has the property Ξ; and*
- *A space which is a countable union of closed subspaces with the property Ξ still has the property Ξ.*

A topological space X is said to a *Ξ-space* if it has the topological property Ξ.

Definition 3.1. Let $\mathcal{P}$ be a family of subsets of a topological space X.

(1) $\mathcal{P}$ is a *network* [1] for X if for every point $x \in X$ and any neighborhood U of x, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$.
(2) $\mathcal{P}$ is *locally finite* if for every $x \in X$, there exists a neighborhood W of x such that W has non-empty intersection with at most finitely many elements of $\mathcal{P}$.
(3) $\mathcal{P}$ is *σ-locally finite* if $\mathcal{P}$ can be expressed as a countable union of locally finite families.

Definition 3.2. Let X be a topological space.

(1) X is called a *σ-space* [25] if it has a σ-locally finite network.
(2) X is called a *semi-stratifiable space* [5] if there exists a function F which assigns to every $n \in \mathbb{N}$ and open set $U \subset X$, a closed set $F(n, U)$ satisfying
   (a) $U = \bigcup_{n\in\mathbb{N}} F(n, U)$; and
   (b) $V \subset U \Rightarrow F(n, V) \subset F(n, U)$.

If also, whenever K is a compact subset of an open set U, there exists an $m \in \mathbb{N}$ such that $K \subset F(m, U)$, then X is k-semistratifiable [21].

Remark 3.3. (1) It is easy to see that the property of being a σ-space satisfies the topological property Ξ.

(2) It was shown in [26] that the property of being a semi-stratifiable space satisfies the topological property Ξ.

(3) Every regular σ-space is a semi-stratifiable space [10, Theorem 5.9]. There exists a submetrizable semi-stratifiable regular space which is not a σ-space [10, Example 9.10].

Now, we show that the class of Ξ-spaces is stable with respect to taking free paratopological groups on submetrizable spaces.

Lemma 3.4. *([29, Theorem 3.2], [14, Theorems 3.2 and 3.3]) Let (X, ϱ) be a metric space, where ϱ is bounded by 1. Let $\widehat{\varrho}$ be the two-sided invariant Graev extension of ϱ to $F_a(X)$. Then the family $\{B_{\widehat{\varrho}}(e, \epsilon) : \epsilon > 0\}$ is a base at the identity e for a metrizable paratopological group topology $\mathcal{T}_\varrho$ on the free group $F_a(X)$ and the restriction of $\mathcal{T}_\varrho$ to X coincides with the topology on X generated by ϱ. For every non-negative integer n, $FP_n(X)$ is closed in $(F_a(X), \mathcal{T}_\varrho)$. The same is true for $AP(X)$.*

A mapping $f : X \to Y$ is perfect [8] if f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

Lemma 3.5. *([16, Propositions 6.3 and 6.4]) Let X be a topological space.*
(1) In the case of $FP(X)$, then the mapping i_n homeomorphically maps $C^*_n(X)$ onto $C_n(X)$.
(2) In the case of $AP(X)$, then the mapping $i_n : C^*_n(X) \to C_n(X)$ is a perfect mapping.

Theorem 3.6. *Let X be a submetrizable space. Then X is a Ξ-space if and only if $FP(X)$ is a Ξ-space.*

Proof. Sufficiency. This is obvious because the class of Ξ-spaces is hereditary with respect to closed subspaces.

Necessity. Suppose that (X, τ) is a submetrizable Ξ-space. Then there exists a topology $\tau' \subset \tau$ such that (X, τ') is metrizable. Let ϱ be a metric on (X, τ') which is compatible with τ' and bounded by 1. Let $id : (X, \tau) \to (X, \tau')$ be the continuous identity mapping. Denote by Y the space (X, τ'). We can extend the mapping id to a continuous homomorphism $\widehat{id} : FP(X) \to FP(Y)$. It follows that $\widehat{id}$ is the identity mapping on the abstract group $F_a(X)$. By Lemma 3.4, ϱ can be extended to a metric $\widehat{\varrho}$ on the abstract group $F_a(X)$ such that $(F_a(X), \widehat{\varrho})$ is a metrizable paratopological group weaker than the topology of $FP(Y)$. Also, for every non-negative integer n, $FP_n(X)$ is closed in $(F_a(X), \widehat{\varrho})$. Since every closed subset of a metrizable space is a G_δ-set, for every non-negative integer n,

$$FP_n(X) = \bigcap_{j\in\mathbb{N}} U_{n,j}$$

with every $U_{n,j}$ open in $(F_a(X), \widehat{\varrho})$. Since $\widehat{id}$ is the continuous identity isomorphism, every $U_{n,j}$ open in $FP(X)$.

Fix $n \in \mathbb{N}$. Let

$$V_{n-1,j} = U_{n-1,j} \cap FP_n(X)$$

for every $j \in \mathbb{N}$. Thus

$$FP_{n-1}(X) = FP_{n-1}(X) \cap FP_n(X) = \bigcap_{j \in \mathbb{N}} (U_{n-1,j} \cap FP_n(X)) = \bigcap_{j \in \mathbb{N}} V_{n-1,j}$$

and

$$C_n(X) = FP_n(X) \setminus FP_{n-1}(X) = \bigcup_{j \in \mathbb{N}} (FP_n(X) \setminus V_{n-1,j}).$$

Obviously, $E_{n,j} = FP_n(X) \setminus V_{n-1,j}$ is closed in $FP(X)$ for every $j \in \mathbb{N}$. Since $i_n^{-1}(E_{n,j})$ is closed in $\widetilde{X}^n$, by Lemma 3.5 (1), $E_{n,j}$ are Ξ-spaces for every $n, j \in \mathbb{N}$. Hence,

$$FP(X) = \{e\} \cup \bigcup_{n \in \mathbb{N}} C_n(X) = \{e\} \cup \bigcup_{n,j \in \mathbb{N}} E_{n,j}$$

is a Ξ-space. □

Corollary 3.7. *Let X be a submetrizable space. Then X is a σ-space if and only if $FP(X)$ is a σ-space.*

Corollary 3.8. *Let X be a submetrizable space. Then X is a semi-stratifiable space if and only if $FP(X)$ is a semi-stratifiable space.*

The following Lemma 3.9 has been probably known.

Lemma 3.9. *Let $f : X \to Y$ be a perfect mapping. If X is a σ-space, then so is Y.*

Proof. Let $\mathcal{A}$ be a σ-locally finite network for the σ-space X. The mapping f being perfect, the family $\{f(A) : A \in \mathcal{A}\}$ is a σ-locally finite network for Y by [8, Lemma 3.10.11]. Hence, Y is a σ-space. □

Lemma 3.10. *([5]) Let $f : X \to Y$ be a closed mapping. If X is a semi-stratifiable space, then so is Y.*

By Lemmas 3.5(2), 3.9, 3.10 and the proof of Theorem 3.6, we obtain following two corollaries.

Corollary 3.11. *Let X be a submetrizable space. Then X is a σ-space if and only if $AP(X)$ is a σ-space.*

Corollary 3.12. *Let X be a submetrizable space. Then X is a semi-stratifiable space if and only if $AP(X)$ is a semi-stratifiable space.*

Since every Hausdorff paracompact σ-space is submetrizable [10], we have the following.

Corollary 3.13. *If X is a Hausdorff paracompact σ-space, then both $FP(X)$ and $AP(X)$ are σ-spaces.*

Question 3.14. *Is the free paratopological group $FP(X)$ or the free Abelian paratopological group $AP(X)$ on a σ-space X a σ-space?*

4. Countable tightness of free paratopological groups on k^*-metrizable k-spaces or k-semistratifiable k-spaces

A new class of generalized metric spaces named *k^*-metrizable spaces* was introduced by means of continuous subproper images of metric spaces in [3]. A mapping $f : X \to Y$ is *subproper* [3] if there exists a subset Z of X such that $f(Z) = Y$ and for any set K with compact closure in Y, the set $Z \cap f^{-1}(K)$ has compact closure in X. Interestingly, it was shown that a Hausdorff space is a k^*-metrizable space if and only if it has a σ-compact-finite cl_1-osed k-network [3]. We shall discuss the countable tightness of free (Abelian) paratopological groups on k^*-metrizable k-spaces or k-semistratifiable k-spaces to complement and improve one of the main results in [16]. Recall a few related notions.

Let X be a topological space. A subset P of X is called a *sequential neighborhood* of $x \in X$ in X if any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x is eventually in P, i.e., $\{x_n : n \geq k_0\} \cup \{x\} \subset P$ for some $k_0 \in \mathbb{N}$. P is called a *sequentially open* subset of X if P is a sequential neighborhood of every point of P in X. The space X is called a *sequential space* [8] if every sequentially open subset of X is open in X. X is called a *k-space* [8] provided that a subset $A \subset X$ is closed in X if and only if $A \cap K$ is closed in K for every compact subset K of X. X is of *countable tightness* if whenever $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subset A$. It is not difficult to check that the property of countable tightness is preserved by quotient mappings and is hereditary. X is *ω_1-compact* if every uncountable subset of X has an accumulation point.

Let $\mathcal{P}$ be a family of subsets of a topological space X. $\mathcal{P}$ is a *k-network* [27] for X if whenever K is a compact subset of an open set U, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset U$. $\mathcal{P}$ is a *cl_1-osed k-network* [3] for X if whenever K is a compact subset of an open set U, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset \mathrm{cl}_1(\cup\mathcal{F}) \subset U$, where $\mathrm{cl}_1(\cup\mathcal{F})$ denotes the set consisting of the limits of convergent sequences of points of $\cup\mathcal{F}$ in X. $\mathcal{P}$ is *compact-finite* if every compact subset of X intersects only finitely many elements of $\mathcal{P}$. $\mathcal{P}$ is *σ-compact-finite* if $\mathcal{P}$ can be expressed as a countable union of compact-finite families.

We still need a series of lemmas.

Lemma 4.1. *Every compact-finite family of an ω_1-compact k-space is countable.*

Proof. Let $\mathcal{P}$ be a compact-finite family of an ω_1-compact k-space X. Suppose $\mathcal{P}$ is not countable. Then there exist a subfamily $\mathcal{F} = \{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$ and a subset $A = \{x_\alpha : \alpha < \omega_1\} \subset X$ satisfying

(1) for every $\alpha < \omega_1$, $x_\alpha \in P_\alpha$; and
(2) for any two distinct $\alpha, \beta < \omega_1$, $x_\alpha \neq x_\beta$ and $P_\alpha \neq P_\beta$.

The space X being ω_1-compact, let x be an accumulation point of A in X, whence the set $A \setminus \{x\}$ is not closed in X. Since X is a k-space, there is a compact subset K of X such that $(A \setminus \{x\}) \cap K$ is not closed in K. Then $(A \setminus \{x\}) \cap K$ is infinite, which contradicts the hypothesis that $\mathcal{P}$ is compact-finite for X. □

Lemma 4.2. *([11]) Every Hausdorff k-space with a point-countable k-network is a sequential space.*

We recall that S_{ω_1} is the quotient space obtained by identifying all limit points of the topological sum of ω_1 convergent sequences homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of the real line $\mathbb{R}$.

Lemma 4.3. *Let X be a Hausdorff k^*-metrizable k-space. If the subspace X' consisting of all non-isolated points in X is not ω_1-compact, then there exists a closed mapping $f : X \to Y$ such that Y contains a closed copy of S_{ω_1}.*

Proof. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-compact-finite cl_1-osed k-network for X. By Lemma 4.2, X is a sequential space. Since the subspace X' consisting of all non-isolated points in X is not ω_1-compact, there exists an uncountable set $D = \{x_\alpha : \alpha \in \Gamma\}$ such that $D \subset X'$ and D has no accumulations in X, where Γ is an index set.

Claim 1. For every $\alpha \in \Gamma$, there exists a non-trivial sequence $\{x_n(\alpha)\}_{n\in\mathbb{N}}$ of points of X converging to x_α in X such that $D \cap \{x_n(\alpha) : n \in \mathbb{N}\} = \emptyset$.

In fact, the space X being a sequential space, for every $\alpha \in \Gamma$, $\{x_\alpha\}$ is not sequentially open in X, whence there exists a non-trivial sequence $\{b_n(\alpha)\}_{n\in\mathbb{N}}$ of points of X converging to x_α in X. Since the set D has no accumulations in X,

$$|D \cap \{b_n(\alpha) : n \in \mathbb{N}\}| < \omega$$

for every $\alpha \in \Gamma$. Thus Claim 1 is proved.

Claim 2. There exists a subset Λ of Γ with $|\Lambda| = \omega_1$ such that for every $\alpha \in \Lambda$, there are a non-trivial sequence $\{z_n(\alpha)\}_{n\in\mathbb{N}}$ of points of X converging to x_α in X and a $P_\alpha \in \mathcal{P}$ satisfying

(a) $\{z_n(\alpha) : n \in \mathbb{N}\} \subset P_\alpha$;
(b) $\{z_n(\alpha) : n \in \mathbb{N}\} \cap \{z_n(\beta) : n \in \mathbb{N}\} = \emptyset$ for any two distinct $\alpha, \beta \in \Lambda$;
(c) $\{P_\alpha : \alpha \in \Lambda\} \subset \mathcal{P}_m$ for some $m \in \mathbb{N}$.

In fact, by Claim 1, since $\mathcal{P}$ is a cl_1-osed k-network for X, for every $\alpha \in \Gamma$, there exists a finite subfamily $\mathcal{F}_\alpha \subset \mathcal{P}$ such that

$$\{x_n(\alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\} \subset \cup\mathcal{F}_\alpha \subset \mathrm{cl}_1(\cup\mathcal{F}_\alpha) \subset X \setminus \{x_\beta : \beta \in \Gamma, \beta \neq \alpha\}.$$

Then for every $\alpha \in \Gamma$, there exist a $P_\alpha \in \mathcal{P}$ and a subsequence $\{x_{n_k}(\alpha)\}_{k\in\mathbb{N}}$ of $\{x_n(\alpha)\}_{n\in\mathbb{N}}$ such that

$$\{x_{n_k}(\alpha) : k \in \mathbb{N}\} \subset P_\alpha \subset \mathrm{cl}_1(P_\alpha) \subset X \setminus \{x_\beta : \beta \in \Gamma, \beta \neq \alpha\}.$$

Thus for any two distinct $\alpha, \beta \in \Gamma$, $x_\beta \notin \mathrm{cl}_1(P_\alpha)$, whence $P_\alpha \neq P_\beta$. Further, there exists some $m \in \mathbb{N}$ such that $\mathcal{P}_m$ contains uncountably many P_α's.

Therefore, there exists a subset Λ of Γ with $|\Lambda| = \omega_1$ such that for every $\alpha \in \Lambda$, there are a non-trivial sequence $\{y_n(\alpha)\}_{n\in\mathbb{N}}$ of points of X converging to x_α in X and a $P_\alpha \in \mathcal{P}$ satisfying $\{y_n(\alpha) : n \in \mathbb{N}\} \subset P_\alpha$ and $\{P_\alpha : \alpha \in \Lambda\} \subset \mathcal{P}_m$ for some $m \in \mathbb{N}$.

Since $\mathcal{P}_m$ is compact-finite in X,

$$\{y_n(\alpha) : n \in \mathbb{N}\} \setminus \bigcup_{\beta\in\Lambda\setminus\{\alpha\}} \{y_n(\beta) : n \in \mathbb{N}\}$$

is infinite and is written by $\{z_n(\alpha) : n \in \mathbb{N}\}$ for every $\alpha \in \Lambda$. Thus Claim 2 is proved.

Now, let Y be the quotient space obtained from X by identifying the subset $E = \{x_\alpha : \alpha \in \Lambda\}$ of X to a point, i.e., $Y = X/E$. Write $Y = \{\infty\} \cup (X \setminus E)$. Let $q : X \to Y$ be the natural quotient mapping. Then the mapping q is closed by [8, Example 2.4.12], whence Y is a T_1-space.

Claim 3. The subspace $C = \{\infty\} \cup \bigcup_{\alpha\in\Lambda}\{z_n(\alpha) : n \in \mathbb{N}\}$ of Y is a closed copy of S_{ω_1}.

Let $Z = \bigcup_{\alpha\in\Lambda}(\{z_n(\alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\})$. Since X is a sequential space and $\mathcal{P}_m$ is compact-finite in X, the subspace Z is closed in X, $q|Z : Z \to q(Z)$ is a closed mapping and

$$Z = \bigoplus_{\alpha\in\Lambda}(\{z_n(\alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\}).$$

Thus the subspace $C = q(Z)$ of Y is a closed copy of S_{ω_1}. This completes the proof of the lemma. □

Lemma 4.4. *([12]) Let X be the product space $S_{\omega_1} \times S_{\omega_1}$. Then the tightness of X is uncountable.*

Lemma 4.5. *([6, Theorem 4.11]) Let X be a topological space and $w = \epsilon_1 x_1 + \epsilon_2 x_2 + \ldots + \epsilon_n x_n$ be a reduced word in $AP_n(X)$, where $x_i \in X$, $\epsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$. Let $\mathcal{B}$ denote the collection of all sets of the form $\epsilon_1 U_1 + \epsilon_2 U_2 + \ldots + \epsilon_n U_n$, where for $i = 1, 2, \ldots, n$, the set U_i is a neighborhood of x_i in X when $\epsilon_i = 1$ and $U_i = \{x_i\}$ when $\epsilon_i = -1$. Then $\mathcal{B}$ is a neighborhood base at the point w in the subspace $AP_n(X)$ of $AP(X)$.*

It was shown in [16,28] that if X is a completely regular space, then $FP(X)$ ($AP(X)$) contains a closed homeomorphic copy of the product space X^n for every $n \in \mathbb{N}$. However, if the separation "completely regular" is weakened to be T_1, we still in $AP(X)$ construct a homeomorphic copy of the product space X^n for every $n \in \mathbb{N}$ by Lemma 4.5.

Lemma 4.6. *Let X be a topological space. Then the following hold.*

(1) $FP(X)$ contains a homeomorphic copy of the product space X^n for every $n \in \mathbb{N}$ [6, Theorem 4.12].

(2) $AP(X)$ contains a homeomorphic copy of the product space X^n for every $n \in \mathbb{N}$.

Proof. Consider the mapping $\Phi : X^n \to AP(X)$ defined by

$$\Phi(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + 2^{n-1} x_n$$

for every $(x_1, x_2, \ldots, x_n) \in X^n$. Put $m = 1 + 2 + \ldots + 2^{n-1}$, whence

$$\Phi(X^n) \subset AP_m(X).$$

From the continuity of the multiplication in $AP(X)$, it follows that Φ is continuous. The set X being a free algebraic basis for $AP(X)$, the mapping Φ is one-to-one. It suffices to show that $\Phi : X^n \to \Phi(X^n)$ is an open mapping. Let U be a non-empty open subset of X^n and $(x_1, x_2, \ldots, x_n) \in U$. Then there exists a neighborhood U_i in X of x_i for every $i \leq n$ such that

$$U_1 + 2U_2 + \ldots + 2^{n-1} U_n = \Phi(U_1 \times U_2 \times \ldots \times U_n) \subset \Phi(U).$$

By Lemma 4.5, $U_1 + 2U_2 + \ldots + 2^{n-1} U_n$ is a neighborhood of the point $\Phi(x_1, x_2, \ldots, x_n)$ in $AP_m(X)$. Thus $\Phi(U)$ is a neighborhood of the point $\Phi(x_1, x_2, \ldots, x_n)$ in $\Phi(X^n)$ and $\Phi(U)$ is open in $\Phi(X^n)$. $\square$

The case for $FP(X)$ in following Lemma 4.7 was proved in [28, Proposition 2.10]. After a careful check, its method also applies to the case for $AP(X)$.

Lemma 4.7. *([28]) Let $f : X \to Y$ be a quotient mapping of topological spaces. Then f admits an extension to continuous open homomorphisms $F(f) : FP(X) \to FP(Y)$ and $A(f) : AP(X) \to AP(Y)$.*

Lemma 4.8. *Let $f : X \to Y$ be a quotient mapping of topological spaces. If $FP(X)$ or $AP(X)$ has countable tightness, then Y contains no closed copy of S_{ω_1}.*

Proof. Suppose Y contains a closed copy of S_{ω_1}. By Lemma 4.7, the mapping f admits an extension to continuous open homomorphisms $F(f) : FP(X) \to FP(Y)$ and $A(f) : AP(X) \to AP(Y)$. Since $FP(X)$ ($AP(X)$) has countable tightness, so does $FP(Y)$ ($AP(Y)$). By Lemma 4.6, $FP(Y)$ ($AP(Y)$) contains a homeomorphic copy of the product space Y^2. Thus $S_{\omega_1} \times S_{\omega_1}$ has countable tightness, which contradicts Lemma 4.4. $\square$

Let X be a topological space. X is called an $\aleph_0$-space [23] if it has a countable k-network. X is called an $\aleph_0'$-space [24] if the subspace X' consisting of all non-isolated points in X is an $\aleph_0$-space.

Theorem 4.9. *Let X be a Hausdorff k^*-metrizable k-space. If $FP(X)$ or $AP(X)$ has countable tightness, then the subspace X' consisting of all non-isolated points in X is ω_1-compact, and hence X is an $\aleph_0'$-space.*

Proof. Suppose X' is not ω_1-compact. By Lemma 4.3, there exists a closed mapping $f : X \to Y$ such that Y contains a closed copy of S_{ω_1}. By Lemma 4.8, Y contains no closed copy of S_{ω_1}. This is a contradiction.

By Lemma 4.1, X is an $\aleph_0'$-space. □

A regular space with a σ-locally finite k-network is called an $\aleph$-*space* [27].

Corollary 4.10. *Let X be a k-and-$\aleph$-space. If $FP(X)$ or $AP(X)$ has countable tightness, then X is an $\aleph_0'$-space.*

A topological space is called a *Lašnev space* if it is a continuous closed image of a metric space. Every Lašnev space is a normal k^*-metrizable k-space [3,19].

Corollary 4.11. *Let X be a Lašnev space. If $FP(X)$ or $AP(X)$ has countable tightness, then X is an $\aleph_0'$-space.*

A family $\mathcal{P}$ of subsets of a topological space X is *star-countable* if for any $P \in \mathcal{P}$, the family $\{B \in \mathcal{P} : B \cap P \neq \emptyset\}$ is countable. Every regular space with a star-countable k-network is k^*-metrizable [20].

Corollary 4.12. *Let X be a regular k-space with a star-countable k-network. If $FP(X)$ or $AP(X)$ has countable tightness, then X is an $\aleph_0'$-space.*

Remark 4.13. As is well known, k-and-$\aleph$-spaces, Lašnev spaces, and regular k-spaces with a star-countable k-network do not imply each other, see e.g. [17,20].

The Abelian case of the following corollary, as one of the main results in [16], was obtained by F. Lin et al.

Corollary 4.14. *Let X be a metrizable space. If $FP(X)$ or $AP(X)$ has countable tightness, then the subspace X' consisting of all non-isolated points in X is ω_1-compact, i.e., X' is separable.*

Around Corollary 4.14, it is natural to pose the following question.

Question 4.15. *Let X be a metrizable space. Does $AP(X)$ have countable tightness if the subspace X' consisting of all non-isolated points in X is ω_1-compact?*

A family $\mathcal{P}$ of subsets of a topological space X is *compact-countable* if every compact subset of X intersects only countably many elements of $\mathcal{P}$.

Question 4.16. *The condition "k^*-metrizable k-space" in Theorem 4.9 can be weakened to the condition "k-space with a compact-countable k-network"?*

Next, we convert to discuss the countable tightness of free (Abelian) paratopological groups on k-semistratifiable k-spaces.

Lemma 4.17. *([18]) Let $f : X \to Y$ be a closed mapping, where X is a Hausdorff k-semistratifiable k-space. Then the set $Frf^{-1}(y)$ is Lindelöf for every $y \in Y$ if Y contains no closed copy of S_{ω_1}.*

Theorem 4.18. *Let X be a Hausdorff k-semistratifiable k-space. If $FP(X)$ or $AP(X)$ has countable tightness, then the subspace X' consisting of all non-isolated points in X is ω_1-compact.*

Proof. Suppose X' is not ω_1-compact. Then there exists an uncountable set D such that $D \subset X'$ and D has no accumulations in X, thus D is a closed discrete subset of X. Further, the space D is not Lindelöf. Put $Y = X/D$. Write $Y = \{\infty\} \cup (X \setminus D)$. Let $q : X \to Y$ be the natural quotient mapping. Then the mapping q is closed by [8, Example 2.4.12].

Claim. $\mathrm{Int}_X q^{-1}(\infty) = \mathrm{Int}_X D = \emptyset$.

Assume $\mathrm{Int}_X D \neq \emptyset$. Pick a point x and an open subset V in X such that $x \in V \subset D$. Then $V \setminus \{x\} \subset D$ is closed in X, and so $\{x\} = V \setminus (V \setminus \{x\})$ is open in X. On the other hand, $\{x\} \subset X'$ is not open in X. This is a contradiction.

Thus $\mathrm{Fr} q^{-1}(\infty) = q^{-1}(\infty) \setminus \mathrm{Int}_X q^{-1}(\infty) = D$ is not Lindelöf. By Lemma 4.17, Y contains a closed copy of S_{ω_1}, which contradicts Lemma 4.8. This completes the proof of the theorem. □

Remark 4.19. (1) There exists a first-countable k-semistratifiable regular space which is not k^*-metrizable [4, Example 9.2].

(2) There exists a first-countable Hausdorff space with a star-countable k-network (hence, with a σ-compact-finite k-network), which is not k-semistratifiable [17, Example 1.5.8].

5. First-countability and strongly Fréchetness of free paratopological groups

It is well known that first-countable compact Hausdorff spaces are not necessarily metrizable. However, we shall show that $FP_2(X)$ is first-countable if and only if $FP_2(X)$ is metrizable for a regular space X, which need a few auxiliary lemmas.

A *quasi-uniformity* $\mathcal{U}$ [9] on a set X is a filter on $X \times X$ such that

(1) every member U of $\mathcal{U}$ contains the diagonal $\triangle_X = \{(x,x) : x \in X\}$ of X;
(2) for every $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V = \{(x,z) \in X \times X : \text{there is } y \in X \text{ such that } (x,y) \in V \text{ and } (y,z) \in V\} \subset U$.

A subfamily $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ on a set X is a *base* for $\mathcal{U}$ if every element of $\mathcal{U}$ contains an element of $\mathcal{B}$. Given a topological space X, a quasi-uniformity $\mathcal{U}$ on the set X is *compatible* with the space X if $\{U(x) : U \in \mathcal{U}\}$ is a neighborhood base at x for every $x \in X$, where $U(x) = \{y \in X : (x,y) \in U\}$. The *fine quasi-uniformity* of a topological space X is defined as the supremum of all quasi-uniformities compatible with the space X.

Let X be a topological space and $\mathcal{FN}(X)$ be the fine quasi-uniformity of X. The cardinal function $fq(X)$ is defined by

$$fq(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } \mathcal{FN}(X)\}.$$

Lemma 5.1. *([7]) Let X be a topological space and $\mathcal{FN}(X)$ be the fine quasi-uniformity of X. Then*

$$\mathcal{B} = \{j_2(U) \cup k_2(U) : U \in \mathcal{FN}(X)\}$$

is a neighborhood base at the identity e in $FP_2(X)$, where $j_2 : X \times X \to F_a(X)$ and $k_2 : X \times X \to F_a(X)$ are defined by $j_2(x,y) = x^{-1}y$ and $k_2(x,y) = yx^{-1}$ for every $(x,y) \in X \times X$ respectively.

Recall that the *character* $\chi(X)$ of a topological space X [8] is defined by

$$\chi(X) = \sup_{x\in X}\chi(x, X),$$

where

$$\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a neighborhood base at } x \in X\}$$

for every $x \in X$.

Lemma 5.2. *Let X be a topological space and κ be an infinite cardinal number. Then the following conditions are equivalent.*

(1) $\chi(FP_2(X)) \leq \kappa$;
(2) $fq(X) \leq \kappa$.

Proof. (1) ⇒ (2). By Lemma 5.1, $\mathcal{B} = \{j_2(U)\cup k_2(U) : U \in \mathcal{FN}(X)\}$ is a neighborhood base at the identity e in $FP_2(X)$. Since $\chi(FP_2(X)) \leq \kappa$, there exists an $\mathcal{S} \subset \mathcal{FN}(X)$ such that $|\mathcal{S}| \leq \kappa$ and $\{j_2(U)\cup k_2(U) : U \in \mathcal{S}\}$ is a neighborhood base at the identity e in $FP_2(X)$. We shall show that $\mathcal{S}$ is a base for $\mathcal{FN}(X)$. In fact, for every $V \in \mathcal{FN}(X)$, there is a $U \in \mathcal{S}$ such that $j_2(U) \cup k_2(U) \subset j_2(V) \cup k_2(V)$ by Lemma 5.1. Thus $U \subset V$ and $\mathcal{S}$ is a base for $\mathcal{FN}(X)$. Hence, $fq(X) \leq \kappa$.

(2) ⇒ (1). Suppose $fq(X) \leq \kappa$. By Lemma 5.1, $\chi(e, FP_2(X)) \leq \kappa$. Since $C_2(X) = FP_2(X) \setminus FP_1(X)$ is homeomorphic to a subspace of $\widetilde{X}^2$ by Lemma 3.5(1) and $C_2(X)$ is open in $FP_2(X)$,

$$\chi(g, FP_2(X)) = \chi(g, C_2(X)) \leq \chi(\widetilde{X}^2) = \chi(\widetilde{X}) \leq fq(X) \leq \kappa$$

for every $g \in C_2(X)$.

Let $\mathbb{Z}$ be the usual discrete addition group consisting of all integers and $f : X \to \mathbb{Z}$ be defined by $f(x) = 1$ for every $x \in X$. Extend f to a continuous homomorphism $\widehat{f} : FP(X) \to \mathbb{Z}$. Hence $\widehat{f}|_{FP_2(X)} : FP_2(X) \to \mathbb{Z}$ is continuous. Put $\varphi = \widehat{f}|_{FP_2(X)}$. Then $\varphi^{-1}(\{1\}) = X$ and $\varphi^{-1}(\{-1\}) = X^{-1}$, whence $X \cup X^{-1}$ is open in $FP_2(X)$.

Therefore,

$$\chi(h, FP_2(X)) = \chi(h, X \cup X^{-1}) \leq \chi(\widetilde{X}) \leq fq(X) \leq \kappa$$

for every $h \in X \cup X^{-1}$.

In a word, $\chi(FP_2(X)) \leq \kappa$. □

Lemma 5.3. *([9]) Let X be a regular space. The fine quasi-uniformity for X has a countable base if and only if X is a metrizable space with only finitely many non-isolated points.*

A topological space X is called a *z-space* [13] if every neighbornet of X is normal. Every topological space with only finitely many non-isolated points is a z-space [9, Proposition 6.25].

Lemma 5.4. *([15]) A topological space X is a z-space if and only if $i_2 : \widetilde{X}^2 \to FP_2(X)$ is a closed mapping.*

Theorem 5.5. *For a regular space X, the following are equivalent.*

(1) *$FP_2(X)$ is first-countable;*
(2) *$FP_2(X)$ is metrizable;*
(3) *X is metrizable and the set of all non-isolated points in X is finite;*

(4) $AP_2(X)$ *is first-countable;*
(5) $AP_2(X)$ *is metrizable;*
(6) $AP_n(X)$ *is first-countable for every* $n \in \mathbb{N}$.

Proof. The equivalence of items (3), (4), (5), and (6) of the theorem was established in [16, Theorem 5.28]. Lemmas 5.2 and 5.3 imply (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). Again by Lemmas 5.2 and 5.3, $FP_2(X)$ is first-countable. Since X is a z-space, the mapping $i_2 : \widetilde{X}^2 \to FP_2(X)$ is a closed mapping by Lemma 5.4. According to the classic Hanai–Morita–Stone Theorem [8, Theorem 4.4.17], $FP_2(X)$ is metrizable.

It is obvious that (2) $\Rightarrow$ (1). □

Question 5.6. *Let X be a regular space and $n \geq 3$. Is the space $FP_n(X)$ or $AP_n(X)$ is metrizable if it is first-countable?*

A topological space X is called a *Fréchet* space [8] if for every $A \subset X$ and every $x \in \overline{A}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of A converging x. A topological space X is said to be *strongly Fréchet* [30] if whenever $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, then there is a point $x_n \in A_n$ for every $n \in \mathbb{N}$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to the point x. Every first-countable space is strongly Fréchet and every strongly Fréchet space is Fréchet.

Lemma 5.7. *([14, Theorem 3.11]) Let X be a completely regular space and A be an arbitrary subset of $FP(X)$. If $A \cap FP_n(X)$ is finite for every non-negative integer n, then A is closed discrete in $FP(X)$. The same is true for $AP(X)$.*

Theorem 5.8. *Let X be a completely regular space. Then $FP(X)$ is strongly Fréchet if and only if the space X is discrete. The same is true for $AP(X)$.*

Proof. Sufficiency. It is evident from the definition of free paratopological groups that if X is discrete, then $FP(X)$ is also discrete and hence strongly Fréchet.

Necessity. Suppose X is not discrete. Then $FP(X)$ is also not discrete.

Claim. *For every $k \in \mathbb{N}$, the identity $e \in \overline{\bigcup_{n\geq k} C_n(X)}$.*

Otherwise, there exist an open neighborhood U in $FP(X)$ of e and an $m \in \mathbb{N}$ such that $U \subset FP_m(X)$. $FP(X)$ being a paratopological group, pick an open neighborhood V of e in $FP(X)$ such that $V^{m+1} \subset U$. Since $FP(X)$ is not discrete, there exists a point $w \in V \setminus \{e\}$, whence $w^{m+1} \in V^{m+1} \subset U \subset FP_m(X)$. This contradicts the fact $FP_m(X)$ consist of all words of reduced length $\leq m$ with respect to the free basis X. Hence, the Claim is proved.

Now, because $\{\bigcup_{n\geq k} C_n(X)\}_{k\in\mathbb{N}}$ is a decreasing sequence and

$$e \in \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} C_n(X)},$$

from the hypothesis that $FP(X)$ is strongly Fréchet, there exists a point $x_k \in \bigcup_{n\geq k} C_n(X)$ for every $k \in \mathbb{N}$ such that the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to e. Then the set $A = \{x_k : k \in \mathbb{N}\}$ is not closed in $FP(X)$. On the other hand, $A \cap FP_k(X)$ is finite for every $k \in \mathbb{N}$, thus A is closed in $FP(X)$ by Lemma 5.7. This is a contradiction.

The argument in the case of $AP(X)$ is exactly the same. □

Question 5.9. *Let X be a completely regular space. Is the space X is discrete if $FP(X)$ or $AP(X)$ is Fréchet?*

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