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A Note on Partial b-Metric Spaces

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Abstract. Let (X, b) be a partial b-metric space with coefficient $s \geq 1$. For each $x \in X$ and each $\varepsilon > 0$, put $B(x, \varepsilon) = \{y \in X : b(x, y) < b(x, x) + \varepsilon\}$ and put $\mathscr{B} = \{B(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. In this brief note, we prove that $\mathscr{B}$ is not a base for any topology on X, which shows that a claim on partial b-metric spaces is not true. However, $\mathscr{B}$ can be a subbase for some topology τ on X. For a sequence in X, we also give some relations between convergence with respect to τ and convergence with respect to b.

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1. Introduction

Recently, partial b-metric spaces were introduced and discussed in [1].

Definition 1.1 [1]. Let X be a non-empty set. A mapping $b : X \times X \longrightarrow [0, +\infty)$ is called a partial b-metric with coefficient $s \geq 1$ and (X, b) is called a partial b-metric space with coefficient $s \geq 1$ if the following are satisfied for all $x, y, z \in X$.

(1) $x = y \Longleftrightarrow b(x, x) = b(y, y) = b(x, y)$.
(2) $b(x, y) = b(y, x)$.
(3) $b(x, x) \leq b(x, y)$.
(4) $b(x, z) \leq s(b(x, y) + b(y, z)) - b(y, y)$.

And the following claim was given in [1] without proof.

Claim 1.2 [1]. *Every partial b-metric "b" on a nonempty set X generates a topology τ_b on X whose base is the family of open b-balls $B_b(x, \varepsilon)$ where $\tau_b = \{B_b(x, \varepsilon) : x \in X, \varepsilon > 0\}$ and $B_b(x, \varepsilon) = \{y \in X : b(x, y) < b(x, x) + \varepsilon\}$.*

In this brief note, we give an example to show that the above Claim 1.2 is not true. More precisely, let (X, b) be a partial b-metric space with coefficient $s \geq 1$. For each $x \in X$ and each $\varepsilon > 0$, put $B(x, \varepsilon) = \{y \in X : b(x, y) < b(x, x) + \varepsilon\}$ and put $\mathscr{B} = \{B(x, \varepsilon) : x \in X$ *and* $\varepsilon > 0\}$. We prove that $\mathscr{B}$ is

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not a base for any topology on X, hence is not a topology on X. However, $\mathscr{B}$ can be a subbase for some topology τ on X. For a sequence in X, we also give some relations between convergence with respect to τ and convergence with respect to b.

2. The Main Results

Example 2.1. Let $X = \{x, y, z\}$ and put $b\colon X \times X \longrightarrow [0, +\infty)$ as follows.

(1) $b(x, x) = b(z, z) = 1$ and $b(y, y) = 0.5$.
(2) $b(x, z) = b(z, x) = 1.5$.
(3) $b(y, z) = b(z, y) = 1$.
(4) $b(x, y) = b(y, x) = 3$.

It is not difficult to check that (X, b) is a partial b-metric space with coefficient $s = 3$. For each $u \in X$ and each $\varepsilon > 0$, put $B(u, \varepsilon) = \{v \in X\colon b(u, v) < b(u, u) + \varepsilon\}$ and put $\mathscr{B} = \{B(u, \varepsilon)\colon u \in X \textit{ and } \varepsilon > 0\}$. We show that $\mathscr{B}$ is not a base for any topology on X as follows.

(1) Since $b(x, z) = 1.5 < 1 + 1 = b(x, x) + 1$, $z \in B(x, 1)$.
(2) For any $\varepsilon > 0$, $B(z, \varepsilon) \nsubseteq B(x, 1)$. In fact, since $b(y, z) = 1 < 1 + \varepsilon = b(z, z) + \varepsilon$, $y \in B(z, \varepsilon)$. On the other hand, $b(x, y) = 3 \nless 2 = 1 + 1 = b(x, x) + 1$, so $y \notin B(x, 1)$.

By the above (1) and (2), $\mathscr{B}$ is not a base for any topology on X, hence $\mathscr{B}$ is not a topology on X.

Remark 2.2. Example 2.1 shows that Claim 1.2 is not true.

Proposition 2.3. *Let (X, b) be a partial b-metric space with coefficient $s \geq 1$. For each $x \in X$ and each $\varepsilon > 0$, put $B(x, \varepsilon) = \{y \in X\colon b(x, y) < b(x, x) + \varepsilon\}$ and put $\mathscr{B} = \{B(x, \varepsilon)\colon x \in X$ and $\varepsilon > 0\}$. Then $\mathscr{B}$ is a subbase for some topology τ on X.*

Proof. Pick $\varepsilon > 0$. Then $b(x, x) < b(x, x) + \varepsilon$ for all $x \in X$. It follows that $X = \bigcup \mathscr{B}$. So $\mathscr{B}$ is a subbase for some topology τ on X. □

Let (X, b) be a partial b-metric space. In this paper, τ denotes the topology on X, $\mathscr{B}$ denotes a subbase for the topology τ and $B(x, \varepsilon)$ denotes the b-ball in (X, b), which are described in Proposition 2.3. In addition, $\mathscr{U}$ denotes the base generated by the subbase $\mathscr{B}$ and $\mathbb{N}$ denote the set of all natural numbers.

Shukla [1] claimed that (X, τ_b) is T_0, but need not be T_1. However, it is necessary to re-examine the separations of (X, τ) by Remark 2.2.

Proposition 2.4. *Let (X, b) be a partial b-metric space. Then (X, τ) is a T_0-space.*

Proof. Let $x, y \in X$ and $x \neq y$. By Definition 1.1(3), $b(x, y) - b(x, x) \geq 0$ and $b(x, y) - b(y, y) \geq 0$. Further, we have $b(x, y) - b(x, x) \neq 0$ or $b(x, y) - b(y, y) \neq 0$ from Definition 1.1(1). So $b(x, y) - b(x, x) > 0$ or $b(x, y) - b(y, y) > 0$. Without loss of generality, we assume that $b(x, y) - b(x, x) > 0$. There is $\varepsilon > 0$ such that $b(x, y) - b(x, x) > \varepsilon$, i.e., $b(x, y) > b(x, x) + \varepsilon$. So $y \notin B(x, \varepsilon) \in \mathscr{B} \subseteq \tau$. This proves that (X, τ) is a T_0-space. □

Remark 2.5. It is well-known that a partial metric space need not to be a T_1-space. So a partial b-metric space (X, b) need not to be T_1.

Let (X, b) be a partial b-metric space. For a sequence in X, we discuss the relations between convergence with respect to τ and convergence with respect to b.

Definition 2.6. Let (X, b) be a partial b-metric space with coefficient $s \geq 1$. A sequence $\{x_n\}$ in X is called to converge to $x \in X$ with respect to b if for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $b(x, x_n) < b(x, x) + \varepsilon$ for all $n > n_0$.

Proposition 2.7. *Let (X, b) be a partial b-metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$ with respect to τ, then $\{x_n\}$ converges to $x \in X$ with respect to b.*

Proof. Let $\{x_n\}$ converge to $x \in X$ with respect to τ. For any $\varepsilon > 0$, since $x \in B(x, \varepsilon) \in \tau$, there is $n_0 \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ for all $n > n_0$. It follows that $b(x, x_n) < b(x, x) + \varepsilon$ for all $n > n_0$. So $\{x_n\}$ converges to $x \in X$ with respect to b. □

The above Proposition 2.7 can not be reversed.

Example 2.8. Let (X, b) be the partial b-metric space described in Example 2.1. For each $n \in \mathbb{N}$, put $u_n = y$, then $\{u_n\}$ is a sequence in X.

Claim 1: $\{u_n\}$ converges to $z \in X$ with respect to b.

In fact, For any $\varepsilon > 0$, $b(z, y) = 1 < 1 + \varepsilon = b(z, z) + \varepsilon$, i.e., $b(z, u_n) < b(z, z) + \varepsilon$ for all $n \in \mathbb{N}$. So $\{u_n\}$ converges to $z \in X$ with respect to b.

Claim 2: $\{u_n\}$ does not converge to $z \in X$ with respect to τ.

Since $b(z, x) = 1.5 > 1+0.2 = b(z, z)+0.2$, $x \notin B(z, 0.2)$. Also, $b(z, y) = 1 < 1 + 0.2 = b(z, z) + 0.2$, hence $y \in B(z, 0.2)$. Note that $z \in B(z, 0.2)$. So $B(z, 0.2) = \{y, z\} \in \mathscr{B} \subseteq \tau$. On the other hand, since $b(x, y) = 3 = 1 + 2 = b(x, x) + 2$, $y \notin B(x, 2)$. Also, $b(x, z) = 1.5 < 3 = b(x, x) + 2$, hence $z \in B(x, 2)$. Note that $x \in B(x, 2)$. So $B(x, 2) = \{x, z\} \in \mathscr{B} \subseteq \tau$. It follows that $\{z\} = B(z, 0.2) \bigcap B(x, 2) \in \tau$. However, $\{u_n\}$ is not eventually in $\{z\}$. So $\{u_n\}$ does not converge to $z \in X$ with respect to τ.

In the end, we raise the following question.

Question 2.9. *Let (X, b) be a partial b-metric space with coefficient $s > 1$. Is there a base $\mathscr{F}$ for some topology $\mathscr{T}$ on X satisfying the following* (1) *and* (2)*?*

(1) *$\mathscr{F}$ consists of some "b-balls type" sets.*
(2) *Topology $\mathscr{T}$ coincides with topology τ.*

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