Topology and its Applications 183 (2015) 90–109

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Free Abelian paratopological groups over metric spaces ☆

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ARTICLE INFO

Article history:
Received 11 November 2013
Received in revised form 8 January 2015
Accepted 9 January 2015
Available online xxxx

MSC:
primary 22A05, 46A03, 4H115
secondary 54E15, 54E20, 54E35, 54C05, 46A22

Keywords:
Asymmetric locally convex spaces
Free paratopological groups
Free Abelian paratopological groups
Metrizable
k-Spaces
Quotient mappings
First-countable

ABSTRACT

Let X be a metrizable space. Let $FP(Y)$ and $AP(X)$ be the free paratopological group over X and the free Abelian paratopological group over X, respectively. Firstly, we use asymmetric locally convex spaces to prove that if Y is a subspace of X then $AP(Y)$ is topological subgroup of $AP(X)$. Then, we mainly prove that:

(a) if the tightness of $AP(X)$ is countable then the set of all non-isolated points in X is separable;
(b) if X is a z-space, then $AP(X)$ is a k-space if and only if X is locally compact, locally countable and the set of all non-isolated points in X is countable;
(c) $AP_2(X)$ is first-countable if and only if the set of all non-isolated points in X is finite.

Moreover, we show that, for a Tychonoff space X, $AP(X)$ has a countable k-network if and only if X is a countable space with a countable k-network. Finally, we give negative answers to three questions which were posed by Arhangel'skiĭ and Tkachenko in [3]. Some questions concerned with free paratopological groups are posed.

1. Introduction

In 1941, free topological groups were introduced by A.A. Markov in [17] with the clear idea of extending the well-known construction of a free group from group theory to topological groups. Now, free topological

☆ The first author is supported by the National Natural Science Foundation of China (Nos. 11201414, 11471153), the Natural Science Foundation of Fujian Province (No. 2012J05013) of China and Training Programme Foundation for Excellent Youth Researching Talents of Fujian's Universities (JA13190); the third author is supported by the National Natural Science Foundation of China (Nos. 11201414, 11171162, 10971185, 11471153).

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http://dx.doi.org/10.1016/j.topol.2015.01.002

groups have become a powerful tool of study in the theory of topological groups and serve as a source of various examples and as an instrument for proving new theorems, see [3].

In 2002, S. Romaguera, M. Sanchis and M.G. Tkachenko in [20] defined free paratopological groups. Recently, A.S. Elfard, F. Lin, P. Nickolas, N.M. Pyrch and A.V. Ravsky have investigated some properties of free paratopological groups, see [6,7,14,15,18,19]. The topological properties of free topological groups over metric spaces were discussed in [2,21,22]. However, the topological properties of free paratopological groups over metric spaces are still unknown.

A relation V on a topological space X is a *neighbornet* of X provided $V(x) = \{y : (x, y) \in V\}$ is a neighborhood of x for each $x \in X$. A sequence $\{V_n : n \in \omega\}$ of neighbornets of a space X is called a *normal sequence* provided $V_{n+l}^2 \subset V_n$ for every $l, n \in \mathbb{N}$. A neighbornet V of X is *normal* if V is a member of a normal sequence of neighbornets of X. A topological space X such that each neighbornet of X is normal, is called a z-space [13].

In this paper we mainly consider the free Abelian paratopological groups over metric spaces. The content is organized as follows:

In Section 3, we prove that if Y is a subspace of a metrizable space X then $AP(Y)$ is topological subgroup of $AP(X)$. In Section 4, we prove that if the tightness of $AP(X)$ over a metrizable space X is countable then the set of all non-isolated points in X is separable. In Section 5, we mainly prove that: (1) If X is a metrizable z-space, then $AP(X)$ is a k-space if and only if X is locally compact, locally countable and the set of all non-isolated points in X is countable; (2) $AP_2(X)$ is first-countable if and only if the set of all non-isolated points in X is finite and X is metrizable. In Section 6, we prove that, for a Tychonoff space X, $AP(X)$ has a countable k-network if and only if X is a countable space with a countable k-network.

2. Preliminaries

All spaces are T_1 unless stated otherwise. The letter e denotes the neutral element of a group. For a space X, we always denote $I(X)$ and $NI(X)$ the set of all isolated points of X and the set of all non-isolated points of X, respectively. Readers may consult [3,9–11] for notations and terminology not explicitly given here.

A *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is continuous.

Definition 2.1. ([20]) Let X be a subspace of a paratopological group G. Assume that

(1) the set X generates G algebraically, that is $\langle X \rangle = G$;
(2) each continuous mapping $f : X \to H$ to a paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group on* X and is denoted by $FP(X)$.

Again, if all the groups in the above definitions are Abelian, then we get the definition of the *Markov free Abelian paratopological group on* X which is denoted by $AP(X)$.

Throughout this paper, we use $PG(X)$ to denote the paratopological group $FP(X)$ or $AP(X)$.

Since X generates the free group $FP_a(X)$, each element $g \in FP_a(X)$ has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \cdots, x_n \in X$ and $\varepsilon_1, \cdots, \varepsilon_n = \pm 1$. This word for g is called *reduced* if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$. It follows that if the word g is reduced and non-empty, then it is different from the neutral element of $FP_a(X)$. In particular, each element $g \in FP_a(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$, where $n \geq 1$, $\varepsilon_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \cdots, n - 1$. Similar assertions are valid for $AP_a(X)$. For every non-negative integer n, denote by $FP_n(X)$ and $AP_n(X)$ the subspace of paratopological group $FP(X)$ and $AP(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X, respectively. We also use $B_n(X)$ to denote the set $FP_n(X)$ or $AP_n(X)$ for every non-negative integer n.

Let X be a space. For each $n \in \mathbb{N}$, denote by i_n the multiplication mapping from $(X \oplus X_d^{-1} \oplus \{e\})^n$ to $B_n(X)$, $i_n(y_1, \cdots, y_n) = y_1 \cdot \cdots \cdot y_n$ for every point $(y_1, \cdots, y_n) \in (X \oplus X_d^{-1} \oplus \{e\})^n$, where X_d^{-1} is the set X^{-1} equipped with discrete topology.

By a *quasi-uniform space* $(X, \mathscr{U})$ we mean the natural analog of a *uniform space* obtained by dropping the symmetry axiom. For each quasi-uniformity $\mathscr{U}$ the filter $\mathscr{U}^{-1}$ consisting of the inverse relations $U^{-1} = \{(y, x) : (x, y) \in U\}$ where $U \in \mathscr{U}$ is called the *conjugate quasi-uniformity* of $\mathscr{U}$.

We also recall that the *universal quasi-uniformity* $\mathscr{U}_X$ of a space X is the finest quasi-uniformity on X that induces on X its original topology. Denote by $\mathscr{U}^\star$ the upper quasi-uniformity on $\mathbb{R}$ the standard base of which consists of the sets

$$U_r = \big\{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x + r\big\},$$

where r is an arbitrary positive real number.

Let X be a topological space. Then X_d denotes X when equipped with the discrete topology in place of its given topology. In [18], the authors proved that X^{-1} is discrete in free paratopological groups $FP(X)$ and $AP(X)$ if X is a T_1-space.

3. Topological monomorphisms between free Abelian paratopological groups

Let X be a real vector space. An *asymmetric seminorm* on X is a positive sublinear function $p : X \longrightarrow [0, \infty)$, that is, for all $x, y \in X$, p satisfies the following conditions:

(AN1) $p(x) \geq 0$;
(AN2) $p(tx) = tp(x)$, $t \geq 0$;
(AN3) $p(x + y) \leq p(x) + p(y)$.

The pair (X, p), where X is a linear space and p is an asymmetric seminorm on X, is called a *space with asymmetric seminorm*.

An asymmetric seminorm p on X generates a topology τ_p on X, having as basis of neighborhoods of a point $x \in X$ the family $\{B_p(x, r) : r > 0\}$ of open p-balls, where each $B_p(x, r) = \{y \in X : p(y - x) < r\}$.

Let now $\mathcal{P}$ be a family of asymmetric seminorms on a real vector space X. Denote by $\mathscr{F}(\mathcal{P})$ the family of all nonempty finite subsets of $\mathcal{P}$, and for $F \in \mathscr{F}(\mathcal{P})$, $x \in X$, and $r > 0$, let

$$B_F(x, r) = \big\{y \in X : p(y - x) < r, p \in F\big\} = \bigcap \big\{B_p(x, r) : p \in F\big\}$$

denote the open multiball of center x and radius r. It is immediate that these multiballs are convex absorbing subsets of X. Letting

$$p_F(x) = \max\big\{p(x) : p \in F\big\}, \quad x \in X,$$

then p_F is an asymmetric seminorm on X and

$$B_F(x, r) = B_{p_F}(x, r).$$

Definition 3.1. ([5]) The *asymmetric locally convex* topology associated to the family $\mathcal{P}$ of asymmetric seminorms on a real vector space X is the topology $\tau_{\mathcal{P}}$ having as basis of neighborhoods of any point $x \in X$ the family $\{B_F(x, r) : F \in \mathscr{F}(\mathcal{P}), r > 0\}$ of convex absorbing open multiballs.

Clearly, each locally convex space is an asymmetric locally convex space. Moreover, it is easy to see that the addition $+ : X \times X \to X$ is continuous [5].

Proposition 3.2. *Let p be an asymmetric seminorm on a real vector space X and τ_p the topology generated by p:*

(a) *for any fixed $x_0 \in X$ the multiplication by scalars $\cdot : \mathbb{R} \to X, \alpha \mapsto \alpha x_0$, is continuous from $\mathbb{R}$ to (X, τ_p), where $\mathbb{R}$ endows with Euclidean topology;*
(b) *the multiplication by scalars is continuous (as a function of two variables) from $\mathbb{R}_+ \times (X, \tau_p)$ to (X, τ_p), where $\mathbb{R}_+ = [0, +\infty)$ as a subspace of Euclidean space $\mathbb{R}$.*

Proof. (a) Let $\alpha_0 \in \mathbb{R}$. For $\varepsilon > 0$ let

$$V_\varepsilon = \{x' \in X : p(x' - \alpha_0 x_0) < \varepsilon\}$$

be a neighborhood of $\alpha_0 x_0$ in X. Let

$$\delta = \frac{\varepsilon}{1 + p(x_0) + p(-x_0)}$$

and

$$U_\delta = \{\alpha \in \mathbb{R} : |\alpha - \alpha_0| < \delta\}.$$

Obviously, U_δ is a neighborhood of α_0 in $\mathbb{R}$. Then

$$0 < \alpha - \alpha_0 < \delta \quad \Rightarrow \quad p(\alpha x_0 - \alpha_0 x_0) = (\alpha - \alpha_0)p(x_0) = |\alpha - \alpha_0|p(x_0),$$

and

$$0 < \alpha_0 - \alpha < \delta \quad \Rightarrow \quad p(\alpha x_0 - \alpha_0 x_0) = p\big((\alpha_0 - \alpha)(-x_0)\big) = (\alpha_0 - \alpha)p(-x_0) = |\alpha_0 - \alpha|p(-x_0),$$

which imply

$$p(\alpha x_0 - \alpha_0 x_0) \le |\alpha_0 - \alpha|\big[p(x_0) + p(-x_0)\big] < \delta\big[p(x_0) + p(-x_0)\big] < \varepsilon.$$

Therefore, we show the continuity of the multiplication at α_0.

(b) Let $\alpha_0 \in \mathbb{R}_+$ and $x_0 \in X$. For any $\varepsilon > 0$, $V_\varepsilon = \{x \in X : p(x - x_0) < \varepsilon\}$ is a neighborhood of $\alpha_0 x_0$.

Case 1: $\alpha_0 > 0$.

There exists a $\frac{\alpha_0}{2} > r_0 > 0$ such that

$$(\alpha_0 + r_0)r_0 + r_0\big[p(x_0) + p(-x_0)\big] < \varepsilon$$

since $\lim_{r \to 0}(\alpha_0 + r)r + r[p(x_0) + p(-x_0)] = 0$. Let $U = \{\alpha \in \mathbb{R}_+ : |\alpha - \alpha_0| < r_0\}$ and $V_{r_0} = \{x \in X : p(x - x_0) < r_0\}$. Then U and V_{r_0} are open neighborhoods of α_0 and x_0 in $\mathbb{R}_+$ and (X, τ_p), respectively. We claim that $UV_{r_0} \subset V_\varepsilon$. Indeed, for each $\alpha \in U$ and $x \in V_{r_0}$, one obtains

$$\begin{aligned} p(\alpha x - \alpha_0 x_0) &\le p(\alpha x - \alpha x_0) + p(\alpha x_0 - \alpha_0 x_0) \\ &\le \alpha p(x - x_0) + |\alpha - \alpha_0|\big[p(x_0) + p(-x_0)\big] \\ &< (\alpha_0 + r_0)r_0 + r_0\big[p(x_0) + p(-x_0)\big] < \varepsilon. \end{aligned}$$

Then we have $UV_{r_0} \subset V_\varepsilon$. Therefore, we show the continuity of the multiplication at (α_0, x_0).

Case 2: $\alpha_0 = 0$.

There exists an $r_0 > 0$ such that

$$r_0\big[p(x_0) + r_0\big] < \varepsilon$$

since $\lim_{r\to 0} r[p(x_0) + r] = 0$. Let $U = \{\alpha \in \mathbb{R}_+ : 0 \le \alpha < r_0\}$ and $V_{r_0} = \{x \in X : p(x - x_0) < r_0\}$. Then U and V_{r_0} are open neighborhoods of 0 and x_0 in $\mathbb{R}_+$ and (X, τ_p), respectively. We claim that $UV_{r_0} \subset V_\varepsilon$. Indeed, for each $\alpha \in U$ and $x \in V_{r_0}$, one obtains

$$p(\alpha x) = \alpha p(x) \le \alpha\big(p(x - x_0) + p(x_0)\big) < r_0\big[r_0 + p(x_0)\big] < \varepsilon.$$

Then we have $UV_{r_0} \subset V_\varepsilon$. Therefore, we show the continuity of the multiplication at $(0, x_0)$.

Therefore, the multiplication by scalars is continuous from $\mathbb{R}_+ \times (X, \tau_p)$ to (X, τ_p). □

From the proof of Proposition 3.2, we have

Proposition 3.3. *Let $(X, \mathcal{P})$ be an asymmetric locally convex space and $\tau_{\mathcal{P}}$ the topology generated by $\mathcal{P}$:*

(a) *for any fixed $x_0 \in X$ the multiplication by scalars $\cdot : \mathbb{R} \to X, \alpha \mapsto \alpha x_0$, is continuous from $\mathbb{R}$ to $(X, \tau_{\mathcal{P}})$, where $\mathbb{R}$ endows with Euclidean topology;*

(b) *the multiplication by scalars is continuous (as a function of two variables) from $\mathbb{R}_+ \times (X, \tau_{\mathcal{P}})$ to $(X, \tau_{\mathcal{P}})$, where $\mathbb{R}_+ = [0, +\infty)$ as a subspace of Euclidean space $\mathbb{R}$.*

Definition 3.4. ([11]) A topological space X is a *stratifiable space* if X is T_1 and, to each open U in X, on can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

(a) $U_n^- \subset U$;

(b) $\bigcup_{n=1}^{\infty} U_n = U$;

(c) $U_n \subset V_n$ whenever $U \subset V$.

Note: Clearly, each metrizable space is stratifiable and each regular stratifiable space is hereditarily paracompact [11].

The proof of the following Theorem 3.6 is quite similar to [4, Theorem 4.3]. However, Theorem 3.6 plays an important role in this paper, thus we give out the proof.

For each open subset U of stratifiable space X and $x \in U$, let $n(U, x)$ be the smallest integer n such that $x \in U_n$, and let

$$U_x = U_{n(U,x)} - \big(X - \{x\}\big)^-_{n(U,x)}.$$

Lemma 3.5. *([4]) For U, V open subsets of stratifiable space X, $x \in U$ and $y \in V$, we have the following:*

(i) *U_x is an open neighborhood of x;*

(ii) *$U_x \cap V_y \ne \emptyset$ and $n(U, x) \le n(V, y)$ implies $y \in U$;*

(iii) *$U_x \cap V_y \ne \emptyset$ implies $x \in V$ or $y \in U$.*

Theorem 3.6. *Let X be a stratifiable space, Y a closed subset of X, E an asymmetric locally convex space, $C(X, E)$ the linear space of continuous functions from X into E, and similarly for $C(Y, E)$. Then there* *exists a mapping*

$$\phi : C(Y, E) \longrightarrow C(X, E)$$

satisfying the following conditions:

(a) *$\phi(f)$ is an extension of f for each $f \in C(Y, E)$;*
(b) *the range of $\phi(f)$ is contained in the convex hull of the range of f, for each $f \in C(Y, E)$.*

Proof. Let $W = X - Y$, and let

$$W' = \{x \in W : x \in U_y \text{ for some } y \in Y \text{ and open } U \text{ containing } y\}.$$

For every $x \in W'$, let

$$m(x) = \max\{n(U, y) : y \in Y \text{ and } x \in U_y\}.$$

We claim that $m(x) < n(W, x)$ for each $x \in W'$. If not, there exists $x \in W'$ such that $m(x) \geq n(W, x)$. Therefore, there are $y \in Y$ and open neighborhood U of y, such that $x \in U_y$ (thus $W_x \cap U_y \neq \emptyset$) and $n(U, y) \geq n(W, x)$; then $y \in W$ by Lemma 3.5(ii), which is impossible.

Obviously, $\{W_x : x \in W\}$ is an open cover of the open subspace W in X. Since W is a paracompact space, $\{W_x : x \in W\}$ has an open locally finite refinement $\mathscr{V}$ with respect to W. Let $\{p_V : V \in \mathscr{V}\}$ be a partition of unity subordinated to $\mathscr{V}$. For every $V \in \mathscr{V}$, choose $x_V \in W$ such that $V \subset W_{x_V}$. If $x_V \in W'$, pick $a_V \in Y$ and open S_V containing a_V such that $x_V \in (S_V)_{a_V}$ and $n(S_V, a_V) = m(x_V)$; if $x_V \notin W'$, let a_V be the fixed point $a_0 \in Y$.

Define $g : X \longrightarrow E$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in Y, \\ \sum_{V \in \mathscr{V}} p_V(x) f(a_V), & \text{if } x \in W. \end{cases}$$

Obviously, $g(X)$ is contained in the convex hull of $f(Y)$ and g is continuous on W by Proposition 3.3. Next we shall show that g is continuous at Y.

Take any point $b \in Y$. Let O be any open subset of E containing $f(b)$. By the local convexity of E, there exists a convex open subset K of E such that $f(b) \subset K \subset O$. Moreover, since f is continuous, there exists an open neighborhood N of b in X such that $f(Y \cap N) \subset K \subset O$. We claim that $g((N_b)_b) \subset O$. Indeed, if $x \in (N_b)_b \cap Y \subset N \cap Y$ then $g(x) = f(x) \in O$. Let $x \in (N_b)_b \setminus Y$. Consider any $V \in \mathscr{V}$ with $x \in V$. Since $b \notin W_{x_V}$ and $x \in (N_b)_b \cap W_{x_V}$, it follows from Lemma 3.5(iii) that $x_V \in N_b$; hence $x_V \in W'$ and $n(N, b) \leq m(x_V) = n(S_V, a_V)$. It follows from Lemma 3.5(ii) that $a_V \in N$ since $x_V \in N_b \cap (S_V)_{a_V}$. Therefore $f(a_V) \in K$ and, by the convexity of K, we have $g(x) \in K \subset O$. Thus $g((N_b)_b) \subset O$. Hence g is continuous on Y. Finally, we only let $\phi(f) = g$. □

Lemma 3.7. *If $(X, \mathscr{U}_X)$ is a regular stratifiable space, then every point of $(X, \mathscr{U}_X^{-1})$ is discrete in $(X, \mathscr{U}_X^{-1})$.*

Proof. Take any point $x_0 \in X$. Since X is a stratifiable space, there exists a point finite open cover $\mathscr{U}$ of X such that $|\mathscr{U}| \geq 2$. Then there exists an $U_0 \in \mathscr{U}$ such that $x_0 \in U_0$. Put $\mathscr{V} = \{U \setminus \{x_0\} : U \in \mathscr{U} \setminus \{U_0\}\} \cup \{U_0\}$. Then $\mathscr{V}$ is also a point finite open cover of X. Let $W = \{(x, y) : x \in X \text{ and } y \in \bigcap C_x\}$, where each $C_x = \bigcap\{V \in \mathscr{V} : x \in V\}$. It follows from [10, Theorem 6.21(d)] that $W \in \mathscr{U}_X$, and hence $W^{-1} \in \mathscr{U}_X^{-1}$. Clearly, we have $W^{-1}(x_0) = \{x_0\}$, and thus x_0 is a discrete point in $(X, \mathscr{U}_X^{-1})$. By the arbitrary of taking the point x_0, every point of $(X, \mathscr{U}_X^{-1})$ is discrete in $(X, \mathscr{U}_X^{-1})$. □

Definition 3.8. A *quasi-pseudometric* d on a set X is a function from $X \times X$ into the set of non-negative real numbers such that for $x, y, z \in X$: (a) $d(x, x) = 0$ and (b) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the additional condition (c) $d(x, y) = 0 \Leftrightarrow x = y$, then d is called a *quasi-metric* on X.

Every quasi-pseudometric d on X generates a topology $\mathscr{F}(d)$ on X which has as a base the family of d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

Definition 3.9. ([15]) Let X be a subspace of a Tychonoff space Y.

(1) The subspace X is *quasi-P-embedded* in Y if each continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ admits a continuous extension from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$.
(2) The subspace X is *quasi-P^*-embedded* in Y if each bounded continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ admits a continuous extension from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$.

Theorem 3.10. *Let Y be a subset of a Tychonoff stratifiable space X. Then Y is quasi-P-embedded. In particular, Y is quasi-P^*-embedded.*

Proof. Let ρ be a continuous quasi-pseudometric defined on from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Denote by M the set Y with the quasi-metric topology induced in the obvious way by ρ and let $f : Y \longrightarrow M$ be the natural continuous projection. It follows from [1] that M is isometrically embedded in an asymmetric seminormed space B. Since X is a Tychonoff stratifiable space, it follows from Theorem 3.6 that each continuous mapping $f : Y \longrightarrow B$ into an arbitrary asymmetric locally convex space B is continuously extendable onto X. Then we can find a continuous extension $\tilde{f} : X \longrightarrow B$ of f into B. Let $\tilde{\rho}(x, y) = \|\tilde{f}(y) - \tilde{f}(x)|$, where $x, y \in X$ and $\| \cdot |$ denotes the asymmetric seminorm in B. Then it is easy to see that $\tilde{\rho}$ is a quasi-pseudometric on X and $\tilde{\rho}|_Y = \rho$. Since X is a Tychonoff stratifiable space, it follows from Lemma 3.7 that $(X, \mathscr{U}_X^{-1})$ is a discrete space. Therefore, it follows from the continuity of $\tilde{f}$ that $\tilde{\rho}$ is a continuous mapping from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$. $\square$

Lemma 3.11. *([15]) Let Y be an arbitrary subspace of a Tychonoff space X. Then the natural mapping $\hat{e}_{Y,X} : AP(Y) \to AP(X)$ is a topological monomorphism if and only if Y is quasi-P^*-embedded in X.*

By Theorem 3.10 and Lemma 3.11, we can easily get the following important theorem.

Theorem 3.12. *Let Y be a subset of a Tychonoff stratifiable space X. Then $AP(Y, X)$ is naturally topologically isomorphic to $AP(Y)$.*

Theorem 3.13. *Let Y be a subset of a metrizable space X. Then $AP(Y, X)$ is naturally topologically isomorphic to $AP(Y)$.*

By Theorem 3.13, it is natural to pose the following question.

Question 3.14. *Let Y be a closed subset of metrizable space X. Is $FP(Y, X)$ naturally topologically isomorphic to $FP(Y)$?*

4. Countable tightness of free Abelian paratopological groups

In this section, we shall show that if the tightness of $AP(X)$ over a metric space X is countable then the set of all non-isolated points in X is separable. To begin, we need the following propositions.

Proposition 4.1. *If X is a Tychonoff space and $n \geq 1$, then we have the following statements:*

(1) *each $i_n(X^n)$ is closed in $PG(X)$;*
(2) *each $i_n|_{X^n} : X^n \longrightarrow i_n(X^n) \subset FP_n(X)$ is a homeomorphism mapping;*
(3) *each $i_n|_{X^n} : X^n \longrightarrow i_n(X^n) \subset AP_n(X)$ is a perfect mapping.*

Proof. Obviously, the canonical embedding $i : X \longrightarrow \beta X$ can be extend to a continuous homomorphism $\hat{i} : PG(X) \longrightarrow PG(\beta X)$. Since $PG(\beta X)$ is algebraically free on βX and the restriction of $\hat{i}$ to X is one-to-one, $\hat{i}$ must be an injective mapping. For each $n \geq 1$, consider the mapping $i_n^* : (\beta X)^n \longrightarrow PG(\beta X)$ defined by formula

$$i_n^*(y_1, y_2, \cdots, y_n) = y_1 y_2 \cdots y_n$$

for all $y_1, y_2, \cdots, y_n \in \beta X$. Obviously, i_n^* is continuous and the restriction of i_n^* to X^n coincides with $\hat{i} \circ j_n$ for each $n \geq 1$, where $j_n = i_n|_{X^n}$.

(1) Clearly, each $i_n^*((\beta X)^n)$ is closed in $PG(\beta X)$. Then

$$\hat{i}\big(j_n\big(X^n\big)\big) = \hat{i}\big(PG(X)\big) \cap i_n^*\big((\beta X)^n\big),$$

thus $\hat{i}(j_n(X^n))$ is closed in $\hat{i}(PG(X))$. Since $\hat{i}$ is one-to-one mapping from $PG(X)$ onto $PG(\beta X)$, $j_n(X^n) = i_n(X^n)$ is closed in $PG(X)$.

(2) Obviously, $i_n|_{X^n} : X^n \longrightarrow i_n(X^n) \subset FP_n(X)$ is a continuous one-to-one mapping and $i_n^* : (\beta X)^n \longrightarrow FP_n(\beta X)$ is a perfect mapping. Let $P_n = \hat{i}(i_n(X^n))$. Then $(i_n^*)^{\leftarrow}(P_n) = X^n$. Therefore, $i_n^*|_{X^n} = \hat{i} \circ i_n|_{X^n}$ is also a perfect mapping [9]. It follows from [9, Proposition 3.7.10] that $i_n|_{X^n} : X^n \longrightarrow i_n(X^n) \subset FP_n(X)$ is a perfect mapping, thus it is a homeomorphism mapping.

(3) By the proof of (2), it is easy to see that each $i_n|_{X^n} : X^n \longrightarrow i_n(X^n) \subset AP_n(X)$ is a perfect mapping. □

Similarly, we can show that

Proposition 4.2. *If X is a space and $n \geq 1$, then we have the following statements:*

(1) *each $i_n((X_d^{-1})^n)$ is closed in $PG(X)$;*
(2) *each $i_n|_{(X_d^{-1})^n} : (X^{-1})^n \longrightarrow i_n((X^{-1})^n) \subset FP_n(X)$ is a homeomorphism mapping;*
(3) *each $i_n|_{(-X_d)^n} : (-X_d)^n \longrightarrow i_n((X^{-1})^n) \subset AP_n(X)$ is a perfect mapping.*

Proposition 4.3. *If X is a Tychonoff space, then the group $AP(X)$ contains a closed homeomorphic copy of X^n, for each positive integer n.*

Proof. If $n = 1$, it is obvious. Let $n \geq 2$ be a positive integer. Consider the mapping $j : X^n \longrightarrow AP(X)$ defined by $f(x_1, x_2, \cdots, x_n) = x_1 + 2x_2 + \cdots + 2^{n-1}x_n$ for each $(x_1, x_2, \cdots, x_n) \in X^n$. Obviously, f is continuous. Apply induction on n along with the fact X is a free algebraic basis for $AP(X)$ to show that f is one-to-one.

Let $m = 2^n - 1$. Let g be the embedding g of X^n to X^m defined by the formula

$$g(x_1, x_2, \cdots, x_n) = (x_1, x_2, x_2, \cdots, x_n, \cdots, x_n),$$

where each x_i appears in the right side of the equality 2^{i-1} times. Then it is easy to see that $g(X^n)$ is closed in X^m and $f = i_m \circ g$, where $i_m : (X \oplus (-X_d) \oplus \{0\})^n \longrightarrow AP_m(X)$. It follows from Proposition 4.1 that

$i_m|_{X^m} : X^m \longrightarrow AP_m(X)$ is perfect, hence the composition $i_m \circ g$ is a closed mapping, and thus f is a homeomorphism. Therefore, it follows from Proposition 4.1 that $i_m(X^m)$ is closed in $AP(X)$, so the image $f(X^n) = i_m(g(X^n))$ is closed in $i_m(X^m)$ and in $AP(X)$. □

Theorem 4.4. *Let X be a regular paracompact sequential space. If the tightness of $AP(X)$ is countable, then the set $NI(X)$ of all non-isolated points in X is separable.*

Proof. Suppose that $NI(X)$ is non-separable. Then $NI(X)$ is not Lindelöf since X is paracompact. Therefore, one can choose an uncountable discrete family $\{U_\alpha : \alpha < \omega_1\}$ of open sets in X such that each U_α contains a point $x_\alpha \in NI(X)$. Since X is sequential, for each $\alpha < \omega_1$, one can choose a non-trivial convergent sequence $C_\alpha \subset U_\alpha$ with the limit point x_α (we may assume each $x_\alpha \in C_\alpha$). Put

$$Y = \bigcup\{C_\alpha : \alpha < \omega_1\}, Y_0 = \{x_\alpha : \alpha < \omega_1\}.$$

It is easy to see that Y is closed in X and is homeomorphic to the product $C \times D(\aleph_1)$, where C is a converging sequence and $D(\aleph_1)$ is the discrete space of cardinality $\aleph_1$. Let Z be the quotient space obtained by identifying the subset Y_0 in X to a point and the subspace $Z_1 = p(Y)$ in Z, where $p : X \longrightarrow Z$ is the projection. Since p is closed, it is easy to see that Z_1 is homeomorphic to S_{ω_1}. Suppose that the tightness of $AP(X)$ is countable. Then the homomorphism $\hat{p} : AP(X) \longrightarrow AP(Z)$ extending quotient mapping $p : X \longrightarrow Z$ is open [18]. Therefore the tightness of $AP(Z)$ is countable. However $AP(Z)$ contains a homeomorphic copy of Z^2 by Proposition 4.3, and hence contains a homeomorphic copy of $S_{\omega_1} \times S_{\omega_1}$. However, the tightness of $S_{\omega_1} \times S_{\omega_1}$ is uncountable [12]. □

Corollary 4.5. *Let X be a metrizable space. If the tightness of $AP(X)$ is countable, then the set $NI(X)$ of all non-isolated points in X is separable.*

Theorem 4.6. *If X is metrizable and $AP(X)$ is a k-space, then the set $NI(X)$ in X is separable.*

Proof. Since X is metrizable, $AP(X)$ is submetrizable [18]. Therefore, every compact subset of $AP(X)$ is metrizable [11], then it is sequential since $AP(X)$ is a k-space. Hence X is sequential and the tightness of $AP(X)$ is countable, and it follows from Theorems 4.4 that the set $NI(X)$ in X is separable. □

5. *k*-Properties of free Abelian paratopological groups over metric spaces

In this section, we shall give characterizations of k-properties of free Abelian paratopological groups over metric spaces.

The *support* of a reduced word $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in PG(X)$ with $x_1, \cdots, x_n \in X$ is defined as follows:

$$\text{supp}(g) = \{x_1, \cdots, x_n\}.$$

Given a subset K of $PG(X)$, we put

$$\text{supp}(K) = \bigcup_{g \in K} \text{supp}(g).$$

A subset Y of a space X is said to be *bounded* in X if each continuous real-valued function on X is bounded on Y.

Lemma 5.1. *([2]) If ϕ is a bounded set in $F(X)$ (in $A(X)$), then* supp(ϕ) *is bounded in X.*

Lemma 5.2. *If ϕ is a bounded set in $FP(X)$ (in $AP(X)$), then* $\mathrm{supp}(\phi)$ *is bounded in X.*

Proof. Obviously, the identity mappings from $FP(X)$ to $F(X)$ and $AP(X)$ to $A(X)$ are continuous, and hence lemma holds by Lemma 5.1. □

Lemma 5.3. *([6]) Let X be a T_1-space and let $w = \epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$ be a reduced word in $AP_n(X)$, where $x_i \in X$ and $\epsilon_i = \pm 1$, for all $i = 1, 2, \cdots, n$, and if $x_i = x_j$ for some $i, j = 1, 2, \cdots, n$, then $\epsilon_i = \epsilon_j$. Then the collection $\mathscr{B}$ of all sets of the form $\epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n$, where, for all $i = 1, 2, \cdots, n$, the set U_i is a neighborhood of x_i in X when $\epsilon_i = 1$ and $U_i = \{x_i\}$ when $\epsilon_i = -1$ is a base for the neighborhood system at w in $AP_n(X)$.*

Let X be a set. Then we define $j_2, k_2 : X \times X \to FP_a(X)$ by $j_2(x, y) = x^{-1}y$ and $k_2(x, y) = yx^{-1}$ for all $(x, y) \in X \times X$. For the Abelian case we define $j_2^* : X \times X \to AP_a(X)$ by $j_2^*(x, y) = y - x$ for all $(x, y) \in X \times X$.

Suppose that $\mathscr{U}_X$ is the finest quasi-uniformity of a space X. Set $\mathscr{M}$ be the family of all countable sequences of mappings $FP_a(X) \to \mathscr{U}_X$. For any $\psi : FP_a(X) \to \mathscr{U}_X$, we define

$$E(\psi) = \bigcup_{g \in FP_a(X)} g\big(j_2\big(\psi(g)\big) \cup k_2\big(\psi(g)\big)\big)g^{-1}.$$

For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the group of permutations of the set $\{1, 2, \cdots, n\}$. Then for each countable sequence $\Psi \in \mathscr{M}$, where $\Psi = (\psi_n)_{n \in \mathbb{N}}$, let

$$E_n(\Psi) = \bigcup_{\pi \in \mathcal{S}_n} E(\psi_{\pi(1)})E(\psi_{\pi(2)}) \cdots E(\psi_{\pi(n)}).$$

Put

$$E(\Psi) = \bigcup_{n \in \mathbb{N}} E_n(\Psi)$$

and then define $\mathscr{W}_F = \{E(\Psi) : \Psi \in \mathscr{M}\}$.

Let $\mathscr{P}$ be the collection of all countable sequences of elements of $\mathscr{U}_X$. For each $P = \{U_1, U_2, \cdots\} \in \mathscr{P}$, let

$$W(P) = \left\{\sum_{i=1}^{n} j_2^*(U_i) : n \in \mathbb{N}\right\}, \quad \text{and}$$

$$\mathscr{W} = \big\{W(P) : P \in \mathscr{P}\big\}.$$

Moreover, fix any $n \in \mathbb{N}$. For each $U \in \mathscr{U}_X$, let

$$W_n(U) = \left\{\sum_{i=1}^{n} j_2^*(x_i, y_i) : (x_i, y_i) \in U\right\},$$

and

$$\mathscr{W}_n = \big\{W_n(U) : U \in \mathscr{U}_X\big\}.$$

Theorem 5.4. *([8]) The collection $\mathscr{W}_F$ as defined above is a neighborhood base at e for the topology of $FP(X)$.*

Theorem 5.5. *([14]) The family $\mathscr{W}$ is a neighborhood base of e in $AP(X)$.*

Theorem 5.6. *([14]) For each $n \in \mathbb{N}$, the family $\mathscr{W}_n$ is a neighborhood base of e in $AP_{2n}(X)$.*

Proposition 5.7. *([7,16]) Let X be a space. Then $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ (or $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X)$) is quotient if and only if X is a z-space.*

A space X is called a *μ-space* if the closure of each bounded subset of X is compact. A mapping $f : X \to Y$ is said to be *compact-covering* if for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

Theorem 5.8. *Let X be a Hausdorff, μ, z-space. For each $n \in \mathbb{N}$, i_n is a compact covering mapping.*

Proof. We only show that each $i_n : (X \oplus X_d^{-1} \oplus \{e\})^n \to AP_n(X)$ is a compact covering-mapping. The proof of analogous assertion for $i_n : (X \oplus X_d^{-1} \oplus \{e\})^n \to FP_n(X)$ is quite similar. (Indeed, we use Theorem 5.4 instead of Theorem 5.6.)

Fix $n \in \mathbb{N}$. Let K a compact subset of $AP_n(X)$. Next we shall show that there exists a compact subset C of $(X \oplus X_d^{-1} \oplus \{e\})^n$ such that $i_n(C) = K$.

For each $g \in K$, we can fix the word g the reduced form

$$g = \eta\big(x(g)_1\big)x(g)_1 + \eta\big(x(g)_2\big)x(g)_2 + \cdots + \eta\big(x(g)_m\big)x(g)_m,$$

where $m \le n$, $x(g)_i \in X$ and $\eta(x(g)_i) = \pm 1$ for $i = 1, 2, \cdots, m$. Let D be the set of X such that for each $x \in D$ there exists $g \in K$ such that $x(g)_i = x$ and $\eta(x(g)_i) = -1$ for some $i \le n$. Then D is a finite set. Suppose not, there exist $1 \le m \le n$ and countable infinite set $E = \{g_k : k \in \mathbb{N}\} \subset AP_m(X) \setminus AP_{m-1}(X)$ satisfy the following conditions (1)–(3):

(1) for each $k \in \mathbb{N}$, we have

$$g_k = -x(g_k)_1 + \eta\big(x(g_k)_2\big)x(g_k)_2 + \cdots + \eta\big(x(g_k)_m\big)x(g_k)_m;$$

(2) for $i \ne j$, we have $x(g_i)_1 \ne x(g_j)_1$;
(3) for $i \ne j$, we have $g_i \ne g_j$.

Since the translation is a homeomorphism in paratopological group, without loss of generalization, we may assume that m is even. Let $m = 2l \le n$.

Claim 1. *We have $e \notin \overline{E}$.*

Indeed, it follows from Theorem 5.6 that we may assume $\sum_{i=1}^{m} \eta(x(g)_i) = 0$. Moreover, for each $k \in \mathbb{N}$, there exists an open neighborhood U_k of $x(g_k)_1$ in X such that

$$U_k \cap \big\{x(g_k)_i : x(g_k)_i \ne x(g_k)_1, i = 2, \cdots, m\big\} = \emptyset.$$

Let $A = X \setminus \{x(g_k)_1 : k \in \mathbb{N}\}$ and

$$U = \bigcup_{k \in \mathbb{N}} \big(\big\{x(g_k)_1\big\} \times U_k\big) \cup \bigcup (A \times X).$$

Since X is a z-space, U is a normal neighbornet. Let $W_l(U) = \{-x_1 + y_1 - \cdots - x_l + y_l : (x_i, y_i) \in U\}$. Then it follows from Theorem 5.6 that $W_l(U)$ is an open neighborhood of e in $AP_m(X)$. We claim that

$W_l(U) \cap E = \emptyset$. Indeed, let $g_k \in W_l(U)$. Then there exists $2 \le j \le m$ such that $x(g_k)_j \in U_k \setminus \{x(g_k)_1\}$, $\eta(x(g_k)_j) = 1$ and $(x(g_k)_1, x(g_k)_j) \in \{x(g_k)_1\} \times U_k$. However, $x(g_k)_j \notin U_k$ by the definition of U_k, which is a contradiction. Therefore, $e \notin \overline{E}$.

Claim 2. *E is closed discrete in $AP_m(X)$.*

Suppose $\overline{E} \setminus E \neq \emptyset$. Take $g \in \overline{E} \setminus E \subset AP_m(X)$. Then $e \in \overline{(-g)E}$. However, it is easy to see that $(-g)E$ satisfies the conditions (1)–(3) in the above, and then $e \notin \overline{(-g)E}$ by Claim 1, which is a contradiction. Therefore, we have E is closed in $AP_m(X)$. By Lemma 5.3, it is easy to see that E is discrete.

By Claim 2, E is an infinite closed discrete subset of compact set K, which is a contradiction. Therefore, D is finite.

Let $A = \text{supp}(K)$. By Lemma 5.2, A is bounded, then the closure $\overline{A}$ of A in X is compact since X is a μ-space. Let $B = \overline{A} \oplus D^{-1} \oplus \{e\}$. Then B is compact, hence $i_n|_{B^n}$ is a closed mapping from B^n onto $i_n(B^n)$. Let $C = (i_n|_{B^n})^{-1}(K)$. Then C is closed in B^n, thus it is compact. Obviously, we have $i_n(C) = K$. □

By Proposition 5.7 and Theorem 5.8, we have the following corollary.

Corollary 5.9. *Let X be a metrizable space. If $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X)$ (resp. $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$) is quotient, then, for each $n \in \mathbb{N}$,*

$$i_n : \left(X \oplus X_d^{-1} \oplus \{e\}\right)^n \to AP_n(X) \quad \left(\textit{resp. } i_n : \left(X \oplus X_d^{-1} \oplus \{e\}\right)^n \to FP_n(X)\right)$$

is a compact covering mapping.

Since each T_1 countable space is a z-space [10, Corollary 6.24], we have the following corollary.

Corollary 5.10. *Let X be a countable regular space. For each $n \in \mathbb{N}$, i_n is a compact covering mapping.*

A space X is said to be a *P-space* if the intersection of countably many open subsets of X is open in X.

Theorem 5.11. *Let X be a μ-space. If X is a P-space, then each i_n is a compact covering mapping.*

Proof. Fix $n \in \mathbb{N}$. Let K be a compact subset of $FP_n(X)$ (or $AP_n(X)$). Put $A = \text{supp}(K)$. By Lemma 5.2, A is bounded, then the closure $\overline{A}$ of A in X is compact since X is a μ-space. Since X is a P-space, $\overline{A}$ is finite, and hence K is finite. Therefore, each i_n is a compact covering mapping. □

The following question is still open.

Question 5.12. *Let X be a Hausdorff μ-space. Is i_2 a compact covering mapping?*

Lemma 5.13. *([9, Theorem 3.3.22]) A continuous mapping $f : X \longrightarrow Y$ of a topological space to a k-space Y is quotient if and only if for each compact set $Z \subset Y$ the restriction $f|_{f^{-1}(Z)} : f^{-1}(Z) \longrightarrow Z$ is quotient.*

By Theorem 5.11 and Lemma 5.13, we have

Proposition 5.14. *Let X be a Hausdorff μ-space. If X is a P-space or a z-space. Then, for each $n \in \mathbb{N}$, the mapping i_n is quotient if $AP_n(X)$ (or $FP_n(X)$) is a k-space.*

Lemma 5.15. *([16, Proposition 3.15]) For arbitrary compact first-countable Hausdorff space X, the mapping i_2 is quotient if and only if X is countable, if and only if X is a z-space.*

Lemma 5.16. *([14]) If Y is a closed subspace of a Tychonoff space X, then the subgroup $PG(Y, X)$ of $PG(X)$ generated by Y is closed in $PG(X)$.*

Theorem 5.17. *If X is a metrizable space and $AP(X)$ is a k-space, then X is locally compact and $NI(X)$ is separable.*

Proof. Suppose that a point $x_0 \in X$ has no neighborhood with compact closure in X. Choose a decreasing countable base $\{U_n : n \in \mathbb{N}\}$ at the point x_0 such that all sets $F_n = \overline{U_n} \setminus U_{n+1}$ $(n \in \mathbb{N})$ are non-compact. For every $n \in \mathbb{N}$ choose an infinite set $X_n = \{x_{n,m} : m \in \mathbb{N}\} \subset F_n$ such that it is closed discrete in X, where $x_{n,m} \neq x_{n,m'}$ if $m \neq m'$. Put

$$M = \{x_{n,m} : n, m \in \mathbb{N}\} \cup \{x_0\}.$$

Obviously, all points of the set M except the point x_0, are isolated in M. It follows from Theorem 3.13 and Lemma 5.16 that $AP(M)$ is homeomorphic to a closed subgroup in $AP(X)$. Next we shall show that $AP(M)$ is not a k-space.

Assign to each pair k, l of positive integers an element

$$h_{k,l} = (-x_0 + x_{k,l}) + (-x_0 + x_{l,1}) + \cdots + (-x_0 + x_{l,k}) \in AP(X)$$

and consider the sets $H_k = \{h_{k,l} : l > k\}$, $k \in \omega$ and $H = \bigcup_{k=0}^{\infty} H_k$. Clearly, we have $e \notin H$.

Claim 1. *For each compact subset K in $AP(X)$, the intersection of H with K is finite.*

Obviously, the length of $h_{k,l}$ equals $2k + 2$, hence $H_k \subset AP(X) \setminus AP_{2k}(X)$. Moreover, it follows from [14, Theorem 3.12] that $K \subset AP_n(X)$ for some $n \in \mathbb{N}$. Therefore, K intersects only finitely many sets H_k, hence it suffices to show that $K \cap H_k$ is finite for each $k \in \mathbb{N}$.

For $k, l \in \mathbb{N}$ with $k < l$, put

$$\operatorname{supp}(h_{k,l}) = \{x_0, x_{k,l}, x_{l,1}, \cdots, x_{l,k}\}, \quad \text{and}$$
$$D_{k,l} = X_k \cap \operatorname{supp}(h_{k,l}).$$

Then $D_{k,l} = \{x_{k,l}\}$ for $k, l \in \mathbb{N}$ with $k < l$, hence the family $\{D_{k,l} : l > k\}$ is disjoint for each $k \in \mathbb{N}$. Therefore, the intersection $X_k \cap \operatorname{supp}(P)$ is infinite for each infinite $P \subset H_k$. Since X_k is closed and discrete in metrizable space X, the subspace $X_k \cap \operatorname{supp}(P)$ is not bounded in X. It follows from Lemma 5.2 that the intersection $K \cap H_k$ is finite. Thus the proof of Claim 1 is complete.

Claim 2. $e \in \overline{H}$.

For each $n \in N$ put

$$V_n = \{x_{k,l} : k \geq n, k, l \in \mathbb{N}\} \cup \{x_0\}.$$

Then the family $\{V_n : n \in \mathbb{N}\}$ is a base of M at the point x_0. For each $n \in N$ put

$$W_n = \Delta \cup (\{x_0\} \times V_n),$$

where $\Delta = \{(x, x) : x \in M\}$. Then it follows from [10, Proposition 2.34] that the family $\{W_n : n \in \mathbb{N}\}$ is a base for the finest quasi-uniformity on M.

Assign to each sequence $P = (p_1, \cdots, p_n, \cdots)$ of naturals a set G_P of all elements in $AP(M)$ of the form $(-y_1 + z_1) + \cdots + (-y_r + z_r)$, where $r \in \mathbb{N}$, $(y_i, z_i) \in W_{p_i}$, $i = 1, \cdots, r$. By Theorem 5.5, the description of neighborhoods of the neutral element in $AP(M)$ implies the family $\{G_P : P \in \mathbb{N}^{\mathbb{N}}\}$ being a base of $AP(M)$ at the neutral element e. To end the proof it suffices to show that each G_P contains a point from H. Fix a $P = (p_1, \cdots, p_n, \cdots) \in \mathbb{N}^{\mathbb{N}}$. Choose naturals k and l in such a way that $k > p_1$, and $l > \max\{k, p_1, \cdots, p_k\}$. Then $(x_0, x_{k,l}) \in W_k \subset W_{p_1}$ and $(x_0, x_{l,i}) \in W_l \subset W_{p_i}$ for each $i \le k$. Therefore, $h_{k,l} \in G_P$ and Claim 2 is proved.

By Claims 1 and 2, it is easy to see that $AP(M)$ is not a k-space. Therefore, X is locally compact. By Theorem 4.6, the set $NI(X)$ is countable. □

Theorem 5.18. *If X is a metrizable z-space and $AP(X)$ is a k-space, then X is locally compact and locally countable. In particular, the set $NI(X)$ is countable.*

Proof. By Theorem 5.17, it suffices to show that X is locally countable. Let V be an open neighborhood in X with compact closure in X. Since each subspace of a z-space is also a z-space, it follows from Lemma 5.15, $\overline{V}$ must be countable, hence V is countable. Therefore, X is locally countable. □

Lemma 5.19. *([19, Theorem 2]) Let X be a functionally Hausdorff space. Then the following conditions are equivalent:*

(1) *$AP(X)$ is a k_ω-space;*
(2) *$FP(X)$ is a k_ω-space;*
(3) *X is a countable k_ω-space.*

Proposition 5.20. *Let X be a metrizable space. If X is locally compact, locally countable and $NI(X)$ is separable, then $AP(X)$ is a k-space.*

Proof. Obviously, the set $NI(X)$ is countable, hence we can choose a countable covering γ of X with open sets such that each element of γ is countable and has compact closure in X. Put $X_0 = \bigcup\{\overline{U} : U \in \gamma\}$. Obviously, X_0 is a countable k_ω-space and $X_1 = X \setminus X_0$ is closed discrete in X. It follows from [18, Proposition 2.13] that $AP(X) = AP(X_0) \times AP(X_1)$. Clearly, $AP(X_1)$ is discrete. By Lemma 5.19, we see that $AP(X_0)$ is a k_ω-space. □

Theorem 5.21. *Let X be a metrizable z-space. Then the following conditions are equivalent:*

(1) *$AP(X)$ is a k-space;*
(2) *$AP(X)$ is homeomorphic to a product of a countable k_ω-space with a discrete space;*
(3) *X is locally compact, locally countable and $NI(X)$ is separable.*

Proof. Obviously, (2) ⇒ (1). By Theorem 5.18 and the proof of Proposition 5.20, we have (1) ⇒ (3) and (3) ⇒ (2), respectively. □

Since $AP(X)$ is a quotient of $FP(X)$, it follows from Theorems 5.17 and 5.18 that we have the following two corollaries.

Corollary 5.22. *Let X be a metrizable space. If $FP(X)$ is a k-space, then X is locally compact and $NI(X)$ is separable.*

Corollary 5.23. *Let X be a metrizable z-space. If $FP(X)$ is a k-space, then X is locally compact, locally countable and $NI(X)$ is separable.*

Naturally, we have the following question.

Question 5.24. *Let X be a metrizable space. If $AP(X)$ (or $FP(X)$) is a k-space, is X a z-space?*

We also define the cardinal function $\mathrm{qu}(X)$ called the finest quasi-uniform weight of X by

$$\mathrm{qu}(X) = \min\{|\mathscr{B}| : \mathscr{B} \text{ is a base for } \mathscr{U}_X\}.$$

The proofs of the following two propositions are similar to the proofs which was given in [22, Lemma 3.1 and Proposition 3.2], respectively. However, we give out the proofs for the completeness.

Proposition 5.25. *Let X be a space and $m, n \in \mathbb{N}$ with $n \leq m$. If B is a neighborhood of e in $FP_{m+n}(X)$ and $g \in FP_n(X)$, then $gB \cap FP_m(X)$ is a neighborhood of g in $FP_m(X)$. The same is true in the Abelian case.*

Proof. Let U be a neighborhood of e in $FP(X)$ such that $U \cap FP_{m+n}(X) \subset B$. Since $gU \cap FP_m(X)$ is a neighborhood of g in $FP_m(X)$, it suffices to show that $gU \cap FP_m(X) \subset gB \cap FP_m(X)$. Take arbitrary point $h \in gU \cap FP_m(X)$. Then there exists $u \in U$ such that $h = gu$. Since the length of $h \leq m$ (we write it $\ell(h) \leq m$.) and $\ell(g) \leq n$, we have $\ell(u) \leq n + m$, and hence $u \in FP_{m+n}(X)$. Therefore, $u \in U \cap FP_{m+n}(X)$ and $= gu \in gB \cap FP_m(X)$. Hence we have $gU \cap FP_m(X) \subset gB \cap FP_m(X)$. □

Proposition 5.26. *Let X be a space, $m, n \in \mathbb{N}$ with $n \leq m$ and κ be a cardinal. Then we have:*

(1) *if $\chi(e, FP_{m+n}(X)) \leq \kappa$, then $\chi(g, FP_m(X)) \leq \kappa$ for each $g \in FP_n(X)$, and*
(2) *if $\chi(0, AP_{m+n}(X)) \leq \kappa$, then $\chi(g, AP_m(X)) \leq \kappa$ for each $g \in AP_n(X)$.*

Proof. Since the proofs of (1) and (2) are similar, we only show (2). Let $\mathscr{U}$ be a neighborhood base at e in $AP(X)$ and $\mathscr{B}_{m+n}$ be a neighborhood base at e in $AP_{m+n}(X)$ such that $|\mathscr{B}_{m+n}| \leq \kappa$. Let $g \in AP_n(X)$ and put

$$\mathscr{B}_m(g) = \{gB \cap AP_m(X) : B \in \mathscr{B}_{m+n}\}.$$

Then each element of $\mathscr{B}_m(g)$ contains g and $|\mathscr{B}_m(g)| \leq \kappa$. For each $U \in \mathscr{U}$, it is easy to see that there exists $B \in \mathscr{B}_{m+n}$ such that $gB \cap AP_m(X)$. On the other hand, Proposition 5.25 shows that each element of $\mathscr{B}_m(g)$ is a neighborhood of g in $AP_m(X)$. Therefore, $\mathscr{B}_m(g)$ is a neighborhood base of g in $AP_m(X)$ and $|\mathscr{B}_m(g)| \leq \kappa$. Hence we have $\chi(g, AP_m(X)) \leq \kappa$ for each $g \in AP_n(X)$. □

Theorem 5.27. *For a space X and a cardinal κ the following are equivalent:*

(1) *$\chi(AP_n(X)) \leq \kappa$ for each $n \in \mathbb{N}$;*
(2) *$\chi(AP_2(X)) \leq \kappa$;*
(3) *$\mathrm{qu}(X) \leq \kappa$.*

Proof. By Theorem 5.6 and Proposition 5.26, we have (1) ⇒ (2) and (3) ⇒ (1). So we shall show that (2) ⇒ (3).

By Theorem 5.6, $\mathscr{W}_1 = \{W_1(U) : U \in \mathscr{U}_X\}$ is a neighborhood base at 0 in $AP_2(X)$. Since $\chi(AP_2(X)) \leq \kappa$, there exists $\mathscr{B} \subset \mathscr{U}_X$ with $|\mathscr{B}| \leq \kappa$ such that $\{W_1(B) : B \in \mathscr{B}\}$ is also a neighborhood base at 0 in $AP_2(X)$. It follows from the definition of $W_1(B)$ that $W_1(U_1) \subset W_1(U_2)$ if and only if $U_1 \subset U_2$. Therefore, $\mathscr{B}$ is a base for $\mathscr{U}_X$, and hence $\mathrm{qu}(X) \leq \kappa$. □

Theorem 5.28. *For a regular space X, then the following are equivalent:*

(1) *$AP_n(X)$ is first-countable for each $n \in \mathbb{N}$;*
(2) *$AP_2(X)$ is metrizable;*
(3) *$AP_2(X)$ is first-countable;*
(4) *the set $NI(X)$ is finite and X is metrizable.*

Proof. Obviously, (1) ⇒ (3), (2) ⇒ (3) and (4) ⇒ (1). It suffices to show that (3) ⇒ (4) and (4) ⇒ (2).

(3) ⇒ (4). It is well-known that the fine quasi-uniformity of a regular space has a countable base if and only if it is a metric space with only finitely many non-isolated points [10, Proposition 2.34]. By Theorem 5.27, it is easy to see that the set $NI(X)$ is finite and X is metrizable.

(4) ⇒ (2). Since $NI(X)$ is finite and X is metrizable, it follows from [10, Proposition 6.25] that X is a z-space, and it follows from Lemma 5.15 that the mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow AP_2(X)$ is a quotient mapping, hence it is closed [16]. It is easy to see that i_2 is a boundary-compact mapping, and therefore, $AP_2(X)$ is metrizable by Hanai–Morita–Stone Theorem. □

The proof of the following theorem is similar to [22, Theorem 4.4]. Hence we give an outline of the proof.

Theorem 5.29. *Let X be a metrizable space such that the set C of all non-isolated points in X is finite. Then $AP_n(X)$ has a σ-disjoint base for each $n \in \mathbb{N}$.*

Proof. Let d be a metric on X which induces the topology on X, and let $B_d(x,r) = \{y : d(x,y) < r\}$ for each real number $r > 0$ and $x \in X$. For each $k \in \mathbb{N}$, put $\mathcal{G}_k = \{B_d(x, \frac{1}{k}) : x \in NI(X)\}$, and put $U_k = \bigcup\{\{x\} \times B_d(x, \frac{1}{k}) : x \in NI(X)\} \cup \Delta_X$, where Δ_X is the diagonal of $X \times X$. By [10, Proposition 2.34], the family $\{U_k : k \in \mathbb{N}\}$ is the countable base of the fine quasi-uniformity for X. Therefore, it follows from Theorem 5.6 that $\mathcal{W}_m = \{W_m(U_k) : k \in \mathbb{N}\}$ is a neighborhood base at 0 in $AP_{2m}(X)$ for each $m \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. For each $g \in AP_n(X)$, put $g = g_{X\setminus C} + g_C$, where $g_{X\setminus C} \in AP_n(X \setminus C)$ and $g_C \in AP_n(C)$, and put $k(g) = \min\{m \in \mathbb{N} : x \notin \bigcup \mathcal{G}_k$ for each $x \in \mathrm{supp}(g_{X\setminus C})\}$. For $k, m \in \mathbb{N}$ with $k \geq m$ and $h \in AP_n(C)$, let

$$\mathscr{B}_{k,m,h} = \big\{\big(g + W_{2n}(U_k)\big) \cap AP_n(X) : g \in AP_n(X), g_C = h \text{ and } k(g) = m\big\}.$$

Put $\mathscr{B} = \bigcup\{\mathscr{B}_{k,m,h} : k \geq m, h \in AP_n(C)\}$. Then $\mathscr{B}$ is a σ-disjoint base for $AP_n(X)$, see [22, Theorem 4.4]. □

Let X be a locally countable metrizable space. If the set $NI(X)$ is finite, it follows from [10, Proposition 6.25] that X is a z-space, and then it follows from Theorem 5.21 that $AP(X)$ is a paracompact σ-space, hence it is perfectly normal. Moreover, it is known that every perfectly normal space with a σ-disjoint base is metrizable. Therefore, we have the following theorem:

Theorem 5.30. *For a locally countable regular space X, the following conditions are equivalent:*

(1) *$AP_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$AP_n(X)$ is first-countable for each $n \in \mathbb{N}$;*

(3) *$AP_2(X)$ is metrizable;*
(4) *$AP_2(X)$ is first-countable;*
(5) *the set $NI(X)$ is finite and X is metrizable.*

6. Generalized metric properties on free paratopological groups

In this section, we shall consider some generalized metric properties of free paratopological groups.

Definition 6.1. Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *k-network* if whenever K is a compact subset of X and $K \subset U \in \tau(X)$, there is a finite subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \cup\mathscr{P}' \subset U$.

Theorem 6.2. *Let X be a Tychonoff space. Then the following are equivalent:*

(1) *$FP(X)$ has a countable k-network;*
(2) *$AP(X)$ has a countable k-network;*
(3) *X is a countable space with a countable k-network.*

Proof. Since X is a T_1-space, X^{-1} is discrete, hence it is easy to see that (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). Next we shall show that (3) $\Rightarrow$ (1). The proof of analogous assertion for (3) $\Rightarrow$ (2) is quite similar.

Suppose that X is a countable space with a countable k-network. Then the product space $(X \oplus X_d^{-1} \oplus \{e\})^n$ has a countable k-network $\mathscr{P}_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, denote $i_n : (X \oplus X_d^{-1} \oplus \{e\})^n \longrightarrow FP_n(X)$ the canonical mapping, and put $\mathscr{B}_n = \{i_n(P) : P \in \mathscr{P}_n\}$. Obviously, X is a μ and z-space, and then it follows from Theorem 5.8 that for each compact set $\phi \subset FP_n(X)$ there exists a compact subset $\phi_1 \subset (X \oplus X_d^{-1} \oplus \{e\})^n$ such that $i_n(\phi_1) = \phi$. Then it is easy to see that $\mathscr{B}_n$ is a countable k-network in $FP_n(X)$ for each $n \in \mathbb{N}$. But every compact $\phi \subset FP(X)$ is in $FP_n(X)$ for some $n \in \mathbb{N}$ by [14], thus $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$ is a countable k-network in $FP(X)$. □

The proofs of the following two propositions are similar to [7, Proposition 3.4] and [16, Theorem 3.4], respectively.

Proposition 6.3. *If X is a space, then the mapping*

$$i_n|_{i_n^{-1}(FP_n(X)\setminus FP_{n-1}(X))} : i_n^{-1}\big(FP_n(X) \setminus FP_{n-1}(X)\big) \longrightarrow FP_n(X) \setminus FP_{n-1}(X)$$

is a homeomorphism.

Proposition 6.4. *If X is a space, then the mapping*

$$i_n|_{i_n^{-1}(AP_n(X)\setminus AP_{n-1}(X))} : i_n^{-1}\big(AP_n(X) \setminus AP_{n-1}(X)\big) \longrightarrow AP_n(X) \setminus AP_{n-1}(X)$$

is an open and closed $n!$ to 1 mapping.

Theorem 6.5. *Let X be a metrizable space. Then $FP(X)$ $(AP(X))$ is σ-closed metrizable and every open covering of $FP(X)$ $(AP(X))$ has a σ-discrete open refinement.*

Proof. Let X be a metrizable space and d is metric on X which is compatible with the topology on X. Then d can be extended to an invariant metric $\hat{d}$ on $FP(X)$. It follows from [14, Theorem 3.2] that, for each $n \in \mathbb{N}$, $C_n = FP_n(X) \setminus FP_{n-1}(X)$ is an open subset of the metric space $(FP_n(X), \hat{d}|_{FP_n(X)})$. Hence we have

$$C_n = \bigcup_{i=1}^{\infty} C_{n,i},$$

where each $C_{n,i}$ is closed in $(FP_n(X), \hat{d}|_{FP_n(X)})$. For each $n, i \in \mathbb{N}$, $C_{n,i}$ is closed in $(FP(X), \hat{d})$, and hence it is closed in $FP(X)$. On the other hand, it follows from Proposition 6.3 that each $C_{n,i}$ is metrizable. Therefore, $FP(X)$ is σ-closed metrizable.

Next, we shall show that every open covering of $FP(X)$ has a σ-discrete open refinement. Let $\mathscr{U}$ be an arbitrary open covering of $FP(X)$. For each $n \in \mathbb{N}$, let

$$\mathscr{U}_n = \{U \cap C_n : U \in \mathscr{U}\},$$

then we can take a σ-discrete closed refinement $\mathscr{H}_n = \bigcup_{i=1}^{\infty} \mathscr{H}_{n,i}$ in $(FP_n(X), \hat{d}|_{FP_n(X)})$ and in $(FP(X), \hat{d})$. Now, for each $n \in \mathbb{N}$ and $H \in \mathscr{H}_n$, choose a set $U(H) \in \mathscr{U}$ such that $H \subset U(H)$. Therefore, for each $n, i \in \mathbb{N}$, there exists a discrete open family $\mathscr{W}_{n,i} = \{W(H) : H \in \mathscr{H}_{n,i}\}$ in $(FP(X), \hat{d})$ such that $H \subset W(H)$ and $W(H) \cap C_n \subset U(H)$ for each $H \in \mathscr{H}_{n,i}$. Let

$$\mathscr{G}_{n,i} = \{W(H) \cap U(H) : H \in \mathscr{H}_{n,i}\} \quad \text{and} \quad \mathscr{G} = \bigcup_{n,i=1}^{\infty} \mathscr{G}_{n,i}.$$

Therefore, it is easy to see that $\mathscr{G}$ is a σ-discrete open refinement of $\mathscr{U}$.

Applying [14, Theorem 3.3] and Proposition 6.4, the proof of analogous assertion for $AP(X)$ is quite similar. □

We don't know if the free paratopological groups $FG(X)$ over metric spaces are regular. Therefore, we have the following question:

Question 6.6. *Let X be an uncountable compact metrizable space. Is $FG(X)$ paracompact? In particular, is $FG([0,1])$ paracompact?*

Definition 6.7. ([3, 7.1.B]) Let X be a Tychonoff space. Then $FP(X)$ and $AP(X)$ are called *free Tychonoff paratopological group* and *free Abelian Tychonoff paratopological group*, respectively.

In [3], A.V. Arhangel'skiĭ and M. Tkachenko posed the following three questions:

Question 6.8. *([3, Open Problem 7.4.3]) Let $FG(X)$ be the free Tychonoff paratopological group of a compact Hausdorff space X. Is $FG(X)$ the direct limit of a countable family of compact spaces? Is $FG(X)$ σ-compact?*

Theorem 6.9. *([3, Theorem 7.5.3]) The following conditions are equivalent for a subset K of the group $G(X)$:*

(1) *K is bounded in $G(X)$;*
(2) *K is precompact in $G(X)$;*
(3) *there exist an integer $n \in \omega$ and bounded subset Y of X such that $K \subset G_n(X, Y)$.*

Question 6.10. *([3, Open Problem 7.5.1]) Can Theorem 6.9 be generalized to the free Tychonoff paratopological group of a Tychonoff space X?*

Question 6.11. *([3, Open Problem 7.5.2]) Is it true that the free Tychonoff paratopological group $FG(X)$ on a Tychonoff space X is σ-bounded if and only if X is σ-bounded?*

Now, we give negative answers to Questions 6.8, 6.10 and 6.11 by the following example.

Example 6.12. Let X be an arbitrary uncountable compact metrizable space. Then we have the following:

(1) Since $-X$ is a closed discrete uncountable subspace in $FP(X)$ or $AP(X)$, $FP(X)$ and $AP(X)$ are not σ-compact and σ-bounded. Of course, it is not the direct limit of a countable family of compact spaces. This give a negative answer to Question 6.8.

(2) Obviously, $-X \subset G_n(X, X)$ and X is bounded, but $-X$ is not bounded. This give a negative answer to Question 6.10.

(3) The space X is bounded and $-X$ is a closed discrete uncountable subspace. Since $-X$ is a closed discrete uncountable subspace in $FP(X)$ or $AP(X)$, $FP(X)$ and $AP(X)$ are not σ-bounded. This give a negative answer to Question 6.11.

The following two questions are still open.

Question 6.13. *Let $FG(X)$ be the Tychonoff free paratopological group of a compact Hausdorff space X. Is $FG(X)$ the direct limit of a countable family of compact spaces? Is $FG(X)$ σ-compact?*

Question 6.14. *Let $FG(X)$ be a Tychonoff free paratopological group of a Tychonoff space X. Are the following conditions equivalent:*

(1) *K is bounded in $FG(X)$;*
(2) *K is precompact in $FG(X)$;*
(3) *there exist an integer $n \in \omega$ and bounded subset Y of X such that $K \subset G_n(X, Y)$.*

Lemma 6.15. *If K is a precompact subset of $FG(X)$, then $Y = \mathrm{supp}(K)$ is bounded in X, and $K \subset G_n(Y, X)$ for some $n \in \mathbb{N}$.*

Proof. By Theorem 6.9 and the continuity of the identity mapping of $FG(X)$ to $G(X)$, it is easy to see that lemma holds. □

By Lemma 6.15, we can obtain the following result which gives a partial answer to Question 6.14.

Theorem 6.16. *Let $FG(X)$ be a Tychonoff free paratopological group of a Tychonoff space X. If K is precompact in $FG(X)$ then there exist an integer $n \in \omega$ and a bounded subset Y of X such that $K \subset G_n(X, Y)$.*

Question 6.17. *Is it true that a Tychonoff free paratopological group $FG(X)$ on a Tychonoff space X is σ-bounded if and only if X is σ-bounded?*

Acknowledgements

The authors wish to thank the reviewers for careful reading preliminary version of this paper and providing many valuable suggestions.

References

[1] A.R. Alimov, The Banach–Mazur theorem for spaces with nonsymmetric distance, Usp. Mat. Nauk 58 (2) (2003) 159–160.

[2] A.V. Arhangel'skiĭ, O.G. Okunev, V.G. Pestov, Free topological groups over metrizable spaces, Topol. Appl. 33 (1989) 63–76.
[3] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., Paris, 2008.
[4] C.J. Borges, On stratifiable spaces, Pac. J. Math. 17 (1) (1966) 1–16.
[5] S. Cobzas, Asymmetric locally convex spaces, Int. J. Math. Math. Sci. 16 (2005) 2585–2608.
[6] A.S. Elfard, P. Nickolas, On the topology of free paratopological groups, Bull. Lond. Math. Soc. 44 (6) (2012) 1103–1115.
[7] A.S. Elfard, P. Nickolas, On the topology of free paratopological groups. II, Topol. Appl. 160 (2013) 220–229.
[8] A.S. Elfard, Neighborhood base at the identity of free paratopological groups, arXiv:1301.2642v1.
[9] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[10] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Marcel Dekker, New York, 1982.
[11] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.
[12] G. Gruenhage, Y. Tanaka, Products of k-spaces and spaces of countable tightness, Trans. Am. Math. Soc. 273 (1982) 299–308.
[13] T.L. Hicks, P.L. Sharma, Properties of z-spaces, Topol. Proc. 4 (1979) 109–113.
[14] F. Lin, A note on free paratopological groups, Topol. Appl. 159 (2012) 3596–3604.
[15] F. Lin, Topological monomorphism between free paratopological groups, Bull. Belg. Math. Soc. Simon Stevin 19 (2012) 507–521.
[16] F. Lin, C. Liu, The mapping i_2 on the free paratopological groups, Publ. Inst. Math. (Belgr.) (2015), in press.
[17] A.A. Markov, On free topological groups, Dokl. Akad. Nauk SSSR 31 (1941) 299–301.
[18] N.M. Pyrch, A.V. Ravsky, On free paratopological groups, Mat. Stud. 25 (2006) 115–125.
[19] N.M. Pyrch, Free paratopological groups and free products of paratopological groups, J. Math. Sci. 174 (2) (2011) 190–195.
[20] S. Romaguera, M. Sanchis, M.G. Tkachenko, Free paratopological groups, Topol. Proc. 27 (2002) 1–28.
[21] V.V. Uspenskij, Free topological groups of metrizable spaces, Math. USSR, Izv. 37 (3) (1991) 657–680.
[22] K. Yamada, Metrizable subspaces of free topological groups on metrizable spaces, Topol. Proc. 23 (1998) 379–409.