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The extensions of paratopological groups ☆

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ABSTRACT

In this paper, we attempt to extend some three space properties in topological groups to paratopological groups. The following results are established: (1) metrizability of compact (resp., sequentially compact, countably compact) subsets is a three space property in the class of k-gentle paratopological groups; (2) let N be a second-countable topological subgroup of an Abelian paratopological group G; if the quotient paratopological group G/N has a countable network, then so does G; (3) let G be an Abelian paratopological group and N a topological subgroup of G; if both N and G/N are first-countable, then so is G.

1. Introduction

Recall that a *semitopological group* is a group with a topology such that the multiplication in the group is separately continuous. A *paratopological group* is a group with a topology such that the multiplication is jointly continuous. If G is a paratopological group and the inverse operation of G is continuous, then G is a *topological group*. The reader can find a lot of recent progress about paratopological and semitopological groups in the survey article [18].

Let $\mathcal{P}$ be a (topological, algebraic, or a mixed nature) property. We consider the following general problem. Let N be a closed invariant subgroup of a paratopological group G such that both N and the quotient paratopological group G/N have $\mathcal{P}$. When can we conclude that G has $\mathcal{P}$? Recall that, for topological

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groups, if the property $\mathcal{P}$ holds in this problem, then we say that $\mathcal{P}$ is a *three space property* [4]. In fact, the list of three space properties in topological groups is quite long. It includes compactness [8], local compactness [16], pseudocompactness [5], metrizability [9], etc. In 2006, M. Bruguera and M. Tkachenko [4] studied some properties of compact, countably compact, pseudocompact, and functionally bounded sets which are preserved or destroyed when taking extensions of topological groups. Recently, the convergence phenomena in the extensions of topological groups were studied in [13].

However, much less is known about three space properties in paratopological groups (see [18]). One of the first results in this direction was established by Ravsky [14] which can now be reformulated as follows: Being a topological group is a three space property in the class of paratopological groups. According to M.I. Graev's theorem that metrizability is a three space property in topological groups [9], it is natural to ask whether the M.I. Graev's theorem can be extended to paratopological groups. The answer is no. There exists a non-metrizable completely regular paratopological Abelian group G which contains a closed discrete subgroup H such that the quotient group G/H is metrizable (see [12, Example 3.3]). However, the following question was posed by A.V. Arhangel'skiĭ and M. Tkachenko [2]:

Question 1.1. *([2, Open problem 7.3.6]) Let f be an open continuous homomorphism of a regular paratopological group G onto a metrizable topological group H such that the kernel of f is metrizable. Is G metrizable?*

In 2013, P. Li and L. Mou [11, Example 3.2] constructed an example[1] which shows that Question 1.1 has a negative answer.

In this paper, we try to study some properties which are preserved or destroyed when taking extensions of paratopological groups. The paper is organized as follows. In Section 2 we investigate the extensions of paratopological groups about compact type sets. We show that the properties of local compactness, compactness, connection, etc, are three space properties in paratopological groups (see Theorem 2.3), and that metrizability of compact (resp., sequentially compact, countably compact) subsets is a three space property in the class of k-gentle paratopological groups (see Theorem 2.7). In Section 3, we study the extensions of Abelian paratopological groups. The following results are established. (1) Let N be a second-countable topological subgroup of an Abelian paratopological group G; if the quotient paratopological group G/N has a countable network, then so does G (see Theorem 3.4). (2) Let G be an Abelian paratopological group and N a topological subgroup of G; if both N and G/N are first-countable, then so is G (see Theorem 3.8).

All spaces are assumed to be Hausdorff, unless otherwise is stated explicitly. Let G be a paratopological group, N a closed (invariant) subgroup of G. Recall that the quotient mapping $\pi : G \to G/N$ is always open. Indeed, for any subset U of G, we have $\pi^{-1}(\pi(U)) = \bigcup\{aN : a \in U\} = UN = \bigcup\{Un : n \in N\}$, and if U is open then so is the union on the right.

The character of a point x (resp., a subset F) in a topological space X is denoted by $\chi(x, X)$ (resp., $\chi(F, X)$), in which $\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base at } x \text{ of } X\} + \omega$, $\chi(F, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a neighborhood base at } F \text{ in } X\} + \omega$. Similar, the character of a topological space X is denoted by $\chi(X)$, in which $\chi(X) = \sup\{\chi(x, X) : x \in X\}$.

2. The extensions of paratopological groups about compact type sets

A topological property $\mathcal{P}$ is called an *inverse fiber property* [4] if (*) $f : X \to Y$ is a continuous and surjective mapping such that both the space Y and the fibers of f have $\mathcal{P}$, then X also has $\mathcal{P}$. If the conclusion in (*) holds under the additional assumption that the domain X is compact (countably compact), we say that $\mathcal{P}$ is an *inverse fiber property for compact (countably compact) sets*.

[1] It is worth mentioning that the authors also independently constructed this example, which was not published.

The following lemma was proved in [4] and [13]. For the sake of completeness we give the proof of the case of sequentially compact sets.

Lemma 2.1. *The first axiom of countability is an inverse fiber property for compact, countably compact [4, Proposition 2.8] and sequentially compact [13, Lemma 2.4] sets.*

Proof. Let $f : X \to Y$ be a continuous and surjective mapping. Suppose that all sequentially compact subsets of the space Y and of the fibers $f^{-1}(y)$ $(\forall y \in Y)$ are first-countable. Take an arbitrary sequentially compact set $C \subset X$. The continuous image K of C is sequentially compact, so that K is first-countable by our assumption. To complete the proof it suffices to show that C is first-countable. Take an arbitrary point $x \in C$, put $y = f(x)$; and let $g = f|_C : C \to K$. From the facts that sequential compactness is inherited by closed sets and that every sequentially compact subset of a first-countable space is closed it follows that g is a closed mapping. The set $C_x = g^{-1}(y) = C \cap f^{-1}(f(x))$ is sequentially compact as a closed subset of C, so that C_x is first-countable by our assumption; as we know that K is first-countable, it follows that $\chi(x, C) \leq \omega$ by [6, 3.7.F]. This proves $\chi(C) \leq \omega$, i.e., C is first-countable. □

Let $\mathcal{P}$ be a topological property. A space X is called $\mathcal{P}$-*compact* if every subset with the property $\mathcal{P}$ of X is compact. The following result was proved in the paper [13]. For the sake of completeness we give its proof.

Lemma 2.2. *If the property $\mathcal{P}$ is preserved by continuous mappings and also inherited by closed sets, then the property $\mathcal{P}$-compact is an inverse fiber property.*

Proof. Let $f : X \to Y$ be a continuous and surjective mapping such that both the space Y and the fibers of f are $\mathcal{P}$-compact. Take an arbitrary $\mathcal{P}$-subset $C \subset X$. The continuous image D of C is a $\mathcal{P}$-subset of Y, so that D is compact by our assumption. Let $g = f|_C : C \to D$. We shall prove that g is a perfect mapping, i.e., g is a closed mapping with compact fibers. For each $y \in D$ the set $g^{-1}(y) = C \cap f^{-1}(y) \subset f^{-1}(y)$ is a $\mathcal{P}$-subset as a closed subset of C, so that $g^{-1}(y)$ is compact by our assumption. On the other hand, if K is closed in C, then K is $\mathcal{P}$-subset in X, so that $f(K) = g(K)$ is $\mathcal{P}$-subset in Y; by our assumption, $g(K)$ is compact, and thus closed in D, because we already know that D is compact. This proves that g is a perfect mappings. From the fact the compactness is inverse invariants of perfect mappings it follows that $C = g^{-1}(D)$ is compact. This completes the proof. □

Theorem 2.3. *The following are three space properties in the class of regular paratopological groups:*

(a) *local compactness;*
(b) *compactness;*
(c) *connectedness;*
(d) *every compact set being first-countable;*
(e) *every countably compact set being first-countable;*
(f) *every sequentially compact set being first-countable;*
(g) *every countably compact set being compact;*
(h) *every sequentially compact set being compact.*

Proof. Let G be a paratopological group and N a closed invariant subgroup of G.

(a) Suppose that both N and the quotient paratopological group G/N are locally compact. It suffices to prove that G is locally compact. From the fact that every locally compact paratopological group is a topological group [18, Theorem 3.3], it follows that both N and G/N are topological groups. Note that the fact that being a topological group is a three space property in the class of paratopological groups [14, Lemma 4], so that G is a topological group. In addition, local compactness is a three space property in the class of topological groups [2, Corollary 3.2.6], so that G is locally compact.

(b) Suppose that both N and the quotient paratopological group G/N are compact. It suffices to prove that G is compact. In view of the proof in (a), we have that G, N and G/N are topological groups. Note that compactness is a three space property in the class of topological groups [2, Corollary 3.2.6], so that G is compact.

(c) Suppose that both N and the quotient paratopological group G/N are connected. It suffices to prove that G is connected. Suppose by contradiction that G is not connected. Thus there exist two non-empty disjoint open sets U, V in G such that $G = V \cup U$. The space N being connected implies that xN is also connected for each $x \in G$, so that either $xN \subset U$ or $xN \subset V$. Let $\pi : G \to G/N$ be the canonical quotient mapping. Thus one readily verifies that $\pi(U)$ and $\pi(V)$ are two non-empty disjoint open sets in G/N such that $G/N = \pi(V) \cup \pi(U)$, which implies that G/N is not connected. This contradiction completes the proof.

The statements of both (d)–(f) and (g)–(h) directly follow from Lemmas 2.1 and 2.2, respectively. □

Remark 2.4.

(1) There exists an example [2, Example 1.5.9] which shows that both (a) and (b) in Theorem 2.3 cannot be extended to semitopological groups.
(2) In view of Theorem 2.3's proof, one can easily extend (c)–(h) to semitopological groups.
(3) Theorem 2.3 improves some results in [2,4,13].

Let $f : X \to Y$ be a mapping. The mapping f is called *k-gentle* if for each compact subset F of X the image $f(F)$ is also compact. A paratopological group G is called k-gentle [1] if the inverse mapping $x \to x^{-1}$ is k-gentle.

Lemma 2.5. *The mapping $g : X \to X^{-1}$ defined by $g(x) = x^{-1}$ is continuous for every k-subspace X of G, where G is a k-gentle paratopological group.*

Proof. Let D be a compact subspace of X. One readily check that the restriction $g_{|D} : D \to D^{-1}$ $(= g(D))$ is continuous, because G is a k-gentle paratopological group. It is well known that a mapping f of a k-space F to a topological space Y is continuous if and only if for every compact subspace $Z \subset F$ the restriction $f_{|Z} : Z \to Y$ is continuous [6, Theorem 3.3.21], so that g is continuous. □

Proposition 2.6. *The following conditions are equivalent for a k-gentle paratopological group G.*

(a) *all compact (resp., sequentially compact, countably compact) subspaces of G are first-countable;*
(b) *all compact (resp., sequentially compact, countably compact) subspaces of G are metrizable.*

Proof. The implication $(b) \Rightarrow (a)$ is obvious; it remains to show that $(a) \Rightarrow (b)$. Suppose that X is an arbitrary non-empty compact (resp., sequentially compact, countably compact) subset of G. By our assumption, X is first-countable, and thus is a k-space; it follows from Lemma 2.5 that the mapping $g : X \to X^{-1}$ defined by $g(x) = x^{-1}$ is continuous, because G is a k-gentle paratopological group. Therefore, the space X^{-1} is compact (resp., sequentially compact, countably compact) as the continuous image of X, so that X^{-1} is first-countable by our assumption. Observe that the compactness, sequential compactness and countable compactness are preserved by the finite Cartesian product in the first-countable spaces, so that XX^{-1} is a compact (resp., sequentially compact, countably compact) subspace as the continuous image of $X \times X^{-1}$, and thus is first-countable by our assumption. Consider the mappings $j : X \times X \to X \times X^{-1}$ and $i : X \times X^{-1} \to XX^{-1}$ defined by $j(x, y) = (x, y^{-1})$ and $i(x, y) = xy$, respectively. One readily check that the mapping $f = i \circ j : X \times X \to XX^{-1}$ is continuous; clearly the identity $e \in XX^{-1}$ with a local countable base in XX^{-1}, we have that $\Delta = f^{-1}(e)$ is a G_δ-set in $X \times X$, i.e., X has a G_δ-diagonal, so that it follows from the fact that every Hausdorff countably compact space with a G_δ-diagonal is metrizable [10, Corollary 7.6] that X is metrizable, because both compactness and sequential compactness imply countable compactness. □

Theorem 2.7. *Metrizability of compact (resp., sequentially compact, countably compact) subsets is a three space property in the class of k-gentle paratopological groups.*

Proof. Let G be a k-gentle paratopological group and N a closed invariant subgroup of G such that all compact (resp., sequentially compact, countably compact) subsets in both N and the quotient paratopological group G/N are metrizable. From Theorem 2.3 it follows that all compact (resp., sequentially compact, countably compact) subsets of G are first-countable, so the statement directly follows from Proposition 2.6. □

Clearly, every topological group is a k-gentle paratopological group, so we have the following:

Corollary 2.8. *Metrizability of compact [4, Theorem 3.2] (resp., sequentially compact [13, (e) of Theorem 2.6], countably compact [4, Corollary 3.3]) subsets is a three space property in the class of topological groups.*

Let X be a topological space. A subset A of X is called *sequentially closed* if no sequence of points of A converges to a point not in A. X is called *sequential* [7] if each sequentially closed subset of X is closed. A space X is called *Fréchet at a point* $x \in X$ if $x \in \overline{A} \subset X$ there is a sequence $\{x_n\}_n$ in A such that $\{x_n\}_n$ converges to x in X. A space X is called *Fréchet* [7] if it is Fréchet at every point $x \in X$. A space X is called *strongly Fréchet at a point* $x \in X$ if whenever $\{A_n\}_n$ is a decreasing sequence of subsets in X and $x \in \bigcap_{n\in\omega} \overline{A_n}$, there exists $x_n \in A_n$ for each $n \in \omega$ such that the sequence $x_n \to x$. A space X is called *strongly Fréchet* [17] if it is strongly Fréchet at every point $x \in X$. Fréchet spaces (resp., strongly Fréchet spaces) are also called Fréchet–Urysohn spaces (resp., strongly Fréchet–Urysohn spaces).

It is obvious that first-countable spaces $\Longrightarrow$ strongly Fréchet spaces $\Longrightarrow$ Fréchet spaces $\Longrightarrow$ sequential spaces.

Lemma 2.9. *([13, Lemma 2.3]) If all countably compact (resp., sequentially compact) subsets of a topological space X are sequential, then all countably compact (resp., sequentially compact) subsets of X are closed.*

Proposition 2.10. *If every compact (resp., countably compact, sequentially compact) subspace of a k-gentle paratopological group G is Fréchet, then every compact (resp., countably compact, sequentially compact) subspace of G is strongly Fréchet.*

Proof. Let A be a compact (resp., countably compact, sequentially compact) subset of G. By our assumption, A is Fréchet, and thus is closed in G by Lemma 2.9. Suppose that $\{A_n\}_{n\in\omega}$ is a decreasing sequence of subsets in A with $a \in \bigcap_{n\in\omega} \overline{A_n}$. We can assume that a is an accumulation point of A. By the Fréchet property of A one can find a sequence $\{a_n\}_{n\in\omega}$ in $A \setminus \{a\}$ converging to a. Clearly the set $B = a^{-1}A$ is closed, compact (resp., countably compact, sequentially compact) and the sequence $\{a^{-1}a_n\}_{n\in\omega}$ converges to e, where e is the identity of G, so that we have, for each $n \in \omega$, the closure of the set $B_n = a^{-1}A_n$ being included in B, $e \in \overline{B_n}$ and $b_n = a^{-1}a_n \in B \setminus \{e\}$; since G is a Hausdorff paratopological group, for each $n \in \omega$ we can find an open set V_n containing e such that $V_n \cap b_nV_n = \emptyset$. By $e \in \overline{B_n}$ we have $e \in \overline{B_n \cap V_n}$;

putting $C_n = b_n(B_n \cap V_n)$, and therefore, $b_n \in \overline{C_n}$, but $e \notin \overline{C_n}$, because $V_n \cap C_n \subset V_n \cap b_n V_n = \emptyset$, for each $n \in \omega$. Put

$$D = \bigcup\{C_n : n \in \omega\}, \quad \text{and} \quad S = \{e\} \cup \{b_n : n \in \omega\}.$$

Then $D \subset \bigcup_{n\in\omega} b_n B_n \subset SB$.

Observe that the Cartesian product of two compact (resp., sequentially compact, countably compact) spaces, if one of which is first-countable, is compact (resp., sequentially compact, countably compact), so that the Cartesian product $S \times B$ of the spaces S and B is compact (resp., countably compact, sequentially compact), because S is compact and metrizable; thus SB is compact (resp., countably compact, sequentially compact) as the continuous image of $S \times B$, furthermore, by our assumption we have that SB is Fréchet, and thus is closed by Lemma 2.9.

By $b_n \in \overline{C_n}$ for each $n \in \omega$ and $b_n \to e$, we have $e \in \overline{D} \subset SB$, so that one can find a sequence $\{d_k\}_{k\in\omega}$ in D converging to e by the Fréchet property of SB; in addition, as we already know $e \notin \overline{C_n}$ for each $n \in \omega$, the set C_n contains only finitely many terms of the sequence $\{d_k\}_{k\in\omega}$; thus we can assume that there is a subsequence $\{C_{n_k}\}_{k\in\omega}$ of the sequence $\{C_n\}_{n\in\omega}$ such that $d_k \in C_{n_k}$ for each $k \in \omega$. By $C_{n_k} \subset b_{n_k} B_{n_k} = b_{n_k} a^{-1} A_{n_k}$, for each $k \in \omega$ we have $d_k = b_{n_k} a^{-1} x_{n_k}$ for some $x_{n_k} \in A_{n_k}$; one readily check that $b_{n_k}^{-1} \to e$, because $b_{n_k} \to e$ and G is a k-gentle paratopological group, so that $x_{n_k} = a b_{n_k}^{-1} d_k \to a$ when $k \to \infty$. Take $x_n = x_{n_k}$ when $n_{k-1} < n < n_k$, then $x_n \in A_n$ for each $n \in \omega$ and $x_n \to a$. Hence, A is strongly Fréchet. □

Lemma 2.11. *([3, Proposition 2.18]) Suppose that X is a regular space, and that $f : X \to Y$ is a closed mapping. Suppose also that $b \in X$ is a G_δ-point in the space $F = f^{-1}(f(b))$ (i.e., the singleton $\{b\}$ is a G_δ-set in the space F) and F is Fréchet at b (resp., strictly Fréchet at b). If the space Y is strongly Fréchet, then X is Fréchet at b.*

Theorem 2.12. *Let G be a regular k-gentle paratopological group and H a closed subgroup of G such that all compact (resp., countably compact, sequentially compact) subsets of the paratopological group H are first-countable. If the quotient space G/H has the following property, then so does the paratopological group G.*

(a) *all compact (resp., countably compact, sequentially compact) subsets are strongly Fréchet.*

Proof. From Lemma 2.11 it follows that all compact (resp., countably compact, sequentially compact) subsets in G are Fréchet. Therefore, the statement directly follows from Proposition 2.10. □

3. Extensions of Abelian paratopological groups

In this section, we investigate the extensions of Abelian paratopological groups. The following fact is known ([2, Problems 4.6.C]).

Let H be a closed and second-countable subgroup of a topological group G. If the quotient space G/H has a countable network, then so does G.

This result cannot be extended to paratopological groups as the following shows.

Example 3.1. There exists an Abelian paratopological group G which does not have a countable network, but it contains a separable and metrizable subgroup H such that the quotient space G/H has a countable network.

Proof. In [11, Example 3.2] it is provided a first-countable, non-metrizable regular Abelian paratopological group G and an open continuous homomorphism f of G onto a separable and metrizable topological group H such that the kernel of f is separable and metrizable paratopological group. Clearly, H is a quotient group of G with a countable network. However, G does not have a countable network, for otherwise it would follow from [15, Proposition 2.3] that G would be second-countable. It is well known that every regular space with a countable base is metrizable, so G is metrizable. This contradiction completes the proof. □

However, we have the following Proposition 3.3 and Theorem 3.4.

Lemma 3.2. *Suppose that G is a paratopological group and H is a separable subgroup of G. If Y is a separable subset of G/H, then $\pi^{-1}(Y)$ is also separable in G, where π is the natural homomorphism of G onto the quotient space G/H.*

Proof. Let B be a countable, dense subset of Y. Since H is separable, there exists a countable, dense subset M_b of $\pi^{-1}(b)$ for each $b \in B$. Put $M = \bigcup_{b\in B} M_b$. Then M is a countable subset of $\pi^{-1}(Y)$, and is dense in $\pi^{-1}(B)$. Since π is an open mapping of G onto G/H, π restricting to $\pi^{-1}(Y)$ is also an open mapping of $\pi^{-1}(Y)$ onto Y. Therefore, $\overline{M} = \overline{\pi^{-1}(B)} = \pi^{-1}(\overline{B}) = \pi^{-1}(Y)$ by [2, Lemma 1.5.22]. Hence, M is dense in $\pi^{-1}(Y)$, i.e., $\pi^{-1}(Y)$ is separable. □

Proposition 3.3. *Let N be a second-countable topological subgroup of an Abelian paratopological group G. If the subspace Y of G/N has a countable network, then so does $\pi^{-1}(Y)$, where π is the natural homomorphism of G onto the quotient space G/N.*

Proof. Let e be the identity of G. Since both N and Y have a countable network, it follows from Lemma 3.2 that $X = \pi^{-1}(Y)$ is separable. We fix a countable, dense subset $\{b_n : n \in \omega\}$ in X. One can find a decreasing sequence $\{U_n : n \in \omega\}$ of symmetric open sets in G satisfying that $U^2_{n+1} \subset U_n$ for each $n \in \omega$ and that $\{U_n \cap N : n \in \omega\}$ is a base at e of open neighbourhoods in N. In fact, since N is a first-countable topological group and G is an Abelian paratopological group, one can easily find a decreasing sequence $\{U'_n : n \in \omega\}$ of open sets in G such that $U'^2_{n+1} \subset U'_n$ for each $n \in \omega$ and that $\{U'_n \cap N : n \in \omega\}$ is a base at e of symmetric open neighbourhoods in N. We put $U_n = U'_n U'^{-1}_n$ for each $n \in \omega$. Since G is Abelian and every element of $\{U'_n \cap N : n \in \omega\}$ is symmetric, one readily check that the sequence $\{U_n : n \in \omega\}$ is the required.

Fix a countable network $\{P_n : n \in \omega\}$ in Y. To finish the proof, it is sufficient to establish the following:

Claim 1. *$\{\pi^{-1}(P_i) \cap U_k b_j : i, j, k \in \omega\}$ is a network in X.*

Take any $g \in X$ and open set V in G such that $g \in V \cap X$. Since G is a paratopological group, one can find an open neighborhood O at e in G such that $O^2 g \subset V$. From the fact that $\{U_m \cap N : m \in \omega\}$ is a local base for N at e it follows that there exists $i \in \omega$ such that $U_i g \cap gN \subset Og$. Since $\{b_n : n \in \omega\}$ is dense in X and $\{P_n : n \in \omega\}$ is a network for Y, there exist $k_1, k_2 \in \omega$ such that $b_{k_1} \in U_{i+2} g$ and $\pi(g) \in P_{k_2} \subset \pi(U_{i+1} g \cap Og) \cap Y$. To proof of Claim 1, it suffices to establish the following:

Claim 2. *$g \in \pi^{-1}(P_{k_2}) \cap U_{i+2} b_{k_1} \subset V \cap X$.*

First, clearly, $g \in \pi^{-1}(P_{k_2})$. Since $U_{i+2} = U^{-1}_{i+2}$ and $b_{k_1} \in U_{i+2} g$, we have $g \in U_{i+2} b_{k_1}$. Thus $g \in \pi^{-1}(P_{k_2}) \cap U_{i+2} b_{k_1}$.

Secondly, take any $z \in \pi^{-1}(P_{k_2}) \cap U_{i+2} b_{k_1}$. Since $P_{k_2} \subset \pi(U_{i+1} g \cap Og) \cap Y$, we have $z \in \pi^{-1}(P_{k_2}) \subset (U_{i+1} g \cap Og)N \cap X = (U_{i+1} \cap O)gN \cap X$. In additional, $z \in U_{i+2} b_{k_1}$ and $b_{k_1} \in U_{i+2} g$, thus, we have $z \in U_{i+2} b_{k_1} \subset U_{i+2} U_{i+2} g \subset U_{i+1} g \subset U^2_{i+1} g \subset U_i g$. It implies that $U^2_{i+1} g \cap (G \setminus U_i g) = \emptyset$, which is equivalent to $U_{i+1} g \cap U_{i+1}(G \setminus U_i g) = \emptyset$ by $U_{i+1} = U^{-1}_{i+1}$. Thus, we have $z \notin U_{i+1}(G \setminus U_i g)$, in particular,

$z \notin (U_{i+1} \cap O)(G \setminus U_i g)$. Since $z \in (U_{i+1} \cap O)gN$, one can easily obtain that $z \in (U_{i+1} \cap O)(gN \cap U_i g)$, which implies that $z \in (U_{i+1} \cap O)(gN \cap U_i g) \cap X \subset O^2 g \cap X \subset V \cap X$ by $(gN \cap U_i g) \subset Og \subset V$. □

According to Proposition 3.3, the following is obvious.

Theorem 3.4. *Let N be a second-countable topological subgroup of an Abelian paratopological group G. If the quotient paratopological group G/N has a countable network, then so does G.*

Remark 3.5. (a) The condition 'N is second-countable' in Theorem 3.4 cannot be weakened to 'N has a countable network', since there is an Abelian topological group G which does not have a countable network, but it contains a closed subgroup H with a countable network such that the quotient paratopological group G/H has also a countable network [19].

(b) From Example 3.1 it follows that the condition 'N is a topological group' cannot be replaced by 'N is a paratopological group' in Theorem 3.4.

Question 3.6. *Can the condition 'Abelian' in Theorem 3.4 be omitted?*

Corollary 3.7. *Let N be a second-countable locally compact subgroup of an Abelian paratopological group G. If the quotient paratopological group G/N has a countable network, then so does G.*

Proof. Since every locally compact paratopological group is a topological group [18, Theorem 3.3], the statement directly follows from Theorem 3.4. □

Theorem 3.8. *Let G be an Abelian paratopological group and N a topological subgroup of G. If both N and G/N are first-countable, then so is G.*

Proof. Let $\pi : G \to G/N$ be a natural quotient mapping and e the identity in G. Since G is a paratopological group and N a first-countable topological group, one can easily find a decreasing sequence $\{W'_n : n \in \omega\}$ of open sets in G such that $W'^2_{n+1} \subset W'_n$ for each $n \in \omega$ and that $\{W'_n \cap N : n \in \omega\}$ is a base at e of symmetric open neighbourhoods in N. We put $W_n = W'_n W'^{-1}_n$ for each $n \in \omega$. Since G is an Abelian paratopological group, the decreasing sequence $\{W_n : n \in \omega\}$ of symmetric open sets satisfies that $W^2_{n+1} \subset W_n$ for each $n \in \omega$ and that $\{W_n \cap N : n \in \omega\}$ is a base at e of open neighbourhoods in N. We also fix a sequence $\{U_n : n \in \omega\}$ at e of open neighbourhoods in G such that $\{\pi(U_n) : n \in \omega\}$ is a base at $\pi(e)$ in G/N. Now put $B_{i,j} = W_i \cap U_j$ for $i, j \in \omega$. To finish the proof, it suffices to establish the following:

Claim. *The family $\eta = \{B_{i,j} : i, j \in \omega\}$ is a base for G at e.*

Take any open neighbourhood O in G at e. Then there is an open set V in G such that $e \in V \subset V^2 \subset O$. Since the sequence $\{W_n \cap N : n \in \omega\}$ is a base at e of open neighborhoods in N, there is $m \in \omega$ such that $W_m \cap N \subset V \cap N \subset V$. Similarly, we can also find $j \in \omega$ such that $\pi(U_j) \subset \pi(W_{m+1} \cap V)$, since π is an open mapping. We claim that $B_{m+1,j} = W_{m+1} \cap U_j \subset O$, which completes the proof of Claim above.

Take any $y \in B_{m+1,j}$. We have $y \in (W_{m+1} \cap V)N$ by $\pi(y) \in \pi(U_j) \subset \pi(W_{m+1} \cap V)$. By $W_{m+1} = W^{-1}_{m+1}$ and $W^2_{m+1} \subset W_m$, we have $y \notin W_{m+1}(G \setminus W_m)$ by $y \in W_{m+1}$. Therefore, $y \in (W_{m+1} \cap V)(W_m \cap N) \subset V^2 \subset O$. □

Corollary 3.9. *Let N be a topological subgroup of an Abelian paratopological group G. If both N and the quotient paratopological group G/N are second-countable, then so is G.*

Proof. From Theorems 3.4 and 3.8 it follows that G is first-countable with a countable network. Thus, the statement directly follows from [15, Proposition 2.3]. □

Corollary 3.10. *Let N be a locally compact subgroup of an Abelian paratopological group G. If both N and the quotient paratopological group G/N are second-countable, then so is G.*

Proof. Since every locally compact paratopological group is a topological group [18, Theorem 3.3], the statement directly follows from Corollary 3.9. □

It is worth mentioning that there exists an Abelian paratopological group G containing a locally compact subgroup H such that both H and the quotient space G/H are metrizable, but G is not metrizable [12]. Therefore, the condition 'second-countable' in Corollaries 3.9 and 3.10 cannot be replaced by 'metrizable'.

Corollary 3.11. *Let G be an Abelian paratopological group and N a locally compact subgroup of G. If both N and G/N are first-countable, then so is G.*

Proof. Since every locally compact paratopological group is a topological group [18, Theorem 3.3], the statement directly follows from Theorem 3.8. □

Question 3.12. *(a) Can the condition 'Abelian' in Theorem 3.8 be omitted? (b) Can the condition 'N is a topological subgroup' be replaced by 'N is a paratopological subgroup' in Theorem 3.8?*

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