Houston Journal of Mathematics

Volume 40, No. 3, 2014

SEQUENCE-COVERING MAPS ON GENERALIZED METRIC SPACES

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Communicated by Yasunao Hattori

Abstract. Let $f : X \to Y$ be a map. f is a *sequence-covering map* [25] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$; f is an *1-sequence-covering map* [14] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. In this paper, we mainly discuss the sequence-covering maps on generalized metric spaces, and give an affirmative answer to a question in [13] and some related questions, which improve some results in [13, 16, 28], respectively. Moreover, we also prove that open and closed maps preserve strongly monotonically monolithity, and closed sequence-covering maps preserve spaces with a σ-point-discrete k-network. Some questions about sequence-covering maps on generalized metric spaces are posed.

1. Introduction

A study of images of topological spaces under certain sequence-covering maps is an important question in general topology [9, 11, 12, 13, 15, 18, 19, 20, 28]. S. Lin and P.F. Yan proved that each sequence-covering and compact map on metric spaces is an 1-sequence-covering map [18] . Recently, the authors proved that each sequence-covering and boundary-compact map on metric spaces is an

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2000 *Mathematics Subject Classification.* 54C10; 54E40; 54E99.

Key words and phrases. Sequence-covering maps; boundary-compact maps; closed maps; point-countable bases; g-metrizable; strongly monotonically monolithic spaces; σ-point-discrete k-network.

Supported by the NSFC (No. 10971185, 11201414) and the Natural Science Foundation of Fujian Province (No. 2012J05013) of China.

1-sequence-covering map [13]. Also, the authors posed the following question in [13] :

Question 1.1. [13, Question 3.6] *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. Is f an 1-sequence-covering map if X is a space with a point-countable base or a developable space?*

In this paper, we shall give an affirmative answer to Question 1.1.

In [16], the second author proved that if X is a metrizable space and f is a sequence-quotient and compact map, then f is a pseudo-sequence-covering map. Recently, the authors proved that if X is a metrizable space and f is a sequence-quotient and boundary-compact map, then f is a pseudo-sequence-covering map [13]. Hence we have the following Question 1.2.

Question 1.2. *Let $f : X \to Y$ be a sequence-quotient and boundary-compact map. Is f a pseudo-sequence-covering map if X is a space with a point-countable base or a developable space?*

On the other hand, P.F. Yan, S. Lin and S.L. Jiang proved that each closed sequence-covering map on metric spaces is an 1-sequence-covering map [28]. Hence we have the following Question 1.3.

Question 1.3. *Let $f : X \to Y$ be a closed sequence-covering map. Is f an 1-sequence-covering map if X is a regular space with a point-countable base or a developable space?*

In this paper, we shall we give an affirmative answer to Question 1.2, which improves some results in [13] and [16], respectively. Moreover, we give an affirmative answer to Question 1.3 when X has a point-countable base or X is g-metrizable. In [27], V.V. Tkachuk introduced the strongly monotonically monolithic spaces. In this paper, we also prove that strongly monotonically monolithities are preserved by open and closed maps, and spaces with a σ-point-discrete k-network are preserved by closed sequence-covering maps.

2. Definitions and terminology

Let X be a space. For $P \subset X$, P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P.

Definition 1. Let $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (b) $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathcal{P}_x$.

(1)$\mathcal{P}$ is called an *sn-network* for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. X is called *snf-countable* [15], if X has an sn-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

(2)$\mathcal{P}$ is called a *weak base* [1] for X if $G \subset X$ is open in X if and only if for each $x \in G$ there is a $P \in \mathcal{P}_x$ with $P \subset G$. X is *g-metrizable* [26] if X is regular and has a σ-locally finite weak base.

Definition 2. Let $f : X \to Y$ be a map.

(1) f is a *compact* (resp. *separable*) map if each $f^{-1}(y)$ is compact (separable) in X;
(2) f is a *boundary-compact*(resp. *boundary-separable*) map if each $\partial f^{-1}(y)$ is compact (separable) in X;
(3) f is a *sequence-covering map* [25] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$;
(4) f is an *1-sequence-covering map* [14] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$;
(5) f is a *sequentially quotient map* [5] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$;
(6) f is a *pseudo-sequence-covering map* [9, 10] if for each convergent sequence L in Y there is a compact subset K in X such that $f(K) = \overline{L}$;

It is obvious that

$$\begin{tikzcd} & & \text{pseudo-sequence-covering maps} \\ \text{1-sequence-covering maps} \arrow[r] & \text{sequence-covering maps} \arrow[ur] \arrow[dr] & \\ & & \text{sequential quotient maps.} \end{tikzcd}$$

Remind readers attention that the sequence-covering maps defined the above-mentioned are different from the sequence-covering maps defined in [9], which is called pseudo-sequence-covering maps in this paper.

Definition 3. [23] Let A be a subset of a space X. We call an open family $\mathcal{N}$ of subsets of X is an *external base* of A in X if for any $x \in A$ and open subset U with $x \in U$ there is a $V \in \mathcal{N}$ such that $x \in V \subset U$.

Similarly, we can define an *external weak base* for a subset A for a space X.

Throughout this paper all spaces are assumed to be Hausdorff, all maps are continuous and onto. The letter $\mathbb{N}$ will denote the set of positive integer numbers. Readers may refer to [6, 8, 15] for unstated definitions and terminology.

3. Sequence-covering and boundary-compact maps

Let Ω be the class of all topological spaces such that, for each compact subset $K \subset X \in \Omega$, K is metrizable and also has a countable neighborhood base in X. Indeed, E.A. Michael and K. Nagami in [23] has proved that $X \in \Omega$ if and only if X is the image of some metric space under an open and compact-covering[1] map. It is easy to see that if a space X is developable or has a point-countable base, then $X \in \Omega$ (see [4] and [27], respectively).

In this paper, when we say an snf-countable space Y, it is always assumed that Y has an sn-network $\mathcal{P} = \cup\{\mathcal{P}_y : y \in Y\}$ such that $\mathcal{P}_y$ is countable and closed under finite intersections for each point $y \in Y$.

Lemma 3.1. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where Y is snf-countable. For each non-isolated point $y \in Y$, there exists a point $x_y \in \partial f^{-1}(y)$ such that whenever U is an open subset with $x_y \in U$, there exists a $P \in \mathcal{P}_y$ satisfying $P \subset f(U)$*

Proof. Suppose not, there exists a non-isolated point $y \in Y$ such that for every point $x \in \partial f^{-1}(y)$, there is an open neighborhood U_x of x such that $P \not\subseteq f(U_x)$ for every $P \in \mathcal{P}_y$. Then $\partial f^{-1}(y) \subset \cup\{U_x : x \in \partial f^{-1}(y)\}$. Since $\partial f^{-1}(y)$ is compact, there exists a finite subfamily $\mathcal{U} \subset \{U_x : x \in \partial f^{-1}(y)\}$ such that $\partial f^{-1}(y) \subset \cup\mathcal{U}$. We denote $\mathcal{U}$ by $\{U_i : 1 \leq i \leq n_0\}$. Assume that $\mathcal{P}_y = \{P_n : n \in \mathbb{N}\}$ and $\mathcal{W}_y = \{F_n = \bigcap_{i=1}^{n} P_i : n \in \mathbb{N}\}$. It is obvious that $\mathcal{W}_y \subset \mathcal{P}_y$ and $F_{n+1} \subset F_n$, for every $n \in \mathbb{N}$. For each $1 \leq m \leq n_0, n \in \mathbb{N}$, it follows that there exists $x_{n,m} \in F_n \setminus f(U_m)$. Then denote $y_k = x_{n,m}$, where $k = (n-1)n_0 + m$. Since $\mathcal{P}_y$ is a network at point y and $F_{n+1} \subset F_n$ for every $n \in \mathbb{N}$, $\{y_k\}$ is a sequence converging to y in Y. Because f is a sequence-covering map, $\{y_k\}$ is an image of some sequence $\{x_k\}$ converging to $x \in \partial f^{-1}(y)$ in X. From $x \in \partial f^{-1}(y) \subset \cup\mathcal{U}$ it follows that there

[1]Let $f : X \to Y$ be a map. f is called a *compact-covering map* [23] if in case L is compact in Y there is a compact subset K of X such that $f(K) = L$.

exists $1 \leq m_0 \leq n_0$ such that $x \in U_{m_0}$. Therefore, $\{x\} \cup \{x_k : k \geq k_0\} \subset U_{m_0}$ for some $k_0 \in \mathbb{N}$. Hence $\{y\} \cup \{y_k : k \geq k_0\} \subset f(U_{m_0})$. However, we can choose an $n > k_0$ such that $k = (n-1)n_0 + m_0 \geq k_0$ and $y_k = x_{n,m_0}$, which implies that $x_{n,m_0} \in f(U_{m_0})$. This contradictions to $x_{n,m_0} \in F_n \setminus f(U_{m_0})$. □

The next lemma is obvious.

Lemma 3.2. *Let $f : X \to Y$ be 1-sequence-covering, where X is snf-countable. Then Y is snf-countable.*

Theorem 3.3. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where X is first-countable. Then Y is snf-countable if and only if f is an 1-sequence-covering map.*

Proof. Necessity. Let y be a non-isolated point in Y. Since Y is snf-countable, it follows from Lemma 3.1 that there exists a point $x_y \in \partial f^{-1}(y)$ such that whenever U is an open neighborhood of x_y, there is a $P \in \mathcal{P}_y$ satisfying $P \subset f(U)$. Let $\{B_n : n \in \mathbb{N}\}$ be a countable neighborhood base at point x_y such that $B_{n+1} \subset B_n$ for each $n \in \mathbb{N}$. Suppose that $\{y_n\}$ is a sequence in Y, which converges to y. Next, we take a sequence $\{x_n\}$ in X as follows.

Since B_n is an open neighborhood of x_y, it follows from the Lemma 3.1 that there exists a $P_n \in \mathcal{P}_y$ such that $P_n \subset f(B_n)$ for each $n \in \mathbb{N}$. Because every $P \in \mathcal{P}_y$ is a sequential neighborhood, it is easy to see that for each $n \in \mathbb{N}$, $f(B_n)$ is a sequential neighborhood of y in Y. Therefore, for each $n \in \mathbb{N}$, there is an $i_n \in \mathbb{N}$ such that $y_i \in f(B_n)$ for every $i \geq i_n$. Suppose that $1 < i_n < i_{n+1}$ for every $n \in \mathbb{N}$. Hence, for each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap B_n, & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

We denote $S = \{x_j : j \in \mathbb{N}\}$. It is easy to see that S converges to x_y in X and $f(S) = \{y_n\}$. Therefore, f is an 1-sequence-covering map.

Sufficiency. It easy to see that Y is snf-countable by Lemma 3.2. □

We don't know whether, in Theorem 3.3, f is an 1-sequence-covering map when X is only first-countable. However, we have the following Theorem 3.6, which gives an affirmative answer to Question 1.1. Firstly, we give some technique lemmas.

Lemma 3.4. [23] *If $X \in \Omega$, then every compact subset of X has a countable external base.*

Lemma 3.5. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. If $X \in \Omega$, then Y is snf-countable.*

Proof. Let y be a non-isolated point for Y. Then $\partial f^{-1}(y)$ is non-empty and compact for X. Therefore, $\partial f^{-1}(y)$ has a countable external base $\mathcal{U}$ in X by Lemma 3.4. Let

$$\mathcal{V} = \{\cup\mathcal{F} : \text{There is a finite subfamily } \mathcal{F} \subset \mathcal{U} \text{ with } \partial f^{-1}(y) \subset \cup\mathcal{F}\}.$$

Obviously, $\mathcal{V}$ is countable. We now prove that $f(\mathcal{V})$ is a countable sn-network at point y.

(1) $f(\mathcal{V})$ is a network at y.

Let $y \in U$. Obviously, $\partial f^{-1}(y) \subset f^{-1}(U)$. For each $x \in \partial f^{-1}(y)$, there exist an $U_x \in \mathcal{U}$ such that $x \in U_x \subset f^{-1}(U)$. Therefore, $\partial f^{-1}(y) \subset \cup\{U_x : x \in \partial f^{-1}(y)\}$. Since $\partial f^{-1}(y)$ is compact, it follows that there exists a finite subfamily $\mathcal{F} \subset \{U_x : x \in \partial f^{-1}(y)\}$ such that $\partial f^{-1}(y) \subset \cup\mathcal{F} \subset f^{-1}(U)$. It is easy to see that $F \in \mathcal{V}$ and $y \in \cup f(\mathcal{F}) \subset U$.

(2) For any $P_1, P_2 \in f(\mathcal{V})$, there exists a $P_3 \in f(\mathcal{V})$ such that $P_3 \subset P_1 \cap P_2$.

It is obvious that there exist $V_1, V_2 \in \mathcal{V}$ such that $f(V_1) = P_1, f(V_2) = P_2$, respectively. Since $\partial f^{-1}(y) \subset V_1 \cap V_2$, it follows from the similar proof of (1) that there exists a $V_3 \in \mathcal{V}$ such that $\partial f^{-1}(y) \subset V_3 \subset V_1 \cap V_2$. Let $P_3 = f(V_3)$. Hence $P_3 \subset f(V_1 \cap V_2) \subset f(V_1) \cap f(V_2) = P_1 \cap P_2$.

(3) For each $P \in f(\mathcal{V})$, P is a sequential neighborhood of y.

Let $\{y_n\}$ be any sequence in Y which converges to y in Y. Since f is a sequence-covering map, $\{y_n\}$ is the image of some sequence $\{x_n\}$ converging to $x \in \partial f^{-1}(y) \subset X$. It follows from $P \in f(\mathcal{V})$ that there exists a $V \in \mathcal{V}$ such that $P = f(V)$. Therefore, $\{x_n\}$ is eventually in V, and this implies that $\{y_n\}$ is eventually in P.

Therefore, $f(\mathcal{V})$ is a countable sn-network at point y. $\square$

Theorem 3.6. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. If $X \in \Omega$, then f is an 1-sequence-covering map.*

Proof. From Lemma 3.5 it follows that Y is snf-countable. Therefore, f is an 1-sequence-covering map by Theorem 3.3. $\square$

By Theorem 3.6, it easily follows the following Corollary 3.7, which gives an affirmative answer to Question 1.1.

Corollary 3.7. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. Suppose also that at least one of the following conditions holds:*

(1) *X has a point-countable base;*

(2) *X is a developable space.*

Then f is an 1-sequence-covering map.

Lemma 3.8. *Let $f : X \to Y$ be a sequence-covering map, where Y is snf-countable and $\partial f^{-1}(y)$ has a countable external base for each point $y \in Y$. Then, for each non-isolated point $y \in Y$, there exists a point $x_y \in \partial f^{-1}(y)$ such that whenever U is an open subset with $x_y \in U$, there exists a $P \in \mathcal{P}_y$ satisfying $P \subset f(U)$*

Proof. Suppose not, there exists a non-isolated point $y \in Y$ such that for every point $x \in \partial f^{-1}(y)$, there is an open neighborhood U_x of x such that $P \not\subseteq f(U_x)$ for every $P \in \mathcal{P}_y$. Let $\mathcal{B}$ be a countable external base for $\partial f^{-1}(y)$. Therefore, for each $x \in \partial f^{-1}(y)$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x$. For each $x \in \partial f^{-1}(y)$, it follows that $P \not\subseteq f(B_x)$ whenever $P \in \mathcal{P}_y$. Assume that $\mathcal{P}_y = \{P_n : n \in \mathbb{N}\}$ and $\mathcal{W}_y = \{F_n = \bigcap_{i=1}^n P_i : n \in \mathbb{N}\}$. We denote $\{B_x \in \mathcal{B} : x \in \partial f^{-1}(y)\}$ by $\{B_m : m \in \mathbb{N}\}$. For each $n, m \in \mathbb{N}$, it follows that there exists $x_{n,m} \in F_n \setminus f(B_m)$. For $n \geq m$, we denote $y_k = x_{n,m}$ with $k = m + n(n-1)/2$. Since $\mathcal{P}_y$ is a network at point y and $F_{n+1} \subset F_n$ for every $n \in \mathbb{N}$, $\{y_k\}$ is a sequence converging to y in Y. Because f is a sequence-covering map, $\{y_k\}$ is an image of some sequence $\{x_k\}$ converging to $x \in \partial f^{-1}(y)$ in X. From $x \in \partial f^{-1}(y) \subset \cup\{B_m : m \in \mathbb{N}\}$ it follows that there exists a $m_0 \in \mathbb{N}$ such that B_{m_0} is an open neighborhood at x. Therefore, $\{x\} \cup \{x_k : k \geq k_0\} \subset B_{m_0}$ for some $k_0 \in \mathbb{N}$. Hence $\{y\} \cup \{y_k : k \geq k_0\} \subset f(B_{m_0})$. However, we can choose a $k \geq k_0$ and an $n \geq m_0$ such that $y_k = x_{n,m_0}$, which implies that $x_{n,m_0} \in f(B_{m_0})$. This is a contradiction to $x_{n,m_0} \in F_n \setminus f(B_{m_0})$. □

Theorem 3.9. *Let $f : X \to Y$ be a sequence-covering and boundary-separable map. If X has a point-countable base and Y is snf-countable, then f is an 1-sequence-covering map.*

Proof. Obviously, $\partial f^{-1}(y)$ has a countable external base for each point $y \in Y$. Therefore, it is easy to see by Lemma 3.8 and the proof of Theorem 3.3. □

Remark We can't omit the condition "Y is snf-countable" in Theorem 3.9. Indeed, the sequence fan[2] S_ω is the image of metric spaces under the sequence-covering s-maps by [15, Corollary 2.4.4]. However, S_ω is not snf-countable, and therefore, S_ω is not the image of metric spaces under an 1-sequence-covering map.

In this section, we finally give an affirmative answer to Question 1.2.

[2] S_ω is the space obtained from the topological sum of ω many copies of the convergent sequence by identifying all the limit points to a point.

Lemma 3.10. [5] *Let $f : X \to Y$ be a map. If X is a Fréchet space*[3]*, then f is a pseudo-open map*[4] *if and only if Y is a Fréchet space and f is a sequentially quotient map.*

Theorem 3.11. *Let $f : X \to Y$ be a boundary-compact map. If $X \in \Omega$, then f is a sequentially quotient map if and only if it is a pseudo-sequence-covering map.*

PROOF. First, suppose that f is sequentially quotient. If $\{y_n\}$ is a non-trivial sequence converging to y_0 in Y, put $S_1 = \{y_0\} \cup \{y_n : n \in \mathbb{N}\}$, $X_1 = f^{-1}(S_1)$ and $g = f|_{X_1}$. Thus g is a sequentially quotient, boundary compact map. So g is a pseudo-open map by Lemma 3.10. Since $X \in \Omega$, let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing neighborhood base of compact subset $\partial g^{-1}(y_0)$ in X_1. Thus $\{U_n \cup \mathrm{Int}(g^{-1}(y_0))\}_{n\in\mathbb{N}}$ is a decreasing neighborhood base of $g^{-1}(y_0)$ in X_1. Let $V_n = U_n \cup \mathrm{Int}(g^{-1}(y_0))$ for each $n \in \mathbb{N}$. Then $y_0 \in \mathrm{Int}(g(V_n))$, thus there exists an $i_n \in \mathbb{N}$ such that $y_i \in g(V_n)$ for each $i \geq i_n$, so $g^{-1}(y_i) \cap V_n \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap V_n, & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = \partial g^{-1}(y_0) \cup \{x_j : j \in \mathbb{N}\}$. Clearly, K is a compact subset in X_1 and $g(K) = S_1$. Thus $f(K) = S_1$. Therefore, f is a pseudo-sequence-covering map.

Conversely, suppose that f is a pseudo-sequence-covering map. If $\{y_n\}$ is a convergent sequence in Y, then there is a compact subset K in X such that $f(K) = \overline{\{y_n\}}$. For each $n \in \mathbb{N}$, take a point $x_n \in f^{-1}(y_n) \cap K$. Since K is compact and metrizable, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. So f is sequentially quotient. □

Corollary 3.12. *Let $f : X \to Y$ be a boundary-compact map. Suppose also that at least one of the following conditions holds:*

(1) *X has a point-countable base;*

(2) *X is a developable space.*

Then f is a sequentially quotient map if and only if it is a pseudo-sequence-covering map.

Question 3.1. *Let $f : X \to Y$ be a sequence-covering and boundary-compact (or compact) map. Is f an 1-sequence-covering map if one of the following conditions is satisfied?*

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[3] X is said to be a *Fréchet space* [7] if $x \in \overline{P} \subset X$, there is a sequence in P converging to x in X.

[4] f is a *pseudo-open map* [3] if whenever $f^{-1}(y) \subset U$ with U open in X, then $y \in \mathrm{Int}(f(U))$.

(1) *Every compact subset of X is metrizable;*
(2) *Every compact subset of X has countable character.*

Remark If X satisfies the conditions (1) and (2) in Question 3.1, then f is an 1-sequence-covering map by Theorem 3.6.

4. Sequence-covering maps on g-metrizable spaces

In this section, we mainly discuss sequence-covering maps on spaces with a special weak base.

Lemma 4.1. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. For each non-isolated point $y \in Y$, there exist a point $x \in \partial f^{-1}(y)$ and a decreasing weak neighborhood base $\{B_{xi}\}_i$ at x such that for each $n \in \mathbb{N}$, there are a $P \in \mathcal{P}_y$ and $i \in \mathbb{N}$ with $P \subset f(B_{xi})$ if X and Y satisfy the following (1) and (2):*

(1) *Y is snf-countable;*
(2) *Every compact subset of X has a countable external weak base in X.*

Proof. Suppose not, there exists a non-isolated point $y \in Y$ such that for every point $x \in \partial f^{-1}(y)$ and every decreasing weak neighborhood base $\{B_{xi}\}_i$ of x, there is an $n \in \mathbb{N}$ such that $P \not\subseteq f(B_{xn})$ for every $P \in \mathcal{P}_y$. Since $\partial f^{-1}(y)$ is compact, it follows that $\partial f^{-1}(y)$ has a countable external weak base $\mathcal{B}$ of X. Without loss of generality, we can assume that $\mathcal{B}$ is closed under finite intersections. Therefore, for each $x \in \partial f^{-1}(y)$, there exists a $B_x \in \mathcal{B}$ such that $P \not\subseteq f(B_x)$ for every $P \in \mathcal{P}_y$. Next, using the argument from the proof of Lemma 3.8, this leads to a contradiction. □

The following Lemma 4.2 is easy to check, and hence we omit it.

Lemma 4.2. *Let X have a compact-countable weak base. Then every compact subset of X has a countable external weak base in X.*

Theorem 4.3. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where X has a compact-countable weak base. Then Y is snf-countable if and only if f is an 1-sequence-covering map.*

Proof. Necessity. Let y be a non-isolated point in Y. Since X has a compact-countable weak base, it follows from Lemmas 4.1 and 4.2 that there exists a point $x_y \in \partial f^{-1}(y)$ and a decreasing countable weak base $\{B_n : n \in \mathbb{N}\}$ at point x_y such that for each $n \in \mathbb{N}$, there is a $P \in \mathcal{P}_y$ satisfying $P \subset f(B_n)$. Suppose that $\{y_n\}$ is a sequence in Y, which converges to y. Then we can take a sequence $\{x_n\}$

in X by the similar argument from the proof of Theorem 3.3. Therefore, f is an 1-sequence-covering map.

Sufficiency. By Lemma 3.2, Y is snf-countable. $\square$

We don't know whether the condition "compact-countable weak base" on X can be replaced by "point-countable weak base" in Theorem 4.3,

Corollary 4.4. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where X is g-metrizable. Then Y is snf-countable if and only if f is an 1-sequence-covering map.*

Each closed sequence-covering map on metric spaces is 1-sequence-covering [28]. Now, we improve the result in the following theorem.

Theorem 4.5. *Let $f : X \to Y$ be a closed sequence-covering map, where X is g-metrizable. Then f is an 1-sequence-covering map.*

PROOF. Since X is g-metrizable and f is a closed sequence-covering map, Y is g-metrizable [21, Theroem 3.3]. Therefore, f is a boundary-compact map by [21, Corollary 2.2]. Hence f is an 1-sequence-covering map by Corollary 4.4. $\square$

Question 4.1. *Let $f : X \to Y$ be a sequence-covering and boundary-compact map. If X is g-metrizable, then is f an 1-sequence-covering map?*

5. Closed sequence-covering maps

Say that a Tychonoff space X is *strongly monotonically monolithic* [27] if, for any $A \subset X$ we can assign an external base $\mathcal{O}(A)$ to the set $\overline{A}$ in such a way that the following conditions are satisfied:

(a) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;

(b) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;

(c) if α is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of X such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\cup_{\beta<\alpha} A_\beta) = \cup_{\beta<\alpha}\mathcal{O}(A_\beta)$.

From [27, Proposition 2.5] it follows that a Tychonoff space with a point-countable base is strongly monotonically monolithic. Moreover, if X is a strongly monotonically monolithic space, then it is easy to see that $X \in \Omega$ by [27, Theorem 2.7].

Lemma 5.1. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a strongly monotonically monolithic space. Then Y contains no closed copy of S_ω.*

PROOF. Suppose that Y contains a closed copy of S_ω, and that $\{y\} \cup \{y_i(n) : i, n \in \mathbb{N}\}$ is a closed copy of S_ω in Y, here $y_i(n) \to y$ as $i \to \infty$. For every $k \in \mathbb{N}$, put $L_k = \bigcup_{n\leq k}\{y_i(n) : i \in \mathbb{N}\}$. Hence L_k is a sequence converging to y. Let M_k be a sequence of X converging to $u_k \in f^{-1}(y)$ such that $f(M_k) = L_k$. We rewrite $M_k = \bigcup_{n\leq k}\{x_i(n,k) : i \in \mathbb{N}\}$ with each $f(x_i(n,k)) = y_i(n)$.

Case 1: $\{u_k : k \in \mathbb{N}\}$ is finite.

There are a $k_0 \in \mathbb{N}$ and an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $M_k \to u_{k_0}$ for every $k \in \mathbb{N}_1$, then X contains a closed copy of S_ω. Hence X is not first countable. This is a contradiction.

Case 2: $\{u_k : k \in \mathbb{N}\}$ has a non-trivial convergent sequence in X.

Without loss of generality, we suppose that $u_k \to u$ as $k \to \infty$. Since X is first-countable, let $\{U_m\}$ be a decreasing open neighborhood base of X at point u with $\overline{U}_{m+1} \subset U_m$. Then $\bigcap_{m\in\mathbb{N}} U_m = \{u\}$. Fix n, pick $x_{i_m}(n,k_m) \in U_m \cap \{x_i(n,k_m)\}_i$. We can suppose that $i_m < i_{m+1}$. Then $\{f(x_{i_m}(n,k_m))\}_m$ is a subsequence of $\{y_i(n)\}$. Since f is closed, $\{x_{i_m}(n,k_m)\}_m$ is not discrete in X. Then there is a subsequence of $\{x_{i_m}(n,k_m)\}_m$ converging to a point $b \in X$ because X is a first-countable space. It is easy to see that $b = u$ by $x_{i_m}(n,k_m) \in U_m$ for every $m \in \mathbb{N}$. Hence $x_{i_m}(n,k_m) \to u$ as $m \to \infty$. Then $\{u\} \cup \{x_{i_m}(n,k_m) : n,m \in \mathbb{N}\}$ is a closed copy of S_ω in X. Thus, X is not first countable. This is a contradiction.

Case 3: $\{u_k : k \in \mathbb{N}\}$ is discrete in X.

Let $B = \{u_k : k \in \mathbb{N}\} \cup \{M_k : k \in \mathbb{N}\}$. Since X is strongly monotonically monolithic, $\overline{B}$ is metrizable. Hence there exists a discrete family $\{V_k\}_{k\in\mathbb{N}}$ consisting of open subsets of $\overline{B}$ with $u_k \in V_k$ for each $k \in \mathbb{N}$. Pick $x_{i_k}(1,k) \in V_k \cap \{x_i(1,k)\}_i$ such that $\{f(x_{i_k}(1,k))\}_k$ is a subsequence of $\{y_i(n)\}$. Since $\{x_{i_k}(1,k)\}_k$ is discrete in $\overline{B}$, $\{f(x_{i_k}(1,k))\}_k$ is discrete in Y. This is a contradiction.

In a word, Y contains no closed copy of S_ω. □

Lemma 5.2. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a strongly monotonically monolithic space. Then $\partial f^{-1}(y)$ is compact for each point $y \in Y$.*

Proof. From Lemma 5.1 it follows that Y contains no closed copy S_ω. Since X is a strongly monotonically monolithic space, every closed separable subset of X is metrizable, and hence is normal. Therefore, $\partial f^{-1}(y)$ is countable compact for each point $y \in Y$ by [21, Theorem 2.6]. From [27, Theorem 2.7] it easily follows that every countable compact subset of X is compact. □

Theorem 5.3. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a strongly monotonically monolithic space. Then f is an 1-sequence-covering map.*

Proof. It is easy to see by Lemma 5.2 and Theorem 3.6. □

Corollary 5.4. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a Tychonoff space with a point-countable base. Then f is an 1-sequence-covering map.*

In fact, we can replace "Tychonoff" by "regular" in Corollary 5.4, and hence we have the following result.

Corollary 5.5. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a regular space with a point-countable base. Then f is an 1-sequence-covering map.*

Proof. Since X has a point-countable base and f is a closed sequence-covering map, Y has a point-countable base by [21, Theorem 3.1]. Therefore, f is a boundary-compact map by [22, Lemma 3.2]. Hence f is an 1-sequence-covering map by Corollary 3.7. □

We don't know whether, in Corollary 5.5, the condition "X has a point-countable base" can be replaced by "$X \in \Omega$". So we have the following question.

Question 5.1. *Let $f : X \to Y$ be a closed sequence-covering map. If $X \in \Omega$ (and X is regular), then is f an 1-sequence-covering map?*

Corollary 5.6. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a strongly monotonically monolithic space. Then f is an almost-open map*[5].

Proof. f is an 1-sequence-covering map by Theorem 5.3. For each point $y \in Y$, there exists a point $x_y \in f^{-1}(y)$ satisfying the Definition 2.2(4). Let U be an open neighborhood of x_y. Then $f(U)$ is a sequential neighborhood of y. Indeed, for each sequence $\{y_n\} \subset Y$ converging to y, there exists a sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to x_y and $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. Obviously, $\{x_n\}$ is eventually in U, and therefore, $\{y_n\}$ is eventually in $f(U)$. Hence $f(U)$ is a sequential neighborhood of y. Since X is first-countable, Y is a Fréchet space. Then $f(U)$ is a neighborhood of y. Otherwise, suppose $y \in Y \setminus \operatorname{int}(f(U))$, and therefore, $y \in \overline{Y \setminus f(U)}$. Since Y is Fréchet, there exists a sequence $\{y_n\} \subset Y \setminus f(U)$ converging to y. This is a contradiction with $f(U)$ is a sequential neighborhood of y. Therefore, f is an almost-open map. □

[5] f is an *almost-open map* [2] if there exists a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of x_y, $f(U)$ is a neighborhood of y in Y.

Remark In [27], V.V. Tkachuk has proved that closed maps don't preserve strongly monotonically monolithic spaces. However, if perfect maps[6] preserve strongly monotonically monolithic spaces, then it is easy to see that closed sequence-covering maps preserve strongly monotonically monolithity by Lemma 5.2. So we have the following two questions.

Question 5.2. *Do closed sequence-covering maps (or an almost open and closed maps) preserve strongly monotonically monolithity?*

Question 5.3. *Do perfect maps preserve strongly monotonically monolithity?*

In [27], V. V. Tkachuk has also proved that open and separable maps preserve strongly monotonically monolithity. However, we have the following result.

Theorem 5.7. *Let $f : X \to Y$ be an open and closed map, where X is a strongly monotonically monolithic space. Then Y is a strongly monotonically monolithic space.*

Proof. From [21, Theorem 3.4] it follows that f is a sequence-covering map. Therefore, $\partial f^{-1}(y)$ is compact for each point $y \in Y$ by Lemma 5.2. Then $\partial f^{-1}(y)$ is metrizable by [27, Theorem 2.7], and hence it is separable, for each point $y \in Y$. For each point $y \in Y$, if y is a non-isolated point, let A_y be a countable dense set in the subspace $\partial f^{-1}(y)$; if y is an isolated point, then we choose a point $x_y \in f^{-1}(y)$ and let $A_y = \{x_y\}$.

Let $B \subset Y$. Put $A_B = \cup\{A_y : y \in B\}$ and $\mathcal{N}(B) = \{f(W) : W \in \mathcal{O}(A_B)\}$. It is easy to see that $\mathcal{N}(B)$ satisfies the conditions (a)-(c) of the definition of strongly monotonically monolithity. Therefore, we only need to prove that $\mathcal{N}(B)$ is an external base for $\overline{B}$. For each point $y \in \overline{B}$, let U be open subset in Y with $y \in U$.

Case 1: y is a non-isolated point in Y.

Since f is an open map, $\emptyset \neq f^{-1}(y) \subset \overline{f^{-1}(B)}$, and hence $\partial f^{-1}(y) \subset \overline{f^{-1}(B)}$. Take any point $x \in \partial f^{-1}(y)$. Then $x \in \overline{A_B}$. Therefore, there exists a $V \in \mathcal{O}(A_B)$ such that $x \in V \subset f^{-1}(U)$. So $W = f(V) \in \mathcal{N}(B)$ and $y \in W \subset U$.

Case 2: y is an isolated point in Y.

It is easy to see that $\{y\} \in \mathcal{N}(B)$, and therefore, $y \in \{y\} \subset U$.

In a word, $\mathcal{N}(B)$ is an external base for $\overline{B}$. $\square$

[6] A map f is called *perfect* if f is a closed and compact map

Let $\mathcal{B} = \{B_\alpha : \alpha \in H\}$ be a family of subsets of a space X. $\mathcal{B}$ is *point-discrete* (or *weakly hereditarily closure-preserving*) if $\{x_\alpha : \alpha \in H\}$ is closed discrete in X, whenever $x_\alpha \in B_\alpha$ for each $\alpha \in H$.

It is well-known that metrizability, g-metrizability, $\aleph$-spaces, and spaces with a point-countable base are preserved by closed sequence-covering maps, see [21, 28]. Next, we shall consider spaces with a σ-point-discrete k-network, and shall prove that spaces with σ-point-discrete k-network are preserved by closed sequence-covering maps. Firstly, we give some technique lemmas.

Lemma 5.8. *Let X be an $\aleph_1$-compact space*[7] *with a σ-point-discrete network. Then X has a countable network.*

Proof. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-point-discrete network for X, where each $\mathcal{P}_n$ is a point-discrete family for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$B_n = \{x \in X : |(\mathcal{P}_n)_x| > \omega\}.$$

Claim 1: $\{P \setminus B_n : P \in \mathcal{P}_n\}$ is countable.

Suppose not, there exist an uncountable subset $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}_n$ and $\{x_\alpha : \alpha < \omega_1\} \subset X$ such that $x_\alpha \in P_\alpha \setminus B_n$. Since $\mathcal{P}_n$ is a point-discrete family and X is $\aleph_1$-compact, $\{x_\alpha : \alpha < \omega_1\}$ is countable. Without loss of generality, we can assume that there exists $x \in X \setminus B_n$ such that each $x_\alpha = x$. Therefore, $x \in B_n$, a contradiction.

Claim 2: For each $n \in \mathbb{N}$, B_n is a countable and closed discrete subspace for X.

For each $Z \subset B_n$ with $|Z| \leq \omega_1$. Let $Z = \{x_\alpha : \alpha \in \bigwedge\}$. By the definition of B_n and Well-ordering Theorem, it is easy to obtain by transfinite induction that $\{P_\alpha : \alpha \in \bigwedge\} \subset \mathcal{P}_n$ such that $x_\alpha \in P_\alpha$ and $P_\alpha \neq P_\beta$ for each $\alpha \neq \beta$. Therefore, Z is a countable and closed discrete subspace for X. Hence B_n is a countable and closed discrete subspace.

For each $n \in \mathbb{N}$, let $\mathcal{P}'_n = \{P \setminus B_n : P \in \mathcal{P}_n\} \cup \{\{x\} : x \in B_n\}$. Then $\mathcal{P}'_n$ is a countable family.

Obviously, $\bigcup_{n\in\mathbb{N}} \mathcal{P}'_n$ is a countable network for X. □

The proof of the following lemma is an easy exercise.

Lemma 5.9. *Let $\{F_\alpha\}_{\alpha\in A}$ be a point-discrete family for X and countably compact subset $K \subset \bigcup_{\alpha\in A} F_\alpha$. Then there exists a finite family $\mathcal{F} \subset \{F_\alpha\}_{\alpha\in A}$ such that $K \subset \cup\mathcal{F}$.*

[7] A space X is called $\aleph_1$-*compact* if each subset of X with a cardinality of $\aleph_1$ has a cluster point.

Lemma 5.10. *Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is a σ-point-discrete wcs^*-network*[8] *for X if and only if $\mathcal{P}$ is a σ-point-discrete k-network*[9] *for X.*

PROOF. Sufficiency. It is obvious. Hence we only need to prove the necessity.

Necessity. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-point-discrete wcs^*-network, where each $\mathcal{P}_n$ is a point-discrete family for each $n \in \mathbb{N}$. Suppose that K is compact and $K \subset U$ with U open in X. For each $n \in \mathbb{N}$, let

$$\mathcal{P}'_n = \{P \in \mathcal{P}_n : P \subset U\}, F_n = \cup\mathcal{P}'_n.$$

Then there exists $m \in \mathbb{N}$ such that $K \subset \bigcup_{k\leq m} F_k$. Suppose not, there is a sequence $\{x_n\} \subset K$ with $x_n \in K - \bigcup_{i\leq n} F_i$. By Lemma 5.8, it is easy to see that K is metrizable. Therefore, K is sequentially compact. It follows that there exists a convergent subsequence of $\{x_n\}$. Without loss of generality, we assume that $x_n \to x$. Since $\mathcal{P}$ is a wcs^*-network, there exist a $P \in \mathcal{P}$, and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$. Therefore, there exists $l \in \mathbb{N}$ such that $P \in \mathcal{P}'_l$. Choose $i > l$, since $P \subset F_l$, $x_{n_i} \in F_l$, a contradiction. Hence there exists $m \in \mathbb{N}$ such that $K \subset \bigcup_{k\leq m} F_k$. By Lemma 5.9, there is a finite family $\mathcal{P}'' \subset \bigcup_{i\leq m} \mathcal{P}'_i$ such that $K \subset \cup\mathcal{P}'' \subset U$. Therefore, $\mathcal{P}$ is a k-network. □

Theorem 5.11. *Closed sequence-covering maps preserve spaces with a σ-point-discrete k-network.*

PROOF. It is easy to see that closed sequence-covering maps preserve spaces with a σ-point-discrete wcs^*-network. Hence closed sequence-covering maps preserve spaces with a σ-point-discrete k-network by Lemma 5.10. □

Question 5.4. *Do closed maps preserve spaces with a σ-point-discrete k-network?*

Acknowledgements. We wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

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[9] A family $\mathcal{P}$ of X is called a *k-network* [24] if whenever K is a compact subset of X and $K \subset U$ with U open in X, there is a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset U$.

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Received February 14, 2012
Revised version received May 29, 2012

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