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A note on pseudobounded paratopological groups

Abstract: Let G be a paratopological group. Then G is said to be *pseudobounded* (resp. *ω-pseudobounded*) if for every neighbourhood V of the identity e in G, there exists a natural number n such that $G = V^n$ (resp. we have $G = \bigcup_{n\in\mathbb{N}} V^n$). We show that every feebly compact (2-pseudocompact) pseudobounded (ω-pseudobounded) premeager paratopological group is a topological group. Also, we prove that if G is a totally ω-pseudobounded paratopological group such that G is a Lusin space, then is G a topological group. We present some examples of paratopological groups with interesting properties:
(1) There exists a metrizable, zero-dimensional and pseudobounded topological group;
(2) There exists a Hausdorff ω-pseudobounded paratopological group G such that G contains a dense subgroup which is not ω-pseudobounded;
(3) There exists a Hausdorff connected paratopological group which is not ω-pseudobounded.

Keywords: Paratopological group; Pseudobounded; ω-pseudobounded; Topological group; Premeager space; Lusin space

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1 Introduction

A *paratopological group* is a group endowed with a topology for which multiplication in the group is jointly continuous. If, additionally, the inversion in a paratopological group is continuous, then it is called a *topological group*.

Let G be a paratopological group and $A \subseteq G$. According to [4], A is a *pseudobounded subset* of G, if for every neighbourhood V of the identity e in G, there exists a natural number n such that $A \subseteq V^n$. If G is pseudobounded in itself, then we say that G is *pseudobounded*.

Following [8], a subset A of a paratopological group G is a *ω-pseudobounded subset* of G, if for every neighbourhood V of the identity e in G, we have that $A \subseteq \bigcup_{n\in\mathbb{N}} V^n$. If G is ω-pseudobounded in itself, then we say that G is *ω-pseudobounded*.

Clearly, every pseudobounded paratopological group is ω-pseudobounded. However, the additive group $(\mathbb{R}, +)$ endowed with the usual topology is a ω-pseudobounded topological group which is not pseudobounded (see [8, Example 3]).

We show that every feebly compact (2-pseudocompact) pseudobounded (ω-pseudobounded) premeager paratopological group is a topological group (see Corollaries 2.3 and 2.10). Also, we prove that if G is a totally

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ω-pseudobounded paratopological group such that G is a Lusin space, then is G a topological group (see Proposition 2.8).

In [8, Proposition 7], the authors showed that each connected topological group is ω-pseudobounded. However, the converse is not true: there exists a metrizable, zero-dimensional and pseudobounded topological group (see Example 2.22). This fact answers Question 9 in [8].

Moreover, we present some examples of pseudoboundedness or ω-pseudoboundedness in the class of paratopological groups.

2 Pseudobounded paratopological groups

A paratopological group G is called *premeager* if for each nowhere dense subset A of G, we have $A^n \neq G$ for each $n \in \mathbb{N}$ (see [8]).

A paratopological group G is *saturated* if for each neighbourhood U of the identity in G, we have $Int(U^{-1}) \neq \emptyset$.

Theorem 2.1. *Let G be a pseudobounded and premeager paratopological group. Then G is saturated.*

Proof. Take a neighbourhood U of the identity e in G. By the continuity of the multiplication in G, there exists a neighbourhood V of e such that $V^2 \subseteq U$. We have that $\overline{V^{-1}} \subseteq U^{-1}$. Suppose that $Int(U^{-1}) = \emptyset$. Since $\overline{V^{-1}} \subseteq U^{-1}$, $Int(\overline{V^{-1}}) = \emptyset$, i.e., $\overline{V^{-1}}$ is a nowhere dense subset of G. Since G is pseudobounded, there exists a natural number n such that $V^n = G$, so $\overline{V^{-1}}^n = G$, which contradicts that G is premeager. We conclude that G is saturated. □

A space X is *feebly compact* if every locally finite family of open sets in X is finite. In the class of Tychonoff spaces, feeble compactness is equivalent to pseudocompactness.

Proposition 2.2. *([12]) Every saturated feebly compact paratopological group is a topological group.*

Corollary 2.3. *Every feebly compact, pseudobounded and premeager paratopological group is a topological group*

Problem 2.4. *If G is feebly compact pseudobounded (Hausdorff) paratopological group, is G a topological group?*

A *Lusin space* is an uncountable space such that every nowhere dense subset of the space is countable. It follows from the definition that every Lusin paratopological group is premeager.

Proposition 2.5. *If G is a ω-pseudobounded and Lusin paratopological group, then G is saturated.*

Proof. Let U a neighbourhood of the identity e in G. Suppose that U^{-1} has empty interior. Take a neighbourhood V of e such that $V^2 \subseteq U$. We have that $\overline{V^{-1}} \subseteq U^{-1}$, so $\overline{V^{-1}}$ is a nowhere dense subset of G. Since G is a Lusin space, V^{-1} is countable. It follows that V^n is countable for every $n \in \omega$, whence, $G = \bigcup_{n\in\mathbb{N}} V^n$ is countable. This contradiction shows that G is saturated. □

Corollary 2.6. *Let G be a feebly compact, ω-pseudobounded, and Lusin paratopological group, then G is a topological group.*

Let G be a paratopological group with topology τ. We define the *conjugate topology* τ^{-1} on G by $\tau^{-1} = \{U^{-1} : U \in \tau\}$. The upper bound $\tau^\star = \tau \vee \tau^{-1}$ is a topological group topology on G and $G^\star = (G, \tau^\star)$ is called *the topological group associated to* G. If $\mathcal{U}$ is local base at the identity in G, then $\mathcal{U}^\star = \{U \cap U^{-1} : U \in \mathcal{U}^\star\}$ is local base at the identity in $G^\star$.

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Suppose that $\mathcal{P}$ is a property. A paratopological group G is *totally* $\mathcal{P}$ if the associated topological group $G^\star$ has the property $\mathcal{P}$.

Lemma 2.7. *([2]) Suppose that G is a paratopological group and not a topological group. Then there exists an open neighbourhood of the identity in G such that is $U \cap U^{-1}$ is nowhere dense in G.*

The following result gives a partial answer to [8, Question 8]: If G a is totally ω-pseudobounded premeager paratopological group, is G a topological group?

Proposition 2.8. *If G is a totally ω-pseudobounded paratopological group such that G is a Lusin space, then is G a topological group.*

Proof. Suppose that G is not a topological group. By Lemma 2.7, there exists an open neighbourhood of the identity in G such that is $U \cap U^{-1}$ is nowhere dense in G. Arguing as in Proposition 2.5, we conclude that $G^\star$ is countable, so G is countable. This contradicts that G is a Lusin space. Therefore G is a topological group. □

We say that a paratopological group G is *2-pseudocompact* if $\bigcap_{n\in\omega} \overline{U_n^{-1}} \neq \emptyset$, for each non-increasing sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of G.

Proposition 2.9. *Every ω-pseudobounded 2-pseudocompact paratopological group is pseudobounded.*

Proof. Let G be a ω-pseudobounded 2-pseudocompact paratopological group. Take a neighbourhood U of the identity e in G. Since G is ω-pseudobounded, $G = \bigcup_{n\in\mathbb{N}} U^n$. Suppose that $G \setminus \overline{U^{-n}}$ is a non-empty open set for every $n \in \mathbb{N}$. The family $\{G \setminus \overline{U^{-n}} : n \in \mathbb{N}\}$ is a non-increasing sequence of non-empty open subsets of G. We have that

$$\bigcap_{n\in\mathbb{N}} \overline{(G \setminus \overline{U^{-n}})^{-1}} \subseteq \bigcap_{n\in\mathbb{N}} \overline{(G \setminus U^{-n})^{-1}} = \bigcap_{n\in\mathbb{N}} (G \setminus U^n) = \emptyset.$$

This contradicts the 2-pseudocompactness of G. Therefore, there exists $n \in \mathbb{N}$ such that $G = \overline{U^{-n}} \subseteq U^{-n-1}$, so $G = U^{n+1}$. □

Corollary 2.10. *Every 2-pseudocompact ω-pseudobounded premeager paratopological group is a topological group.*

Proof. Let G be a 2-pseudocompact ω-pseudobounded premeager paratopological group. By Proposition 2.9 G is pseudobounded. According to [12], every 2-pseudocompact paratopological group is feebly compact. We finish the proof applying Corollary 2.3. □

Problem 2.11. *Let G be a 2-pseudocompact pseudobounded paratopological group. Is G a topological group?*

Given a pseudobounded subset A of a paratopological group G, in general, $\overline{A}$ and A^{-1} are not pseudobounded subsets of G. We have the next result in this direction.

Proposition 2.12. *If a paratopological group G contains a pseudobounded (ω-pseudobounded) dense subgroup, then G is pseudobounded (ω-pseudobounded)*

Proof. Let H be a pseudobounded dense subgroup of G. Take an open neighbourhood U of the identity e in G. Since H is a pseudobounded subset of G, there exists $n \in \mathbb{N}$ such that $H \subseteq U^n$, equivalently, $H \subseteq U^{-n}$. Hence, $G = \overline{H} \subseteq HU^{-1} \subseteq U^{-n-1}$, whence, $G = U^{n+1}$. Using a similar argument, we can prove that if H is an ω-pseudobounded dense subgroup of a paratopological group G, then G is ω-pseudobounded. □

Lemma 2.13. *([14]) Suppose that H is dense subgroup of a paratopological group G. Let $n \in \mathbb{N}$ and suppose that $W_1, \dots, W_{2n}$ are open neighbourhoods of the identity e in G. Then, for every neighbourhood O of e and*

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every choice $w_i \in W_i$ for $i = 1, \dots, 2n$ satisfying $w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1} \in H$, there exist elements $u_i \in H \cap w_i O$, for $i = 1, \dots, 2n$ such that

$$u_1 u_2^{-1} \cdots u_{2n-1} u_{2n}^{-1} = w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1}.$$

Here, we present the converse of Proposition 2.12 for topological groups.

Theorem 2.14. *If H is dense subgroup of a pseudobounded (ω-pseudobounded) topological group G, then H is pseudobounded (ω-pseudobounded).*

Proof. Suppose that G is ω-pseudobounded. Take a neighbourhood U of the identity e in H, there exists V neighbourhood of e in G such that $U = V \cap H$. We can find a symmetric neighbourhood W of e in G such that $W^2 \subseteq V$. Since G is ω-pseudobounded, $G = \bigcup_{n\in\mathbb{N}} W^n$. Let us show that $H = \bigcup_{n\in\mathbb{N}} U^n$. Take $h \in H$, there exists $n \in \mathbb{N}$ such that $h \in W^{2n}$. Since W is a symmetric neighbourhood W of e in G, we can put $h = w_1 w_2^{-1} \dots w_{2n-1} w_{2n}^{-1}$, where $w_i \in W$, for $i = 1, \dots, 2n$. By Lemma 2.13, there exist elements $u_i \in H \cap w_i W \subseteq H \cap W^2 \subseteq H \cap V = U$, for $i = 1, \dots, 2n$ such that

$$h = w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1} = u_1 u_2^{-1} \cdots u_{2n-1} u_{2n}^{-1}.$$

We have that $u_i^{-1} \in H \cap W^{-1} w_i^{-1} \subseteq H \cap W^2 \subseteq H \cap V = U$, for $i = 1, \dots, 2n$. Therefore, $h = u_1 u_2^{-1} \cdots u_{2n-1} u_{2n}^{-1} \in U^{2n}$. It follows that $H = \bigcup_{n\in\mathbb{N}} U^n$. The proof of the case pseudobounded is similar. □

We need the following lemma to construct examples of Hausdorff paratopological groups.

Lemma 2.15. *([10, Proposition 1.1]) Let G be a group, and let $\mathfrak{U}$ be a family of subsets of G containing the identity e in G. Suppose that the family $\mathfrak{U}$ satisfies the following conditions (called the Pontryagin's conditions):*
i) for every $U, V \in \mathfrak{U}$, there is $W \in \mathfrak{U}$ such that $W \in U \cap V$;
ii) for every $U \in \mathfrak{U}$ and each $x \in U$, there exists $V \in \mathfrak{U}$ such that $Vx \subseteq U$;
iii) for each $U \in \mathfrak{U}$ and $x \in G$, there exists $V \in \mathfrak{U}$ such that $xVx^{-1} \subseteq U$;
iv) for every $U \in \mathfrak{U}$, there is an element $V \in \mathfrak{U}$ such that $V^2 \subseteq U$.

Then there exists a topology τ on G such that (G, τ) is a paratopological group and the family $\mathfrak{U}$ is a local base at the identity e in G. In addition, if the family $\mathfrak{U}$ satisfies $\bigcap_{U\in\mathfrak{U}} UU^{-1} = \{e\}$, then the paratopological group (G, τ) is Hausdorff.

The following example shows that Theorem 2.14 is false in the class of Hausdorff ω-pseudobounded paratopological groups.

Example 2.16. *There exists a Hausdorff ω-pseudobounded paratopological group G such that G contains a dense subgroup which is not ω-pseudobounded.*

Proof. Consider de additive group $(\mathbb{R}, +)$. Fix a natural number k and put $U_n(k) = k(\mathbb{N} \cup \{0\}) + (-\frac{1}{n}, \frac{1}{n})$ for each $n \in \mathbb{N}$. Let us show that the family $\mathfrak{U} = \{U_n(k) : k, n \in \mathbb{N}\}$ satisfies conditions i)–iv) in Lemma 2.15. To prove i), take $k_1, k_2, n_1, n_2 \in \mathbb{N}$. Put $k = k_1 k_2$ and $n = n_1 n_2$. Clearly, $U_n(k) \subseteq U_{n_1}(k_1) \cap U_{n_2}(k_2)$. Let us check ii). Choose $x \in U_n(k)$, for some $n, k \in \mathbb{N}$. We can find $s \in \mathbb{N} \cup \{0\}$ such that $x \in ks + (-\frac{1}{n}, \frac{1}{n})$, so $x - ks \in (-\frac{1}{n}, \frac{1}{n})$. There exists $m \in \mathbb{N}$ satisfying $x - ks + (-\frac{1}{m}, \frac{1}{m}) \subseteq (-\frac{1}{n}, \frac{1}{n})$, whence, $x + (-\frac{1}{m}, \frac{1}{m}) \subseteq ks + (-\frac{1}{n}, \frac{1}{n})$. Therefore, $x + U_m(k) \subseteq U_n(k)$. Item iii) is trivial. To verify iv), take $U_n(k)$. We can find $m \in \mathbb{N}$ such that $\frac{2}{m} < \frac{1}{n}$. We have that $U_m(k) + U_m(k) = k(\mathbb{N} \cup \{0\}) + (-\frac{1}{m}, \frac{1}{m}) + k(\mathbb{N} \cup \{0\}) + (-\frac{1}{m}, \frac{1}{m}) \subseteq k(\mathbb{N} \cup \{0\}) + (-\frac{2}{m}, \frac{2}{m}) \subseteq U_n(k)$. By Lemma 2.15, there exists a topology σ on $\mathbb{R}$ such that $G = (\mathbb{R}, \sigma)$ is a paratopological group and the family $\mathfrak{U}$ is a local base at 0 in G.

It is easy to check that $U_n(k) - U_n(k) = k\mathbb{Z} + (-\frac{2}{n}, \frac{2}{n})$ for each $k, n \in \mathbb{N}$. Fix $k \in \mathbb{N}$, we have that $\bigcap_{n\in\mathbb{N}}(U_n(k) - U_n(k)) = k\mathbb{Z}$. It follows that $\bigcap_{k,n\in\mathbb{N}}(U_n(k) - U_n(k)) = \{0\}$, so G is a Hausdorff space. Clearly, G is ω-pseudobounded.

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Let α be a positive irrational number. Denote by H the subgroup of G generated by α. Let us show that H is dense in G. Take a non-empty open set O in G. We can find $a \in O$ and $n, k \in \mathbb{N}$ such that $a + U_n(K) \subseteq O$. There exists $r \in \mathbb{N}$ satisfying $r\alpha > a$. Put $p = r\alpha$. Let $\mathbb{T}$ be the circle group endowed with the usual topology. Consider the continuous homomorphism $f\colon G \to \mathbb{T}$ such that $f(x) = e^{\frac{2\pi i x}{k}}$, for each $x \in G$. Put $z = f(a)$. Clearly, $f(a + U_n(k)) = zf(-\frac{1}{n}, \frac{1}{n})$ is an open set in $\mathbb{T}$. Since the set $\{f(sp) : s \in \mathbb{N}\}$ is dense in $\mathbb{T}$, there exists $m \in \mathbb{N}$ such that $f(mp) \in (a + U_k(n))$. It follows that $mp \in a + U_k(n) + k\mathbb{Z} = a + k\mathbb{Z} + (-\frac{1}{n}, \frac{1}{n})$. Since $p > a$, we have that $mp > a$. Hence,

$$mr\alpha = mp \in a + k(\mathbb{N} \cup \{0\}) + (-\frac{1}{n}, \frac{1}{n}) = a + U_n(k) \subseteq O.$$

We conclude that H is dense in G. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$, then the elements of $U_n(1) \cap H$ are non-negative real numbers, so $\bigcup_{\eta \in \mathbb{N}} \eta(U_n(1) \cap H) \neq H$. It follows that H is not ω-pseudobounded. □

We have the following two questions.

Problem 2.17. *Let G be a regular ω-pseudobounded paratopological and H is a dense subgroup of G. Is H ω-pseudobounded?*

Problem 2.18. *Suppose that G is a pseudobounded paratopological and H is a dense subgroup of G. Is H pseudobounded?*

The multiplication mapping of a paratopological group G is said to *locally closed at e* if there exists an open neighborhood U of e in G satisfying the following condition:

(a) For each open neighborhood V of e in G and $n \in \mathbb{N}$, if $\overline{V}^n \subset U$ then $\overline{V}^n$ is closed in G.

Note. Obviously, if the multiplication map of a paratopological group is closed then it is locally closed at e.

Example 2.19. *There exists a topological group G which multiplication mapping is locally closed at e and not closed.*

Proof. Let G be the set $\mathbb{R} \setminus \{0\}$ of all non-zero real numbers with the usual multiplication and usual topology. Then G is a topological group. Obviously, the product map is locally closed at 1. Obviously, both the sets $\mathbb{Z}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are closed in G. Then the product of $\mathbb{Z}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ is $\mathbb{Q}$ which is proper dense subset of G, and hence the product of $\mathbb{Z}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed. Therefore, the product map is not a closed map. □

The following gives a partial answer to Problem 2.17 and Problem 2.18.

Theorem 2.20. *Suppose that G is a pseudobounded (resp. ω-pseudobounded) regular paratopological group and H is a dense subgroup of G. If the multiplication mapping $H \times H$ into H is locally closed at e, then H is pseudobounded (resp. ω-pseudobounded).*

Proof. We shall show the case of ω-pseudoboundedness. The proof of analogous assertion for pseudoboundedness is quite similar.

Let U be an open neighborhood of e in H. Then there exists an open neighborhood V of e in G such that $V \cap H \subset \overline{V}^G \cap H \subset U$. Since G is ω-pseudobounded, we have $G = \bigcup_{n \in \mathbb{N}} V^n$, and then $\bigcup_{n \in \mathbb{N}} (\overline{V \cap H})^n = G$ since H is dense in G.

Next we shall show that $\bigcup_{n \in \mathbb{N}} (\overline{V \cap H}^H)^n = H$. Indeed, pick any $h \in H$. Then there exists $n \in \mathbb{N}$ such that $h \in V^n$. Then h has the form $h = g_1 \cdots g_n$, where $g_i \in V$ for each $i = 1, \cdots, n$.

Claim: $h \in (\overline{V \cap H}^H)^n$.

Let W be an any open neighborhood of h in H, and therefore there exists an open neighborhood O of e in G such that $W = O \cap H$. Then, for each $i = 1, \cdots, n$, there exist open neighborhoods W_i of g_i in G such that $\prod_{i=1}^{i=n} W_i \subset O$. For each $i = 1, \cdots, n$, we have $W_i \cap V \cap H \neq \emptyset$ since $g_i \in \overline{V \cap H}$. Therefore, $(\prod_{i=1}^{i=n} W_i) \cap (V \cap H)^n \neq$

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$\emptyset$, and thus $O \cap (V \cap H)^n \neq \emptyset$. Therefore,

$$W \cap (V \cap H)^n = (O \cap H) \cap (V \cap H)^n = O \cap (V \cap H)^n \neq \emptyset.$$

Hence $h \in \overline{(V \cap H)^n}^H$. Since the product map $H \times H$ into H is locally closed at e, we have

$$\overline{(V \cap H)^n}^H \subset (\overline{V \cap H}^H)^n,$$

and then $h \in (\overline{V \cap H}^H)^n$.

By the arbitrary of h, it follows from Claim that we have

$$H = \bigcup_{n \in \mathbb{N}} (\overline{V \cap H}^H)^n \subset \bigcup_{n \in \mathbb{N}} (\overline{V} \cap H)^n \subset \bigcup_{n \in \mathbb{N}} U^n,$$

that is, H is ω-pseudobounded. □

In [8], the authors posed the following questions:

A) Is every first-countable and pseudobounded paratopological group a topological group? (see [8, Question 2])

B) Is every first-countable and pseudobounded paratopological group metrizable?

The following example answers questions A) and B) in the negative.

Example 2.21. *There exists a normal first-countable and pseudobounded paratopological group which is non-metrizable.*

Proof. Consider $S^1 = \{x \in \mathbb{C} : |x| = 1\}$ and put $U_n = \{e^{i\theta} : 0 \leq \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Let $\mathbb{K}$ be the circle group endowed with the Sorgenfrey topology, i.e., the topology on $S^1 = \{x \in \mathbb{C} : |x| = 1\}$ such that the family $\{U_n : n \in \mathbb{N}\}$ is a local base at 1 in $\mathbb{K}$. It is easy to see that $\mathbb{K}$ is a normal space. Clearly, $\mathbb{K}$ is a first-countable pseudobounded paratopological group which is not a topological group. Let us show that $\mathbb{K}$ is non-metrizable. Suppose the contrary, i.e., $\mathbb{K}$ is metrizable. Since G is separable, G is second countable. According to [13, Corollary 3.3], the associated topological group $\mathbb{K}^{\star}$ is second countable too. On the other hand, $\mathbb{K}^{\star}$ is a discrete uncountable topological group. This contradiction shows that $\mathbb{K}$ is non-metrizable. □

Every connected topological group is ω-pseudobounded (see [8, Proposition 7]). The following example shows that the converse is false. This answers Question 9 in [8]. Also, Example 2.24 shows that [8, Proposition 7] can not be extended to Hausdorff paratopological groups, answering [8, Question 10] for the Hausdorff case.

Example 2.22. *There exists a metrizable, zero-dimensional and pseudobounded topological group.*

Proof. Let $\mathbb{T}$ be the circle group endowed with the usual topology. Let G be the torsion subgroup of $\mathbb{T}$, i.e., G consists of the elements of $\mathbb{T}$ of finite order. Clearly, G is metrizable and zero-dimensional. We know that G is a dense subgroup of the topological group $\mathbb{T}$. By Theorem 2.14, we conclude that G is pseudobounded. □

Consider a paratopological group (G, τ). Let τ^{-1} the conjugate topology on G. Then $\tau_{\star} = \tau \wedge \tau^{-1}$ is the finest group topology on G weaker than τ and $G_{\star} = (G, \tau_{\star})$ is called the *group reflection* of G (see [14]).

Proposition 2.23. *([10]) If G is an Abelian paratopological group and $\mathcal{U}$ is local base at the identity in G, then $\mathcal{U}_{\star} = \{UU^{-1} : U \in \mathcal{U}\}$ is local base at the identity in $G_{\star}$.*

Example 2.24. *There exists a Hausdorff connected paratopological group which is not ω-pseudobounded.*

Proof. Consider de additive group $(\mathbb{R}, +)$. Fix a natural number k and put $U_n(k) = \{0\} \cup (k\mathbb{N} + (-\frac{1}{n}, \frac{1}{n}))$ for each $n \in \mathbb{N}$. Arguing as in Example 2.16, we can conclude that the family $\mathcal{U} = \{U_n(k) : k, n \in \mathbb{N}\}$ satisfies conditions i)–iv) in Lemma 2.15, so there exists a topology τ on $\mathbb{R}$ such that $H = (\mathbb{R}, \tau)$ is a paratopological group and the family $\mathcal{U}$ is a local base at 0 in H. According to Proposition 2.23, $\mathcal{U}_{\star} = \{U_n(k) - U_n(k) : k, n \in \mathbb{N}\}$

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is local base at 0 in $H_\star$. It is easy to check that $U_n(k)-U_n(k) = k\mathbb{Z}+(-\frac{2}{n}, \frac{2}{n})$ for each $k, n \in \mathbb{N}$. We conclude that the topology $\tau_\star$ is weaker than the usual topology in $\mathbb{R}$, so the space $H_\star$ is connected. Put $W_n(k) = k\mathbb{Z}+(-\frac{1}{n}, \frac{1}{n})$, then the family $\{W_n(k) : k, n \in \mathbb{N}\}$ is local base at 0 in $H_\star$. Fix $k \in \mathbb{N}$, we have that $\bigcap_{n\in\mathbb{N}}(U_n(k)-U_n(k)) = k\mathbb{Z}$. It follows that $\bigcap_{k,n\in\mathbb{N}}(U_n(k) - U_n(k)) = \{0\}$, so $H_\star$ and H are Hausdorff spaces.

By the definition of the topology τ, we have that $W_n(k) \subseteq \overline{U_n(k)}^\tau$ for every $k, n \in \mathbb{N}$. Let us show that H is connected. Suppose the contrary, then there exists non-empty open sets A and B in H such that $A\cup B = H$ and $A\cap B = \emptyset$. We claim that A and B are open sets in $H_\star$. Take $a \in A$, there exists $k, n \in \mathbb{N}$ such that $a+U_n(k) \subseteq A$. Since A is closed in H, we have that $a + W_n(k) \subseteq a + \overline{U_n(k)}^\tau \subseteq A$. This implies that A is open in $H_\star$. Using a similar argument we conclude that B is open in $H_\star$. In addition, $A \cup B = H_\star$ and $A \cap B = \emptyset$. This contradicts the connectedness of $H_\star$, so H is connected. Clearly, H is not ω-pseudobounded. □

Proposition 2.25. *Let $f\colon G \to H$ be a continuous homomorphism from the paratopological group G onto the paratopological group H. If G is pseudobounded (ω-pseudobounded), then H is ω-pseudobounded.*

Proof. Suppose that G is ω-pseudobounded. Take V a open neighbourhood of the identity in H. Put $U = f^{-1}(V)$. Since f is a continuous homomorphism, U is a neighbourhood of the identity in G. By hypothesis, $G = \bigcup_{n\in\mathbb{N}} U^n$.We conclude, $H = \bigcup_{n\in\mathbb{N}} f(U^n) = \bigcup_{n\in\mathbb{N}} f(U)^n = \bigcup_{n\in\mathbb{N}} V^n$, so H is ω-pseudobounded. The proof of the pseudobounded case is similar. □

Denote by $FP(X)$ and $AP(X)$ the *free paratopological group* and the *free Abelian paratopological group* on a space X, respectively. In what follows, we use $PG(X)$ to denote the paratopological group $F(X)$ or $A(X)$. We use the argument in the proof of [3, Proposition 7.1.12] to prove the following result.

Proposition 2.26. *For every space X, the following conditions are equivalent:*
a) $PG(X)$ *is pseudobounded;*
b) $PG(X)$ *is ω-pseudobounded;*
c) $X = \emptyset$.

Proof. Let X be a non-empty space. Define a function f to the discrete group $\mathbb{Z}$ by $f(x) = 1$ for each $x \in X$. Then f is continuous, so it admits an extension to a continuous homomorphism $\widehat{f}\colon PG(X) \to Z$. Clearly, $\widehat{f}(PG(X)) = Z$. By Proposition 2.25, $\mathbb{Z}$ is ω-pseudobounded. This contradiction shows that $X = \emptyset$. □

Problem 2.27. *Is every pseudobounded paratopological (topological) group a precompact or ω-narrow?*

For a Hausdorff paratopological group G with the identity e the *Hausdorff number* [15] of G, denoted by $Hs(G)$, is the minimum cardinal number κ such that for every neighborhood U of e in G, there exists a family γ of neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

Theorem 2.28. *Let G be a Hausdorff paratopological group of countable pseudocharacter. If $Hs(G) \leq \omega$ and G is saturated, then it is submetrizable.*

Proof. Suppose that $\{U_n : n \in \mathbb{N}\}$ is a sequence of open neighborhoods of e in G such that $\bigcap_{n\in\mathbb{N}} U_n = \{e\}$. Let $\mathcal{B}_e$ be a local base at e in G, and let

$$\sigma = \{U \subset G : \text{There exists a } V \in \mathcal{B}_e \text{ such that } xVV^{-1} \subset U \text{ for each } x \in U\}.$$

Since G is saturated, it follows from [5, Theorem 3.2] that (G, σ) is a topological group. Obvious, (G, σ) is T_1 since (G, τ) is Hausdorff, and hence (G, σ) is regular. For each $n \in \mathbb{N}$, since $Hs(G) \leq \omega$, there exists a countable subfamily $\mathcal{B}_n \subset \mathcal{B}_e$ such that $\bigcap_{V\in\mathcal{B}_n} VV^{-1} \subset U_n$. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n \subset \mathcal{B}_e$. Then we have

$$\bigcap_{V\in\mathcal{B}} VV^{-1} \subset \bigcap_{n\in\mathbb{N}} U_n = \{e\}.$$

Therefore, topological group (G, σ) is of countable pseudocharacter, and thus it is submetrizable by [3, Theorem 3.3.16]. Therefore, (G, τ) is submetrizable. □

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By Theorem 2.1, Proposition 2.5 and Theorem 2.28, we have the following proposition.

Proposition 2.29. *Let G be a Hausdorff paratopological group of countable pseudocharacter. If G satisfies (1) or (2), then it is submetrizable.*

(1) The group G is a pseudobounded premeager paratopological group with $Hs(G) \leq \omega$;

(2) The group G is an ω-pseudobounded Lusin paratopological group with $Hs(G) \leq \omega$.

It is easy to see that a Hausdorff first-countable paratopological group has a countable Hausdorff number. Hence we have the following corollary.

Corollary 2.30. *Let G be a Hausdorff pseudobounded and premeager paratopological group. If G is first-countable, then it is submetrizable.*

However, the following problem is still open.

Problem 2.31. *Let G be a Hausdorff pseudobounded and premeager paratopological group. If G is of countable pseudocharacter, is it submetrizable?*

In [9], the authors show that there exists a regular developable paratopological group which is not metrizable. However, that example is not ω-pseudobounded. Therefore, we have the following problem.

Problem 2.32. *Is every regular developable and pseudobounded paratopological group metrizable?*

Problem 2.33. *Is every regular pseudobounded paratopological group with a uniform base metrizable?*

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