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Remainders in compactifications of semitopological and paratopological groups ☆

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ABSTRACT

In this paper, the remainders of semitopological and paratopological groups are investigated. We mainly establish that if G is a non-locally compact semitopological group and bG is a compactification of G such that $Y = bG \setminus G$ has locally a point-countable base, then bG is separable and metrizable. This gives a positive answer to a question posed in Wang and He (2014) [25]. We also show that if G is a non-locally compact $\mathbb{R}_1$-factorizable paratopological group and $Y = bG \setminus G$ is a local $\aleph$-space, then bG is separable and metrizable. Some questions in [14] are answered.

1. Introduction

"A space" in this paper stands for a Tychonoff topological space. A remainder of a space X is the space $bX \setminus X$, where bX is a Hausdorff compactification of X.

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The question when a space X has a Hausdorff compactification with the remainder belonging to a given class of spaces is important. A famous classical result in this direction is the theorem of M. Henriksen and J. Isbell [13]:

Theorem 1.1. *A space X is of countable type if and only if the remainder in any (in some) compactification of X is Lindelöf.*

Recall that a space X is of *countable type* if every compact subspace F of X is contained in a compact subspace $K \subseteq X$ with a countable base of open neighborhoods in X.

Recall that a *semitopological group* (resp., *paratopological group*) is a group with a topology such that the multiplication in the group is separately continuous (resp., jointly continuous). If G is a paratopological group and the inverse operation of G is continuous, then G is called a *topological group*. The reader can find a lot of recent progress about paratopological (or semitopological) groups in the survey article [24].

A series of results on remainders of topological groups have been obtained in [2,4,6,7,17]. They show that remainders of topological groups are much more sensitive to the topological properties of groups than the remainders of topological spaces are in general. However, much less is known about remainders of paratopological (semitopological) groups [24]. The reader can find some recent progress in this direction in [9,14,18,25–27]. In this paper, we will continue to study how the generalized metrizability of remainders affects the paratopological (semitopological) groups.

First, we recall some concepts [1,12].

A base $\mathcal{B}$ for a space X is said to be *uniform* if for each injective sequence $(B_n) \subseteq \mathcal{B}$ and every $x \in \bigcap_{n\in\omega} B_n$, the sequence (B_n) is a base at x.

A base $\mathcal{B}$ for a space X is said to be *weakly uniform* if for each countably infinite family $\mathcal{U} \subseteq \mathcal{B}$ and for each $x \in X$, if $x \in U$ for each $U \in \mathcal{U}$, then $\{x\} = \bigcap \mathcal{U}$.

A base $\mathcal{B}$ for a space X is said to be *sharp* if for every $x \in X$ and every sequence (U_n) of pairwise distinct elements of $\mathcal{B}$ with $x \in U_n$ for all $n \in \omega$, the collection $\{\bigcap_{i\le n} U_i : n \in \omega\}$ forms a base at x.

Recall that a space X has a *base of countable order* (BCO) if X has a base $\mathcal{B}$ such that whenever $x \in X$ and a strictly decreasing sequence (B_n) of elements of $\mathcal{B}$ is such that $x \in \bigcap_{n\in\omega} B_n$, then (B_n) is a base at x.

Let $(\mathcal{U}_n)$ be a sequence of open covers of a space X. Recall that, for every $x \in X$ and n, $\mathrm{st}(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$.

A sequence of open covers $(\mathcal{U}_n)$ of a space X is called:

• A *G_δ-diagonal sequence*, if for every $x \in X$, $\bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{U}_n) = \{x\}$. A space with a G_δ-diagonal sequence is called a space with a *G_δ-diagonal*.

• A *weak development*, if for every $x \in X$ and the sequence (U_n) such that $x \in U_n \in \mathcal{U}_n$ for every n, the sequence $(\bigcap_{i\le n} U_i)$ is a base at x. A space with a weak development is called a *weakly developable space*.

• A *development*, if for every $x \in X$, the sequence $(\mathrm{st}(x, \mathcal{U}_n))$ is a base at x. A space with a development is called a *developable space*.

The implications of the following diagram have been established in [1, Theorem 3.5].

$$\begin{tikzcd}
 & X \text{ has a sharp base} \arrow[dr] & & \\
X \text{ has a uniform base} \arrow[ur] \arrow[dr] & & X \text{ is weakly development} \arrow[r] & X \text{ has BCO} \\
 & X \text{ is development} \arrow[ur] & &
\end{tikzcd}$$

If X is metacompact, then all the five assertions above are equivalent [1, Theorem 3.5].

The question whether a non-locally compact topological group G is separable and metrizable if G has a BCO remainder is still open [17, Question 14]. Arhangel'skiĭ [6] proved that if the remainder of the compactification bG of a non-locally compact topological group G has a point-countable base, then bG is separable and metrizable. Inspired by this, Lin and Shen [15] asked whether this result can extend to k-gentle paratopological groups. In 2012, Liu [18] gave a positive answer to this question.

Recall that a map $f : X \to Y$ is called *k-gentle* if for each compact subset F of X the image $f(F)$ is also compact. A paratopological group G is called k-gentle [5] if the inverse map $x \to x^{-1}$ is k-gentle. Recently, Wang and He [25] proved that if the remainder of the compactification bG of a non-locally compact paratopological group G has a point-countable base, then bG is separable and metrizable. Also, they posed the following question:

Question 1.2. *([25]) Suppose that G is a non-locally compact semitopological group and bG is a compactification of G such that $Y = bG \setminus G$ has a point-countable base. Is bG separable and metrizable?*

The following questions are posed in [14].

Question 1.3. *([14, Questions 3.3 and 3.4]) Let G be a non-locally compact paratopological group. If the remainder $Y = bG \setminus G$ has (locally) a sharp base, are G and bG separable and metrizable?*

Question 1.4. *([14, Question 3.6]) Let G be a non-locally compact paratopological group. If the remainder $Y = bG \setminus G$ has a weakly uniform base, are G and bG separable and metrizable?*

Question 1.5. *([14, Question 4.2]) Let G be a non-locally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a local $\aleph$-space, are G and bG separable and metrizable?*

Recall that a paratopological group G is called *$\mathbb{R}_1$-factorizable* [22] if G is a T_1-space and for every continuous real-valued function f on G, one can find a continuous homomorphism $p : G \to K$ onto a second-countable paratopological group K satisfying T_1 separation axiom, and a continuous real-valued function g on K such that $f = g \circ p$. More about $\mathbb{R}_1$-factorizable paratopological group one can refer to [28].

In this paper, we give positive answers to Questions 1.2, 1.3 and 1.5. Also, Question 1.4 is answered partially. The following result was proved by Arhangel'skiĭ [4].

Theorem 1.6. *([4]) If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

Throughout this paper, all the undefined topological concepts can be found in [10,12].

2. Remainders of semitopological groups

First we give a positive answer to Question 1.2. This also gives answers to [14, Questions 3.5 and 4.3] and show even more. Lemma 2.1 is easy, so we omit its proof. Lemma 2.2 was proved in [23], which plays an important role in the proof of Theorem 2.3.

Lemma 2.1. *Suppose that Y is dense in X and that Y has a countable π-base. Then X has also a countable π-base.*[1]

Lemma 2.2. *([23]) Every ω-narrow semitopological group of countable π-character has countable π-base.*

[1] A *π-base* of a space at a point x of X is a family γ of non-empty open subsets of X such that every open neighborhood of x contains at least one element of γ. Put $\pi\chi(x, X) = \min\{|\gamma| : \gamma \text{ is a } \pi\text{-base at } x\} + \omega$. Then the *$\pi$-character* of X is $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.

Theorem 2.3. *Suppose that G is a non-locally compact semitopological group and bG is a compactification of G such that $Y = bG \setminus G$ has locally a point-countable base. Then bG is separable and metrizable.*

Proof. Since Y has locally a point-countable base, for each $y \in Y$, there exists an open set U in bG such that $y \in U \cap Y$ and that $\overline{U \cap Y}^Y$ has a point-countable base.

Claim. *G has countable π-character.*

Proof of Claim. It is well known that every countably compact space with a point-countable base is compact and metrizable, and therefore, $\overline{U \cap Y}^Y$ is not countably compact by G being nowhere locally compact. Suppose that $A = \{a_n : n \in \omega\}$ is infinitely closed discrete in $\overline{U \cap Y}^Y$. Clearly, there is a cluster point $g \in \overline{U}^{bG} \cap G$ for A. Since $U \cap Y$ is open in Y and $U \cap Y$ is dense in $\overline{U \cap Y}^Y$, one can take a sequence $\{U_{n,i} : i \in \omega\}$ of open sets in bG such that $\{U_{n,i} \cap (U \cap Y) : i \in \omega\}$ is a local π-base at a_n in $\overline{U \cap Y}^Y$ for each $n \in \omega$. One can easily check that $\{U_{n,i} \cap (G \cap U) : i, n \in \omega\}$ is a local π-base at g in G. The proof of Claim is finished.

Since every space with a point-countable base is of countable type, Y is of local countable type. From [25, Lemma 2.2] it follows that Y is of countable type. Thus, by Theorem 1.1, the group G is Lindelöf. As we all know, every Lindelöf semitopological group is ω-narrow. From Claim and Lemma 2.2, it follows that G has a countable π-base.

From Lemma 2.1 it follows that Y has also a countable π-base, and therefore, $\overline{U \cap Y}^Y$ is separable. From [12, Theorem 7.2] it follows that $\overline{U \cap Y}^Y$ has a countable base, since $\overline{U \cap Y}^Y$ has a point-countable base. Therefore, $\overline{U \cap Y}^Y$ is a Lindelöf p-space. According to Theorem 1.6 the set $\overline{U \cap Y}^{bG} \cap G = \overline{U}^{bG} \setminus \overline{U \cap Y}^Y$ is a Lindelöf p-space. In addition, every Hausdorff semitopological group of countable π-character has a G_δ-diagonal [5, Corollary 2.5], so $\overline{U \cap Y}^{bG} \cap G$ is separable and metrizable by [12, Corollary 3.8]. Therefore, G is of local countable type, and G is of countable type by [25, Lemma 2.2]. This implies that Y is Lindelöf by Theorem 1.1. Since we have proved that Y is locally metrizable, Y is separable and metrizable. In particular, Y is a Lindelöf p-space, so G is also a Lindelöf p-space by Theorem 1.6. Hence, G is separable and metrizable by [12, Corollary 3.8], since G has a G_δ-diagonal. Since Y and G are separable and metrizable and $bG = Y \cup G$ is compact, one can easily check that bG is separable and metrizable. □

Corollary 2.4. *([6, Theorem 10]) Let G be a non-locally compact topological group and bG a compactification of G such that the remainder $Y = bG \setminus G$ has a point-countable base. Then bG is separable and metrizable.*

The following result gives a positive answer to Question 1.3.

Corollary 2.5. *Let G be a non-locally compact semitopological group, and let bG be a compactification of G such that the remainder $Y = bG \setminus G$ has (locally) a sharp base. Then bG is separable and metrizable.*

Proof. Since every T_1-space with a sharp base has a point-countable base [3, Theorem 5], the statement follows from Theorem 2.3. □

The following result improves [18, Theorem 3.2].

Corollary 2.6. *Let G be a non-locally compact semitopological group. If the remainder $Y = bG \setminus G$ has σ-locally countable base, then bG is separable and metrizable.*

Recall that a space X is called a *q-space* [21] if, for every point $x \in X$, there exists a sequence (U_n) of open neighborhoods of x satisfying that: if $x_n \in U_n$, then $\{x_n\}$ has a cluster point. The following gives a partial answer to Question 1.4.

Corollary 2.7. *Let G be a non-locally compact semitopological group and bG a compactification of G such that the remainder $Y = bG \setminus G$ has (locally) a weakly uniform base and is a q-space. Then bG is separable and metrizable.*

Proof. Since G is non-locally compact semitopological group, G is nowhere locally compact. Thus $Y = bG \setminus G$ is dense in bG. Since bG is a T_1-space, one can easily check that Y has no isolated points. It is well known that every q-space X such that every point is a G_δ-set is first-countable, so Y is first-countable. Since every weakly uniform base in a first-countable space is point-countable at each nonisolated point [3, Lemma 6], it follows that bG is separable and metrizable by Theorem 2.3. □

In the following, we discuss the cardinal invariants in compactifications of semitopological groups. We need the following lemma.

Lemma 2.8. *Let X be a non-locally compact homogeneous space and the remainder $Y = bX \setminus X$ has a G_δ-diagonal and countable π-character. Then X has countable π-character.*

Proof. It is well known that every countably compact T_2-space with a G_δ-diagonal is a compact metrizable space. Since X is a non-locally compact homogeneous space, Y is not countably compact. Therefore, there exists an infinitely discrete closed set $A = \{y_n : n \in \omega\}$ in Y. Clearly, there is a cluster point b of A such that $b \in X$. Take countable open sets $\{U_{n,i} : i \in \omega\}$ of bX such that $\{U_{n,i} \cap Y\}$ is a locally π-base at y_n for each $n \in \omega$. Then one can easily obtain that $\{U_{n,i} \cap X : n, i \in \omega\}$ is a locally π-base at b in X. Thus, X has countable π-character by the homogeneity of X. □

Clearly, every developable space is weakly developable. A space is weakly developable if and only if it is a p-space with a G_δ-diagonal [1, Theorem 2.4]. Thus the following theorem improves [25, Theorem 3.1].

Theorem 2.9. *Let G be a non-locally compact semitopological group and bG a compactification of G such that the remainder $Y = bG \setminus G$ is weakly developable. If G is a Σ-space, then $nw(G) = \pi w(G) = \pi w(Y) = \omega$.*

Proof. Clearly, Y is first-countable. Since G is a non-locally compact semitopological group, from Lemma 2.8 it follows that G has a countable π-character, i.e., $\pi w(G) = \omega$. Since G is dense in bG, one can easily obtain that $\pi w(Y) = \omega$. Therefore, G has a G_δ-diagonal by [5, Corollary 2.5]. Since every Σ-space with a G_δ-diagonal is a σ-space [12, Theorem 4.15], G is a σ-space, i.e., G has a σ-discrete closed network. Since Y is weakly developable, Y is a p-space. It is well known that every p-space is of countable type, so G is Lindelöf by Theorem 1.1. Therefore, $nw(G) = \omega$. □

3. Remainders of paratopological groups

For a non-locally compact topological group G, if the remainder $bG \setminus G$ has a G_δ-diagonal, then G is separable and metrizable [6, Theorem 5]. However, this result cannot extend to paratopological groups. In fact, Alexandorff's double-arrow space is a Hausdorff compactification of Sorgenfrey line, its remainder is still a copy of Sorgenfrey line, so the remainder has a regular G_δ-diagonal, but Sorgenfrey line is not metrizable. This motivated Liu and Lin [19] to pose the following question.

Question 3.1. *([19, Question 5.1]) Let G be a non-locally compact paratopological group. Suppose that the remainder $Y = bG \setminus G$ has a regular G_δ-diagonal. Does G have a regular G_δ-diagonal?*

In the same paper, Liu and Lin gave a partial answer to this question, i.e., for a non-locally compact Abelian paratopological group G in which every compact subset is first-countable, if the remainder $Y =$

$bG \setminus G$ has a regular G_δ-diagonal, then G has a regular G_δ-diagonal [19, Theorem 5.1]. In fact, the condition 'Abelian' can be dropped and the condition 'every compact subset is first-countable' can be weaken to 'every compact subset has countable π-character'.

Theorem 3.2. *Let G be a nonlocally compact paratopological group in which every compact subset has countable π-character. Suppose that the remainder $Y = bG \setminus G$ has a G_δ-diagonal. Then G has a regular G_δ-diagonal.*

Proof. Since every space with a G_δ-diagonal is Ohio complete,[2] there exists a G_δ-set Z of bG such that $Y \subseteq Z$ and every $z \in Z \setminus Y$ is separated from Y by a G_δ-set of bG.

Case 1. $Z \setminus Y \neq \emptyset$. Take a point $x \in Z \setminus Y$. Then there exists a G_δ-set F of bG such that $x \in F$ and $F \cap Y = \emptyset$. One can easily find a sequence $\{V_n : n \in \omega\}$ such that $\overline{V_{n+1}}^{bG} \subseteq V_n$ for each $n \in \omega$ and that $x \in L = \bigcap_{n\in\omega} V_n \subseteq F$. Clearly, L is a compact set with a countable local base in G by the compactness of bG. Since every compact subset in G has countable π-character, x has a countable π-base in G by [25, Lemma 2.1]. Thus, G has countable π-character by the homogeneity of G. From [23, Theorem 2.25] it follows that G has a regular G_δ-diagonal.

Case 2. $Z \setminus Y = \emptyset$. Then Y is a G_δ-set of bG. Since every point $y \in Y$ is a G_δ-set of Y, every point $y \in Y$ is a G_δ-set of bG. Therefore, Y is first-countable by the compactness of bG. Therefore, according to Lemma 2.8 G has countable π-character. From [23, Theorem 2.25] it follows that G has a regular G_δ-diagonal. $\square$

The following result is obvious by Theorem 3.2.

Corollary 3.3. *Let G be a non-locally compact paratopological group in which every point is a G_δ-set. Suppose that the remainder $Y = bG \setminus G$ has a G_δ-diagonal. Then G has a regular G_δ-diagonal.*

Theorem 3.4. *Let G be a non-locally compact paratopological group. Suppose that the remainder $Y = bG \setminus G$ is a weakly developable space. Then $nw(G) = \pi w(G) = \pi w(Y) = \omega$.*

Proof. Since Y is weakly developable, Y is a p-space [1, Theorem 2.4]. Thus Y is Ohio complete [4]. Therefore, there exists a G_δ-set Z of bG such that $Y \subseteq Z$ and every $z \in Z \setminus Y$ is separated from Y by a G_δ-set of bG. We split our proof in two cases.

Case 1. $Z \setminus Y \neq \emptyset$. Take a point $x \in Z \setminus Y$. Then there exists a G_δ-set F of bG such that $x \in F$ and $F \cap Y = \emptyset$. One can easily find a sequence $\{V_n : n \in \omega\}$ such that $\overline{V_{n+1}}^{bG} \subseteq V_n$ for each $n \in \omega$ and that $x \in L = \bigcap_{n\in\omega} V_n \subseteq F$. Clearly, L is a compact set with a countable local base in G by the compactness of bG. It is known (see [8, Proposition 4.1]) that if a paratopological group H has a compact subset K of countable character in H, then H is of countable type; apply this fact to see that G is of countable type, and therefore, Y is Lindelöf by Theorem 1.1. In addition, Y is a p-space with a G_δ-diagonal, so Y is separable and metrizable. By Theorem 2.3, one can easily obtain that bG is separable and metrizable, and therefore, $nw(G) = \pi w(G) = \pi w(Y)$.

Case 2. $Z \setminus Y = \emptyset$. Then Y is a G_δ-set in bG. Therefore, Y has countable character. By the proof of Theorem 3.2, we obtain that G has countable π-character. From [23, Theorem 2.25] it follows that G has a regular G_δ-diagonal. Since G is σ-compact, G is the union of countable many of separable metrizable spaces. Thus, $nw(G) = \omega$. From Lemma 2.2 it follows that $\pi w(G) = \omega$. Since G is dense in bG, one can easily obtain that $\pi w(Y) = \omega$. $\square$

[2] Recall that a space X is *Ohio complete* [4] if in every compactification bX of X there exists a G_δ-subset Z such that $X \subseteq Z$ and every $y \in Z \setminus X$ is separated from X by a G_δ-subset of Z.

We have the following by Theorem 3.4.

Corollary 3.5. *([25, Theorem 3.2]) Let G be a non-locally compact paratopological group, and let bG be a compactification of G such that the remainder $Y = bG \setminus G$ is developable. Then $nw(G) = \pi w(G) = \pi w(Y) = \omega$.*

Recently, Lin, Liu and Xie [14] proved that for a non-locally compact paratopological group G, if the remainder $Y = bG \setminus G$ is a developable and meta-Lindelöf space, then bG is separable and metrizable. Now we can weaken the condition 'developable' to 'weakly developable'.

Corollary 3.6. *Let G be a non-locally compact paratopological group. Suppose that the remainder $Y = bG \setminus G$ is a weakly developable and meta-Lindelöf space. Then bG is separable and metrizable.*

Proof. According to Theorem 3.4 we obtain that Y is separable. In addition, Y is meta-Lindelöf space, and therefore, Y is Lindelöf. Since Y is weakly developable, Y is a p-space with a G_δ-diagonal [1, Theorem 2.4]. Therefore, Y is separable and metrizable. The statement directly follows from Theorem 2.3. □

Next, we discuss the remainders of k-gentle paratopological groups.

Proposition 3.7. *Let G be a non-locally compact k-gentle paratopological group, and bG a compactification of G such that the remainder $Y = bG \setminus G$ has a G_δ-diagonal. Then either Y is Čech-complete, or bG is separable and metrizable.*

Proof. Since G is a non-locally compact k-gentle paratopological group, the remainder $Y = bG \setminus G$ is either pseudocompact or Lindelöf [8, Theorem 4.4].

Case 1. Y is pseudocompact. Since every pseudocompact space with a G_δ-diagonal is Čech-complete [3, Lemma 20], Y is Čech-complete.

Case 2. Y is Lindelöf. From [8, Corollary 4.5] it follows that G is a topological group, so bG is separable and metrizable by [6, Theorem 5]. □

Theorem 3.8. *Let G be a non-locally compact k-gentle paratopological group, and bG a compactification of G such that the remainder $Y = bG \setminus G$ has a G_δ-diagonal. Then $nw(G) = \pi w(G) = \pi w(Y) = \omega$.*

Proof. From Proposition 3.7 it follows that either Y is Čech-complete, or bG is separable and metrizable. Therefore, we suppose that Y is Čech-complete. Then Y is a p-space and G is σ-compact. From Lemma 2.8 it follows that G has countable π-character. Therefore, G has a G_δ-diagonal by [5, Corollary 2.5]. Since G is a σ-compact, G is the union of countable many of compact metrizable spaces, and therefore, $nw(G) = \omega$. From Lemma 2.2 it follows that $\pi w(G) = \omega$. Since G is dense in bG, one can easily obtain that $\pi w(Y) = \omega$. □

Recall that a space (X, τ) is called a *k-semistratifiable space* [20] if there exists a function $\mathcal{S} : \mathbb{N} \times \tau \to \tau^c$ such that:

(a) for each $U \in \tau$, $U = \bigcup\{\mathcal{S}(n, U) : n \in \mathbb{N}\}$;
(b) if $U, V \in \tau$ and $U \subseteq V$, then $\mathcal{S}(n, U) \subseteq \mathcal{S}(n, V)$ for each $n \in \mathbb{N}$;
(c) for each compact subset K of X and open neighborhood U of K, there exists an $n \in \mathbb{N}$ such that $K \subseteq \mathcal{S}(n, U)$.

Corollary 3.9. *Let G be a non-locally compact k-gentle paratopological group, and bG a compactification of G such that the remainder $Y = bG \setminus G$ is a k-semistratifiable space. Then bG is separable and metrizable.*

Proof. It is well known that every k-semistratifiable space is a σ-space [11,16]. Hence, Y has a G_δ-diagonal [12] and every point of Y is a G_δ-set in Y. We split our proof in the following two cases.

Case 1. Y is first-countable.

Since every first-countable k-semistratifiable space is paracompact [20], Y is a paracompact σ-space. From Theorem 3.8 we know that Y is separable. It is known that every separable paracompact σ-space has a countable network [12, Theorem 4.4]. Thus Y has a countable network. Therefore, $bG = Y \cup G$ has a countable network. By the compactness of bG, bG is separable and metrizable.

Case 2. Y is not first-countable.

Since every point of Y is a G_δ-set in Y, there exists a point $y \in Y$ and a G_δ-set F of bG such that $F \cap Y = \{y\}$ and $F \cap G \neq \emptyset$. Therefore, one can find a nonempty closed G_δ-set F' of bG such that $F' \subseteq F$ and $F' \subseteq G$. By the compactness of bG the set F' has a countable base of open neighborhoods in bG and hence in G. Since G is a paratopological group, from [8, Proposition 4.1] it follows that G is of countable type. Therefore, Y is Lindelöf by Theorem 1.1. Thus Y is also a paracompact σ-space. The remainder of the proof is the same as Case 1. □

Finally, we shall give a positive answer to Question 1.5.

Theorem 3.10. *Let G be a non-locally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a local $\aleph$-space, then bG is separable and metrizable.*

Proof. Since Y is a local $\aleph$-space, for every point $y \in Y$ there exists an open set U in bG such that $y \in V = U \cap Y$ is an $\aleph$-space. Since every $\aleph$-space is a σ-space, V is a σ-space and have a G_δ-diagonal. Therefore, every point of Y is a G_δ-set.

Case 1. Y is first-countable.

Since every first-countable $\aleph$-space is metrizable [12, Theorem 11.4], Y is a locally metrizable space. Since every metrizable space has a point countable base, from Theorem 2.3 it follows that bG is separable and metrizable.

Case 2. Y is not first-countable.

There exists a point $y_0 \in Y$ such that y_0 is not a G_δ-set in bG. One can find a G_δ-set F of bG such that $F \cap Y = \{y_0\}$ and $F \cap G \neq \emptyset$. Therefore, one can find a nonempty closed G_δ-set F' of bG such that $F' \subseteq F$ and $F' \subseteq G$. By the compactness of bG the set F' has a countable base of open neighborhoods in bG and hence in G. Since G is a paratopological group, from [8, Proposition 4.1] it follows that G is of countable type. Therefore, Y is Lindelöf by Theorem 1.1. Since V is a σ-space, one can find an open set W in Y such that $y \in W \subseteq \overline{W}^Y \subseteq V$ and $\overline{W}^Y$ is a σ-space. Therefore, $\overline{W}^Y$ has a countable network. This implies that Y has a countable network, since Y is Lindelöf. Thus G is first-countable from [14, Theorem 2.1]. Since every first-countable $\mathbb{R}_1$-factorizable paratopological group is totally ω-narrow [22, Proposition 3.5] and every first-countable totally ω-narrow paratopological group has a countable base [24], G is separable and metrizable. Thus $bG = Y \cup G$ has a countable network. By the compactness of bG, the space bG is separable and metrizable. □

Theorem 3.11. *Let G be a non-locally compact k-gentle paratopological group. If the remainder $Y = bG \setminus G$ is a locally $\aleph$-space, then bG is separable and metrizable.*

Proof. In the proof of Theorem 3.10 we have shown that G is first-countable. Thus G is a k-space. Since G is k-gentle paratopological group, G is a topological group. It is well known that if a non-locally compact topological group H has a G_δ-diagonal remainder $bH \setminus H$, then bH is separable and metrizable. Thus bG is separable and metrizable. □

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References

[1] B. Alleche, A.V. Arhangel'skiĭ, J. Calbrix, Weak developments and metrization, Topol. Appl. 100 (2000) 23–38.
[2] A.V. Arhangel'skiĭ, Some connections between properties of topological groups and of their remainders, Mosc. Univ. Math. Bull. 54 (1999) 1–6.
[3] A.V. Arhangel'skiĭ, W. Just, E.A. Rezniczenko, P.J. Szeptycki, Sharp bases and weakly uniform bases versus point-countable bases, Topol. Appl. 100 (2000) 39–46.
[4] A.V. Arhangel'skiĭ, Remainders in compactifications and generalized metrizability properties, Topol. Appl. 150 (2005) 79–90.
[5] A.V. Arhangel'skiĭ, E.A. Reznichenko, Paratopological and semitopological groups versus topological groups, Topol. Appl. 151 (2005) 107–119.
[6] A.V. Arhangel'skiĭ, More on remainders close to metrizable spaces, Topol. Appl. 154 (2007) 1084–1088.
[7] A.V. Arhangel'skiĭ, Two types of remainders of topological groups, Comment. Math. Univ. Carol. 49 (2008) 119–126.
[8] A.V. Arhangel'skiĭ, M.M. Choban, On remainders of rectifiable spaces, Topol. Appl. 157 (2010) 789–799.
[9] A.V. Arhangel'skiĭ, M.M. Choban, Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis, Topol. Proc. 37 (2011) 33–60.
[10] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, 1989.
[11] Z.M. Gao, On g-function separation, Quest. Answ. Gen. Topol. 4 (1986) 47–57.
[12] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, pp. 423–501.
[13] M. Henriksen, J. Isbell, Some properties of compactifications, Duke Math. J. 25 (1958) 83–106.
[14] F. Lin, C. Liu, L.H. Xie, Remainders of semitopological groups or paratopological groups, Ukr. Math. J. 4 (2014) 500–509.
[15] F. Lin, R. Shen, On rectifiable spaces and paratopological groups, Topol. Appl. 158 (2011) 597–610.
[16] S. Lin, A note on k-semistratifiable spaces, J. Suzhou Univ. (Nat. Sci. Ed.) 4 (1988) 357–363.
[17] C. Liu, Remainders in compactification of topological groups, Topol. Appl. 156 (2009) 849–854.
[18] C. Liu, Metrizability of paratopological (semitopological) groups, Topol. Appl. 159 (2012) 1415–1420.
[19] C. Liu, S. Lin, Generalized metric spaces with algebraic structures, Topol. Appl. 157 (2010) 1966–1974.
[20] D.J. Lutzer, Semimetrizable and stratifiable spaces, Topol. Appl. 1 (1971) 43–48.
[21] E.A. Michale, A quintuple quotient quest, Gen. Topol. Appl. 2 (1972) 91–138.
[22] M. Sanchis, M. Tkachenko, $\mathbb{R}$-Factorizable paratopological groups, Topol. Appl. 157 (2010) 800–808.
[23] I. Sánchez, Cardinal invariants of paratopological groups, Topol. Algebra Appl. 1 (2013) 37–45.
[24] M. Tkachenko, Paratopological and semitopological groups vs topological groups, in: K.P. Hart, J. van Mill, P. Simon (Eds.), Recent Progress in General Topology III, Springer Atlantis Press, 2013, pp. 803–859.
[25] H. Wang, W. He, Remainders and cardinal invariants, Topol. Appl. 164 (2014) 14–23.
[26] L.H. Xie, S. Lin, The Baire property in the remainders of semitopological groups, Bull. Aust. Math. Soc. 88 (2013) 301–308.
[27] L.H. Xie, S. Lin, Remainders of topological and paratopological groups, Topol. Appl. 160 (2013) 648–655.
[28] L.H. Xie, S. Lin, M. Tkachenko, Factorization properties of paratopological groups, Topol. Appl. 160 (2013) 1902–1917.