Northeastern Math. J.
6(3)(1990), 287--290

Spaces with σ-hereditarily Closure-preserving Pseudobases*)

Lin Shou (林 寿)
(Department of Mathematics, Ningde Teachers' College, Fujian, 352100)

Abstract In this paper the relationship between pseudobases and k-networks is discussed. It is shown that a regular and T_1 space with σ-hereditarily closure-preserving pseudobases is an $\aleph$-space, and that a regular and T_1 space is a space with σ-hereditarily closure-preserving closed pseudobases if and only if it is either an $\aleph_0$-space or a σ-closed discrete space, of which all compact subsets are finite.

Key Words Pseudobase; k-network; $\aleph_0$-space; $\aleph$-space; Hereditarily Closure-preserving Collection

All spaces considered in this paper are regular and T_1.

Let X be a topological space, and let $\mathscr{P}$ be a collection of subsets of X. $\mathscr{P}$ is closure-preserving (abbr. CP) if $\overline{\bigcup\mathscr{P}'}=\bigcup\{\overline{P}: P\in\mathscr{P}'\}$ for each subcollection $\mathscr{P}'$ of $\mathscr{P}$. $\mathscr{P}$ is locally finite (abbr. LF) if $\{\overline{P}: P\in\mathscr{P}\}$ is point-finite and CP. $\mathscr{P}$ is hereditarily closure-preserving (abbr. HCP) if $\{H(P): P\in\mathscr{P}\}$ is CP for each $H(P)\subset P\in\mathscr{P}$. $\mathscr{P}$ is weakly hereditarily closure-preserving (abbr. WHCP) if $\{\{x(P)\}: P\in\mathscr{P}\}$ is CP for each $x(P)\in P\in\mathscr{P}$. Obviously, LF collections are HCP, and HCP collections are CP and WHCP. $\mathscr{P}$ is a pseudobase (k-network) if whenever K is a compact subset of an open set U of X, then $K\subset P\subset U$ ($K\subset\bigcup\mathscr{P}'\subset U$) for some $P\in\mathscr{P}$ (finite subfamily $\mathscr{P}'\subset\mathscr{P}$). Spaces with countable pseudobases are $\aleph_0$-spaces. Spaces with σ-LF k-networks are $\aleph$-spaces.

Let $\mathscr{P}$ be a k-network for X. Put

$$\mathscr{P}'=\{\bigcup\mathscr{F}: \mathscr{F} \text{ is a finite subfamily of } \mathscr{P}\}.$$

Then $\mathscr{P}'$ is a pseudobase for X. But, if $\mathscr{P}$ is HCP, then $\mathscr{P}'$ can not be HCP. It is natural to raise the following question: What is the relationship between spaces with σ-HCP pasudobases and $\aleph$-spaces? In this paper it is shown that spaces with σ-HCP pseudobases are $\aleph$-spaces (Theorem 1), and that a space with σ-HCP closed pseudobases if and only if it is either an $\aleph_0$-space or a σ-closed discrete space, of which all compact subsets are finite (Theorem 2).

§ 1. Spaces with σ-HCP Pseudobases

A space X is $\aleph_1$-compact if every closed discrete subspace of X is countable.

Received Dec. 9, 1987.
*) Project Supported Partly by the Science Fund of Fujian Province.

Lemma 1 An $\aleph_1$-compact space with σ-WHCP k-networks is an $\aleph_0$-space.

Proof Suppose X is an $\aleph_1$-compact space with σ-WHCP k-network $\mathscr{P}$. Let $\mathscr{P}=\bigcup\{\mathscr{P}_n: n\in N\}$, where $\mathscr{P}_n\subset\mathscr{P}_{n+1}$ and $\mathscr{P}_n$ is WHCP. For each $n\in N$, put

$$A_n=\{x\in X: \mathscr{P}_n \text{ is not point countable at } x\}.$$

Then $\{P-A_n: P\in\mathscr{P}_n\}$ is countable, and A_n is a countable closed discrete subspace of X. In fact, if $\{P-A_n: P\in P_n\}$ were not countable, then there would exist $\{P_a: a<\omega_1\}$ such that the (P_a-A_n)'s are distinct and non-empty. For each $a<\omega_1$, take a point $x_a\in P_a-A_n$. Since $\mathscr{P}_n$ is WHCP and X is $\aleph_1$-compact, $\{x_a: a<\omega_1\}$ is countable. So there exist an uncountable subset A of ω_1 and $x\notin A_n$ such that $x_a=x$ for each $a\in A$, a contradiction. Hence $\{P-A_n: P\in\mathscr{P}_n\}$ is countable. If $Z=\{z_h\in A_n: h\in H\}$ with $|H|\leqslant\aleph_1$, since $\mathscr{P}_n$ is not point countable at point z_h, by well-ordering principle and transfinite induction we can obtain a subcollection $\{P_h: h\in H\}$ of $\mathscr{P}_n$ such that $z_h\in P_h$ and the P_h's are distinct. Since $\mathscr{P}_n$ is WHCP, Z is a closed discrete subspace of X. By the $\aleph_1$-compactness of X, A_n is a countable closed discrete subspace of X. Therefore

$$\mathscr{P}'_n=\{P-A_n: P\in\mathscr{P}_n\}\cup\{\{x\}: x\in A_n\}$$

is a countable collection. If K is a compact subset of an open set U of X, then there exist $n\in N$ and a finite subcollection $\mathscr{P}_n^*$ of $\mathscr{P}_n$ such that $K\subset\bigcup P_n^*\subset U$. Since every closed discrete subspace of a compact space is finite, $K\cap A_n$ is finite, so

$$\mathscr{P}_n^{**}=\{P-A_n: P\in\mathscr{P}_n^*\}\cup\{\{x\}: x\in K\cap A_n\}$$

is a finite subcollection of $\mathscr{P}'_n$, and $K\subset\bigcup\mathscr{P}_n^{**}\subset U$. Hence $\bigcup\{\mathscr{P}'_n: n\in N\}$ is a countable k-network for X. This completes the proof of the Lemma.

Lemma 2 A space with σ-WHCP pseudobases is either an $\aleph_0$-space or the space whose all compact subsets are finite.

Proof Suppose the space X has σ-WHCP pseudobases. Let $\mathscr{P}=\bigcup\{\mathscr{P}_n: n\in N\}$ be a σ-WHCP pseudobase for X, where each $\mathscr{P}_n$ is WHCP. If X has an infinite compact subset, we shall show that X is an $\aleph_0$-space. Let K be an infinite compact subset of X. Then $\{P\cap K: P\in\mathscr{P}\}$ is a σ-WHCP pseudobase for K because $\mathscr{P}$ is a σ-WHCP pseudobase for X. So K is a compact $\aleph_0$-space by Lemma 1. Thus K is a compact metrizable subset of X (cf. [1]). Therefore there exists mutually distinct $x_n\in K$ such that the sequence $\{x_n\}$ converges to $x\in K-\{x_n: n\in N\}$. If X is not $\aleph_1$-compact, there exists a closed discrete subspace $A\subset X$ with $|A|=\aleph_1$. Then $|A-K|=\aleph_1$ because $K\cap A$ is finite. Let $A-K=\{z_a: a<\omega_1\}$. For each $a<\omega_1$, let

$$V_a=X-\{z_b: a\neq b<\omega_1\}.$$

Then V_a is an open subset of X and $K\cup\{z_a\}\subset V_a$. So there exist $n(a)\in N$ and $P_a\in\mathscr{P}_{n(a)}$ such that $K\cup\{z_a\}\subset P_a\subset V_a$. Thus there exist an uncountable subset Λ of ω_1 and $m\in N$ such that $n(a)=m$ when $a\in\Lambda$, i.e., $P_a\in\mathscr{P}_m$. For distinct $a, b\in\Lambda$, the P_a's are distinct because $z_a\in P_a\subset X-\{z_b\}$. So $\{P_a: a\in\Lambda\}$ is WHCP. Now, by the uncountability of Λ and $\{x_n: n\in N\}\subset\bigcap\{P_a: a\in\Lambda\}$, $\{x_n: n\in N\}$ is a closed discrete subspace of X, a contradiction. Therefore X is an $\aleph_1$-compact space, and so X is an $\aleph_0$-space by Lemma 1. This completes the proof of the Lemma.

By Lemma 2, we have the following Theorem.

Theorem 1 Spaces with σ-HCP pseudobases are $\aleph$-spaces.

§ 2. Spaces with σ-HCP Closed Pseudobases

In this section we give a characterization of spaces with σ-HCP closed pseudobases.

Theorem 2 A space X has a σ-HCP closed pseudobase if and only if it is either an $\aleph_0$-space or a σ-closed discrete space, of which all compact subsets are finite.

Proof Necessity. Suppose X is a space with σ-HCP closed pseudobases. By Lemma 2, X is either an $\aleph_0$-space or a space whose all compact subspaces are finite. Now, suppose X is a space whose all compact subspaces are finite. We shall prove that X is a σ-closed discrete space. Let $\mathscr{P} = \bigcup\{\mathscr{P}_n: n\in N\}$ be a σ-HCP closed pseudobase for X, where each $\mathscr{P}_n$ is an HCP collection of closed subsets of X. We can assume that $X\in\mathscr{P}_n\subset\mathscr{P}_{n+1}$. For each $n\in N$, $x\in X$, put

$$(\mathscr{P}_n)_x = \{P \in \mathscr{P}_n: x \in P\},$$

$$Y = \{x \in X: \bigcap (\mathscr{P}_n)_x = \{x\} \text{ for some } n \in N\}.$$

The proof of necessity can be completed by the following two steps.

Ⅰ. Y is a σ-closed discrete subspace for X.

Ⅱ. $X-Y$ is countable.

First, we prove Ⅰ. For each $n\in N$, put

$$Y_n = \{x \in X: \bigcap (\mathscr{P}_n)_x = \{x\}\}.$$

Then $Y=\bigcup\{Y_n: n\in N\}$ and each Y_n is a closed discrete subspace for X. In fact, for each $x\in X$, put

$$U = X - \bigcup\{P \in \mathscr{P}_n: x \notin P\}.$$

Then U is an open neighborhood of x, because $\mathscr{P}_n$ is a CP collection of closed subsets of X. If $y\in U\cap Y_n$, then $\bigcap(\mathscr{P}_n)_x\subset\bigcap(\mathscr{P}_n)_y$. Otherwise there would exists $z\in\bigcap(\mathscr{P}_n)_x-\bigcap(\mathscr{P}_n)_y$. Since $z\notin\bigcap(\mathscr{P}_n)_y$, there exists $P'\in\mathscr{P}_n$ such that $y\in P'$ and $z\notin P'$. By the fact that $z\in\bigcap(\mathscr{P}_n)_x$, $x\notin P'$. Hence

$$y \in U \cap (\bigcap (\mathscr{P}_n)_y) \subset (X - P') \cap P' = \phi,$$

a contradiction. Thus

$$x \in \bigcap (\mathscr{P}_n)_x \subset \bigcap (\mathscr{P}_n)_y = \{y\}$$

by $y\in Y_n$, and $x=y$. This means that the open neighborhood U of x contains at most one point of Y_n, so Y_n is a closed discrete subspace of X, and Y is a σ-closed discrete subspace of X, which completes the proof of Ⅰ.

For the proof of Ⅱ, we prove $X-Y$ is an $\aleph_0$-space. We assert that each $\mathscr{P}_n$ is point finite at $X-Y$, i.e., $(\mathscr{P}_n)_x$ is finite for each $x\in X-Y$. Otherwise $(\mathscr{P}_i)_x$ is infinite for some $i\in N$ and $x\in X-Y$. Since $x\notin Y$, $\bigcap(\mathscr{P}_j)_x-\{x\}\neq\phi$ for each $j\in N$. Take $x_i\in\bigcap(\mathscr{P}_i)_x-\{x\}$ and $P_i\in(\mathscr{P}_i)_x$ with $x_i\in P_i$. For each $m>i$, if, when $i<j\leqslant m$, we have chosen $x_j\in X-\{x\}$ and $P_j\in(P_i)_x-\{P_i,\cdots,P_{j-1}\}$ with $x_j\in P_j$, then take

$$x_{m+1} \in \bigcap (\mathscr{P}_{m+1})_x - \{x\} \subset \bigcap (\mathscr{P}_i)_x$$

and

$$P_{m+1} \in (\mathscr{P}_i)_x - \{P_i, \cdots, P_m\}$$

with $x_{m+1} \in P_{m+1}$. Thus we obtain a sequence $\{x_j\}_{j \geqslant i}$ of $X - \{x\}$ and a subfamily $\{P_j: j \geqslant i\}$ of $(\mathscr{P}_i)_x$ with $x_j \in P_j \cap (\cap (\mathscr{P}_j)_x)$. Now, $\{x_j: j \geqslant i\}$ is closed because $\mathscr{P}_i$ is HCP, so there exist $m \geqslant i$ and $P \in \mathscr{P}_m$ such that

$$x \in P \subset X - \{x_j: j \geqslant i\} \subset X - \{x_m\},$$

which contradicts $x_m \in \cap (\mathscr{P}_m)_x \subset P$. Therefore every $(\mathscr{P}_n)_x$ is finite, and $\{P - Y: P \in \mathscr{P}\}$ is a point countable pseudobase for subspace $X - Y$ of X. Since spaces with point countable pseudobases are $\aleph_0$-spaces (cf. [2]), $X - Y$ is an $\aleph_0$-space. Now, there exist a separable metrizable space M and a continuous, surjective mapping (cf. [1]) $f: M \to X - Y$. Take $z(x) \in f^{-1}(x)$ for each $x \in X - Y$, and put

$$Z = \{z(x): x \in X - Y\}.$$

Then Z is a separable metrizable subspace of M, and $f|_Z: Z \to X - Y$ is one-to-one and surjective. Since $X - Y$ is a space whose all compact subsets are finite, so is Z. In addition, Z is first countable, so Z is a separable discrete space. Hence Z is countable. Thus $X - Y$ is countable, which completes the proof of Ⅱ.

Sufficiency. If X is an $\aleph_0$-space, then X has a σ-HCP closed pseudobase. Suppose X is a σ-closed discrete space, of which all compact subspaces are finite. Let $X = \bigcup \{X_n: n \in N\}$, where each X_n is a closed discrete subspace of X. For each $n \in N$, put

$$\mathscr{P}_n = \{P \subset \bigcup \{X_i: i \leqslant n\}: |P| \leqslant n\}.$$

Then $\mathscr{P}_n$ is an HCP collection of closed subsets of X because $\bigcup \{X_i: i \leqslant n\}$ is a closed discrete subspace of X. Since all compact subspaces of X are finite, $\bigcup \{\mathscr{P}_n: n \in N\}$ is a σ-HCP closed pseudobase of X. This completes the proof of Theorem 2.

Question Does a space with σ-HCP pseudobases have a σ-HCP closed pseudobase?

Recently, we have obtained an affirmative answer to the above question (cf. [3]).

References

[1] Michael, E., $\aleph_0$-spaces, *J. Math. Mech.*, **15**(1966), 983—1002.

[2] Lin Shou, A note on pseudo-bases (in Chinese), *J. Suzhou University (Natural Science)*, **5**(1989), 209—210.

[3] Lin Shou, A study of pseudobases, *Questions and Answers in General Topology*, **6**(1988), 81—97.

Edited by Qu Tianzhen