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ABOUT REMAINDERS IN COMPACTIFICATIONS OF PARATOPOLOGICAL GROUPS

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Abstract. In this paper, we prove a dichotomy theorem for remainders in compactifications of paratopological groups: every remainder of a paratopological group G is either Lindelöf and meager or Baire. Furthermore, we give a negative answer to a question posed in [D. Basile and A. Bella, About remainders in compactifications of homogeneous spaces, *Comment. Math. Univ. Carolin.* **50** (2009), no. 4, 607–613]. Some questions about remainders in compactifications of paratopological groups are posed.
Keywords: Remainders, paratopological groups, topological groups, homogeneous spaces, Baire spaces.
MSC(2010): Primary: 54A25; Secondary: 22A05, 54B05, 54D35, 54D40.

1. Introduction

By a remainder of a space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. Remainders in compactifications of topological spaces have been studied by some topologists in the last few years. A famous classical result in this study is the following theorem of M. Henriksen and J. Isbell's:

Theorem 1.1. (M. Henriksen and J. Isbell) [9] *A space X is of countable type if and only if the remiander in any (in some) compactification of X is Lindelöf.*

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Since topological groups are much more sensitive to the properties of their remainders than topological spaces in general, topologists are mainly interested in the remainders of topological groups or paratopological groups. For instance, A.V. Arhangel'skiĭ proved the following two dichotomy theorems about remainders in compactifications of topological groups.

Theorem 1.2. [1] *Let G be a topological group. If some remainder of G is not pseudocompact, then every remainder of G is Lindelöf.*

Theorem 1.3. [2] *Suppose that G is a non-locally compact topological group. Then either every remainder of G has the Baire property or every remainder of G is σ-compact.*

In 2009, D. Basile and A. Bella proved a dichotomy theorem about remainders in compactifications of homogeneous spaces, see Theorem 1.4.

Theorem 1.4. [6] *The remainder of a homogeneous space is either Baire or meager and real compact.*

Furtherly, D. Basile and A. Bella posed the following question.

Question 1.5. [6] *Let X be a homogeneous space and let bX be a compactification of X. Is it true that the remainder $bX \setminus X$ is either pseudocompact or realcompact and meager?*

In [6], D. Basile and A. Bella proved that none of A.V. Arhangel'skiĭ's dichotomy theorems can be extended to homogeneous spaces. In [4], A.V. Arhangel'skiĭ gave an example to show that Theorem 1.2 cannot be extended to paratopological groups. Naturally, the following two questions arise.

Question 1.6. *What about the dichotomy theorem of the remainders in compactifications of paratopological groups?*

Question 1.7. *Can Theorem 1.3 be extended to paratopological groups?*

In this paper, we show that, for a paratopological group G, every remainder of G is either Lindelöf and meager or Baire, which gives an answer to Question 1.6. Moreover, we give a partial answer to Question 1.7. Finally, we give a negative answer to Question 1.5.

2. **Preliminaries**

Recall that a *semitopological group* G is a group G with a topology such that the product map of $G\times G$ into G is separately continuous. A *quasitopological group* G is a group G with a topology such that it is a semitopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x\in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product map of $G\times G$ into G associating xy with arbitrary $(x,y)\in G\times G$ is jointly continuous. A *topological group* G is a group G with a topology such that it is a paratopological group and the inverse map of G onto itself is continuous.

Recall that a space is *Baire* if the intersection of a sequence of open and dense subsets is dense. Moreover, a space is called *meager* if it can be represented as the union of a sequence of nowhere dense subsets.

Let us call a map f of a space X into a space Y *k-gentle* [4] if for every compact subset F of X the image $f(F)$ is compact. A semitopological group G will be called *k-gentle* [4] if the inverse map $(x\mapsto x^{-1},\forall x\in G)$ is k-gentle.

A family $\mathcal{A}$ of open subsets of a space X is called *a base of X at a set* A if $A=\cap\mathcal{A}$ and for any neighborhood U of A, there is a $V\in\mathcal{A}$ such that $A\subset V\subset U$. If $\mathcal{A}$ is countable, then we say that A has countable character in X. A space X is of *countable type* [8] if every compact subspace F of X is contained in a compact subspace $K\subset X$ with a countable base of open neighborhoods in X.

Throughout this paper, all spaces are assumed to be Tychonoff. Denote $\mathbb{N}$ by the set of positive natural numbers. We refer the readers to [3, 8] for notations and terminology which have not been given here explicitly.

3. **Remainders of paratopological groups**

First, we give a lemma.

Lemma 3.1. [4] *Let G be a paratopological group. If there exists a nonempty compact subset K of G such that K has countable character in G, then G is of countable type.*

Now, we give a dichotomy theorem for remainders in compactifications of paratopological groups.

Theorem 3.2. *Let G be a non-locally compact paratopological group. Then either every remainder of G has the Baire property or every remainder of G is meager and Lindelöf.*

Proof. Suppose that bG is a compactification of G such that the remainder $Y = bG \setminus G$ does not have the Baire property. Clearly, Y is dense in bG because G is nowhere locally compact. Next, we shall prove that Y is Lindelöf and meager.

Since Y does not have the Baire property, there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of Y such that $\bigcap\{U_n : n \in \mathbb{N}\}$ is not dense in Y. Then, for each $n \in \mathbb{N}$, there exists an open subset V_n of bG such that $U_n = V_n \cap Y$. Let $\gamma = \{V_n : n \in \mathbb{N}\}$. Therefore, we can find a non-empty open subset U of bG such that $(\bigcap \gamma) \cap (U \cap Y) = \emptyset$. It follows from [8, Theorem 3.9.6] that the subspace $Z = (\bigcap \gamma) \cap (U \cap G) = (\bigcap \gamma) \cap U$ is Čech-complete in $U \cap G$. It is well known that every Čech-complete space is of countable type, hence there exists a non-empty compact subset F of Z with countable character in Z. Because Z is dense in the open subspace $U \cap G$ of G, F has countable character in $U \cap G$ [3], hence F has countable character in G. Obviously, F is compact in G. Then G is of countable type by Lemma 3.1, and thus Y is Lindelöf by M. Henriksen and J. Isbell's theorem. Moreover, Y is meager by Theorem 1.4. □

Remark 3.3. *Observe that a remainder Y of a non-locally compact paratopological group G cannot have the Baire property and be Lindelöf and meager at the same time. Indeed, it is easy to see that the failure of the Baire property is equivalent to the existence of some non-empty open meager subset. Thus we have the following two corollaries.*

Corollary 3.4. *Let X be neither Baire nor meager. Then X cannot be a remainder in compactifications of any paratopological group.*

Corollary 3.5. *Let X be neither Baire nor Lindelöf. Then X cannot be a remainder in compactifications of any paratopological group.*

Remark 3.6. *D. Basile and A. Bella proved that there exists a homogeneous space such that the remainder of some compactification is neither Baire nor Lindelöf, see* [6, Example 3.3]. *Hence Theorem 3.2 cannot be extended to homogeneous spaces. However, we have the following question.*

Question 3.7. *Let X be a non-locally compact semitopological group or quasitopological group, and let bX be a compactification of X. Is it true* *that the remainder $bX \setminus X$ has the Baire property or is Lindelöf and meager?*

Next, we obtain two corollaries from Theorem 3.2. First, we show that Theorem 1.3 can be generalized to the case of k-gentle paratopological groups, which gives a partial answer to Question 1.7.

Lemma 3.8. [4] *Let G be a k-gentle paratopological group such that some remainder of G is Lindelöf. Then G is a topological group.*

Corollary 3.9. *Let G be a non-locally compact k-gentle paratopological group. Then either every remainder of G has the Baire property or every remainder of G is σ-compact.*

Proof. Suppose that bG is a compactification of G. Put $Y = bG \setminus G$. By Theorem 3.2, Y either has the Baire property or is meager and Lindelöf. Suppose that Y does not have the Baire property. Then Y is Lindelöf, and hence G is a topological group by Lemma 3.8. Then Y is σ-compact by Theorem 1.3. $\square$

A space X is called *metacompact* if each open covering of X can be refined by a point-finite open covering.

It follows from [1] that a remainder in some compactification of a topological group is metacompact if and only if it is Lindelöf, if and only if it is realcompact. Therefore, we pose the following question.

Question 3.10. *Assume that G is a non-locally compact paratopological group and $Y = bG \setminus G$, where bG is a Hausdorff compactification of G. Are the following conditions equivalent?*

(1) *Y is metacompact;*
(2) *Y is Lindelöf;*
(3) *Y is realcompact.*

A space X is called *ccc* if every disjoint family of open subsets of X is countable.

Lemma 3.11. [7] *Every point-finite open collection in a ccc Baire space is countable.*

The next corollary gives a partial answer to Question 3.10.

Corollary 3.12. *Assume that G is a non-locally compact paratopological group and $Y = bG \setminus G$, where bG is a Hausdorff compactification of G. If Y is metacompact and ccc, then Y is Lindelöf.*

Proof. By Theorem 3.2, Y either has the Baire property or is meager and Lindelöf. Suppose that Y has the Baire property. Then Y is Lindelöf by Lemma 3.11. Hence Y is Lindelöf. □

The following Example 3.13 gives a negative answer to Question 1.5.

Example 3.13. *There exists a paratopological group X such that some compactification bX of X has a remainder which is neither pseudocompact nor meager.*

Proof. Let $Z = X \cup Y$ be the two-arrows space of P. S. Alexandroff and P. S. Urysohn [8, Exercise 3.10. C], where $X = \{(x, 0) : 0 < x \leq 1\}$ and $Y = \{(x, 1) : 0 \leq x < 1\}$. The space X is the arrow space which is homeomorphic to the Sorgenfrey line, see [8, Example 1.2.2]. Clearly, Z is a Hausdorff compactification of Sorgenfrey line X, and its remainder Y is a copy of Sorgenfrey line. Moreover, there exists a natural structure of an Abelian group on Y such that the multiplication $(u, v) \mapsto u \cdot v$ is continuous, that is, the space Y admits a structure of a paratopological group. For example, if $u = (x, 1)$ and $v = (y, 1)$ are two points in Y, then $u \cdot v = (x + y, 1)$ if $x + y < 1$, and $u \cdot v = (x + y - 1, 1)$ if $x + y \geq 1$. However, Sorgenfrey line is non-pseudocompact; otherwise, Sorgenfrey line is a compact space since it is a Lindelöf space, which is a contradiction. Moreover, since X has the Baire property [5], X is non-meager. Therefore, Y is neither pseudocompact nor meager. □

Remark 3.14. *It follows from Example 3.13 that, in Question 1.5, the answer is negative if we replace the "homogeneous space" by "paratopological group".*

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