Houston Journal of Mathematics
© 2013 University of Houston
Volume 39, No. 4, 2013

CONVERGENT-SEQUENCE SPACES AND SEQUENCE-COVERING MAPPINGS

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Communicated by Yasunao Hattori

Abstract. Based on ideas of mappings characterizing spaces, this paper characterizes the space X, where each sequence-covering mapping (resp. sequentially-quotient mapping, pseudo-sequence-covering mapping) onto X is an almost open mapping (resp. bi-quotient mapping, 1-sequence-covering mapping, almost weak-open mapping, almost *sn*-open mapping), and describes some properties of a space in which each point has a neighborhood consisting of a convergent sequence, which answers a question posed by F. Siwiec.

1. Introduction

In 1961, the first Topological Symposium, International Conference on General Topology and its Relations to Modern Analysis and Algebra was held in Prague organized by International Mathematical Union and the Czechoslovak Academy of Sciences. In this conference, P. S. Alexandroff took a famous report for topological spaces and continuous mappings [1]. He raised a conjecture of researching spaces by mappings, i.e., various classes of spaces are tied together by mappings as a link, and spaces (resp. mappings) are researched according to their differences in categories. In the past 50 years, papers and surveys on discussion for theory of images of metric spaces were published in large amounts. This issue has become a typical research direction in developments of general topology, which has made outstanding contributions for progress and prosperity of this subject [3].

2010 *Mathematics Subject Classification.* 54B15; 54C10; 54D55.

Key words and phrases. Locally convergent sequence spaces; sequence-covering mappings; sequentially-quotient mappings; pseudo-sequence-covering mappings.

This project was supported by the NSFC (No. 10971185, 11171162, 11061004).

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Taking spaces as images of metric spaces, we deal with structures of metric spaces. Usually, there are two ways for us to construct these metric spaces. One is the Ponomarev-systems [15], the other is the topological sums of the family consisting of all convergent sequences including its limit point [7]. We use the first way to characterize first countable spaces, i.e., each first countable space is an open image of a metric space; and use the second way to characterize sequential spaces, i.e., each sequential space is a quotient image of a metric space.

In 1975, F. Siwiec [22, Pages 32 and 33] raised the following question in a summarization for mapping characterizations of weak first countable spaces.

Question 1.1. *How characterize a space X? Here, each sequence-covering mapping onto X is a bi-quotient mapping.*

Question 1.2. *How characterize a space X? Here, each sequentially-quotient mapping onto X is a bi-quotient mapping.*

Question 1.3. *How characterize the images of topological sums of convergent sequence spaces under bi-quotient mappings or almost open mappings?*

Question 1.1 was solved by M. Sakai [19], and Question 1.2 was solved by R. Shen and S. Lin [20]. In investigations for these questions, an interesting property of a space X was introduced in their papers: "for each nonisolated point $x \in X$, there is an open neighborhood U of x in X such that U consists of a nontrivial sequence converging to x." A space with this property is called a locally convergent sequence space (see Definition 2.1). In this paper, we investigate the structure and mapping properties of locally convergent sequence spaces, and give some answers for Question 1.3. In addition, we discuss weak neighborhoods and sequential neighborhoods of a point in a space. Taking these, satisfactorily, some classes of spaces are characterized by the property: "each sequence-covering mapping (resp. pseudo-sequence-covering mapping, sequentially-quotient mapping) onto X is an almost open mapping (resp. bi-quotient mapping, 1-sequence-covering mapping, almost weak-open mapping, almost sn-open mapping, sn-bi-quotient mapping)."

Throughout this paper, all spaces are T_2 topological spaces and all mappings are continuous and onto. For undefined notations and terminology, one may refer to [6].

2. Sequence-Covering Mappings

At first, we recall some known concepts. Let X be a space and $x \in X$. A subset P of X is called a *sequential neighborhood* of x if whenever a sequence $\{x_n\}$ in X converging to x, there is $m \in \mathbb{N}$ such that $\{x_n : n > m\} \bigcup \{x\} \subset P$. A family $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ of subsets of X is called a *weak base* of X [2] if $\mathscr{P}$ satisfies: (a) $x \in \bigcap \mathscr{P}_x$ for each $x \in X$; (b) if $P_1, P_2 \in \mathscr{P}_x$, then $P \subset P_1 \bigcap P_2$ for some $P \in \mathscr{P}_x$; (c) U is an open subset of X if and only if for each $x \in U$, there is $P \in \mathscr{P}_x$ such that $P \subset U$. Here, $\mathscr{P}_x$ is called a *weak neighborhood base* at x in X and a subset V of X is called a *weak neighborhood* of x if $P \subset V$ for some $P \in \mathscr{P}_x$. It is clear that each weak neighborhood of x is a sequential neighborhood of x [13].

Definition 2.1. A space X is called a *locally convergent sequence space* (resp. *weak-locally convergent sequence space*, *sn-locally convergent sequence space*) if for each $x \in X$, there is a sequence $\{x_n\}$ in X converging to $\{x\}$ such that the set $\{x_n : n \in \mathbb{N}\} \bigcup \{x\}$ is an open neighborhood (resp. weak neighborhood, sequential neighborhood) of x.

Locally convergent sequence spaces and weak-locally convergent sequence spaces were investigated by M. Sakai in [17, 19] for characterizations of certain images of metric spaces. K. Li [11] gave some properties of *sn*-locally convergent sequence spaces.

Remark 2.2. The following hold.

(1) Metric spaces $\not\Longrightarrow$ locally convergent sequence spaces (for example, the real line $\mathbb{R}$).

(2) Locally convergent sequence spaces $\not\Longrightarrow$ metric spaces (for example, the ordinal space $[0, \omega_1)$).

(3) Locally convergent sequence space $\Longrightarrow$ locally compact, first countable space.

(4) Isbell-Mrówka space $\Psi(\mathbb{N})$ [5, Example 4.4] is a locally convergent sequence space; the Arens space S_2 is a weak-locally convergent sequence space; $\beta\mathbb{N}$ is an *sn*-locally convergent sequence space.

The definition of weak neighborhoods is complex and lengthy. The following lemma shows that there are some simple descriptions for weak neighborhoods in classes of some weak first countable spaces.

Lemma 2.3. *The following hold.*

(1) Weak-locally convergent sequence spaces $\Longleftrightarrow$ sequential, sn-locally convergent sequence spaces [17].

(2) Locally convergent sequence spaces $\Longleftrightarrow$ Fréchet, sn-locally convergent sequence spaces [11].

In this section, we characterize locally convergent sequence spaces by sequence-covering mappings.

Definition 2.4. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a *sequence-covering mapping* [21] if whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(2) f is called an *1-sequence-covering mapping* [12] if for each $y \in Y$ there is $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(3) f is called an *almost open mapping* [6] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y for each neighborhood U of x.

(4) f is called a *bi-quotient mapping* [14] if for each $y \in Y$ and each family $\mathscr{U}$ of open subsets of X covering $f^{-1}(y)$, there is a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $f(\bigcup \mathscr{U}')$ is a neighborhood of y in Y.

(5) f is called a *pseudo-open mapping* [6] if for each $y \in Y$ and each open subset U of X, $f^{-1}(y) \subset U$ implies $f(U)$ is a neighborhood of y in Y.

(6) f is called a *quotient mapping* [6] if U is open in Y if and only if $f^{-1}(U)$ is open in X.

Remark 2.5. It is clear that the following implications hold.

(1) 1-sequence-covering mappings $\Longrightarrow$ sequence-covering mappings.

(2) Almost open mappings $\Longrightarrow$ bi-quotient mappings $\Longrightarrow$ pseudo-open mappings $\Longrightarrow$ quotient mappings.

A closed mapping defined on a compact metric space need not be a sequence-covering mapping [21, Example 2.6]. However, we have the following result.

Lemma 2.6. *Each space is an image of a locally convergent sequence metric space under a sequence-covering mapping.*

Proof. Let X be a space, and $\mathscr{S}$ be the family of all convergent sequences containing its limit point in X. Let $S \in \mathscr{S}$. If S is a finite set, then S is endowed with the discrete topology. If S is an infinite set, then S is endowed with a topology which is homeomorphic to $\{0\} \bigcup \{1/n : n \in \mathbb{N}\}$, where the limit point of S is corresponding to 0. Then S is a locally convergent sequence metric space. Put $M = \bigoplus \mathscr{S}$ and $f : M \longrightarrow X$ is the natural mapping. Then M is a locally convergent sequence metric space and f is a sequence-covering mapping. $\square$

How characterize *sn*-locally convergent sequence spaces by mappings? The following theorem gives an answer.

Theorem 2.7. *Let X be a space. Then the following are equivalent.*

(1) X is an sn-locally convergent sequence space.

(2) X is an image of a locally convergent sequence (metric) space under an 1-sequence-covering mapping.

(3) Each sequence-covering mapping onto X is an 1-sequence-covering mapping.

PROOF. (1) $\Longrightarrow$ (3). Let $f : Z \longrightarrow X$ be a sequence-covering mapping, where X is an sn-locally convergent sequence space. For each $x \in X$, there is a sequential neighborhood S of x such that S is a sequence in X converging to x. Since f is a sequence-covering mapping, there is a sequence L in Z converging to a point $b \in f^{-1}(x)$ such that $f(L) = S$. If S' is a sequence in X converging to x, then S' is eventually in S because S is a sequential neighborhood of x. Without loss of generality, we assume that $S' \subset S$. Put $L' = L \bigcap f^{-1}(S')$, then L' converges to b and $f(L') = S'$. So f is an 1-sequence-covering mapping.

(3) $\Longrightarrow$ (2). It holds by Lemma 2.6.

(2) $\Longrightarrow$ (1). Let $f : Z \longrightarrow X$ be an 1-sequence-covering mapping, where Z is a locally convergent sequence space. For each $x \in X$, there is $z_x \in f^{-1}(x)$ satisfying the condition that f is an 1-sequence-covering mapping. Since Z is a locally convergent sequence space, there is an open neighborhood U_x of z_x such that U_x is a sequence in Z converging to z_x. Put $S_x = f(U_x)$, then S_x is a sequential neighborhood of x and S_x is a sequence in X converging to x. So X is an sn-locally convergent sequence space. $\square$

Lemma 2.8 ([22]). *A space is X is a sequential space if and only if each sequence-covering mapping onto X is a quotient mapping.*

Theorem 2.9. *Let X be a space. Then the following are equivalent.*

(1) X is a weak-locally convergent sequence space.

(2) X is an image of a locally convergent sequence (metric) space under an 1-sequence-covering, quotient mapping.

(3) Each sequence-covering mapping onto X is an 1-sequence-covering, quotient mapping.

PROOF. (1) $\Longrightarrow$ (3). Let $f : Z \longrightarrow X$ be a sequence-covering mapping, where X is a weak-locally convergent sequence space. By Lemmas 2.3(1) and 2.8, f is a quotient mapping. It follows that f is an 1-sequence-covering mapping from Theorem 2.7.

(3) $\Longrightarrow$ (2). It holds by Lemma 2.6.

(2) $\Longrightarrow$ (1). Let $f : Z \longrightarrow X$ be an 1-sequence-covering, quotient mapping, where Z is a locally convergent sequence space. Since quotient mappings preserve sequential spaces, X is a sequential space. By Theorem 2.7, X is an sn-locally

convergent sequence space. It follows that X is a weak-locally convergent sequence space from Lemma 2.3(1). $\square$

Lemma 2.10. *Let $f : X \longrightarrow Y$ be an 1-sequence-covering mapping. If Y is a Fréchet space, then f is an almost open mapping.*

Proof. For each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition that f is an 1-sequence-covering mapping. If U is a neighborhood of x in X, but $f(U)$ is not a neighborhood of y in Y, i.e., $y \in \overline{Y - f(U)}$. Since Y is a Fréchet space, there is a sequence $\{y_n\}$ in $Y - f(U)$ converging to y, and hence there is a sequence $\{x_n\}$ in X converging to x such that $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. Thus, $x_m \in U$ for some $m \in \mathbb{N}$. It follows that $y_m \in f(U)$. This contradicts that $\{y_n\}$ is contained in $Y - f(U)$. So f is an almost open mapping. $\square$

Theorem 2.11. *Let X be a space. Then the following are equivalent.*

(1) X is a locally convergent sequence space.

(2) Each sequence-covering mapping onto X is an almost open mapping [11, Theorem 3].

(3) Each sequence-covering mapping onto X is a bi-quotient mapping [19, Theorem 3.4].

(4) X is an image of a locally convergent sequence (metric) space under a bi-quotient mapping.

(5) X is an image of a locally convergent sequence (metric) space under an almost open mapping.

Proof. (1) $\Longrightarrow$ (2). It holds by Theorem 2.7, Lemmas 2.10 and 2.3(2).

(2) $\Longrightarrow$ (5). It holds from Lemma 2.6.

(5) $\Longrightarrow$ (4). It holds from Remark 2.5(2).

(4) $\Longrightarrow$ (1). It is not difficult to be checked.

(2) $\Longrightarrow$ (3). It holds from Remark 2.5(2).

(3) $\Longrightarrow$ (4). It holds from Lemma 2.6. $\square$

Remark 2.12. (1) Locally convergent sequence metric spaces $\Longleftrightarrow$ The topological sums of a family consisting of convergent sequences. In fact, Let X be a locally convergent sequence metric space. Then there is an open cover $\mathscr{V} = \{V_\alpha : \alpha < \gamma\}$ of X such that each V_α is a finite set or a nontrivial convergent sequence. Without loss of generality, we assume that $\mathscr{V}$ is locally finite. For each $\alpha < \gamma$, put $S_\alpha = V_\alpha - \bigcup_{\beta<\alpha} V_\beta$. Then $\{S_\alpha : \alpha < \gamma, S_\alpha \neq \emptyset\}$ is a cover consisting of mutually disjoint clopen subsets of X, where each S_α is a finite set or a nontrivial convergent sequence. So X is a topological sum of a family consisting of convergent sequences.

(2) Thus, images of locally convergent sequence (metric) spaces under bi-quotient mappings (resp. pseudo-open mappings, quotient mappings) characterize strong Fréchet spaces (resp. Fréchet spaces, sequential spaces).

(3) (1)$\Longleftrightarrow$ (3) in Theorem 2.11 answers F. Siwiec's Question 1.1, which was proved by M. Sakai in [19]. (1)$\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) in Theorem 2.11 answers F. Siwiec's Question 1.3.

(4) Images of metric spaces under almost open mappings characterize first countable spaces, and a first countable space need not be a locally convergent sequence space, so "locally convergent sequence (metric) space" in Theorem 2.7, Theorem 2.9 and Theorem 2.11 cannot be replaced by "metric space".

Lemma 2.13 ([23])**.** *If a regular sequential space X has point-G_δ property, then X is a Fréchet space if and only if X contains no closed copy of S_2.*

Theorem 2.14. *Let X be a regular weak-locally convergent sequence space. If X has no closed copy of S_2, then it is a locally convergent sequence space.*

PROOF. For each point $z \in X$, fix a sequential neighborhood L_z of z which is homeomorphic to a convergent sequence, where $L_z = \{z\}$ if z is isolated in X. For a subset $A \subset X$, let $E(A) = \bigcup\{L_z : z \in A\}$, and inductively let $E^1(A) = E(A)$ and $E^{n+1}(A) = E(E^n(A))$. Fix a non-isolated point $x \in X$ and put $L_x = \{x\} \cup \{x_n : n \in \mathbb{N}\}$. Consider the set $N_x = \bigcup\{E^n(\{x\}) : n \in \mathbb{N}\}$. Then N_x is sequentially open in X. Since X is sequential, N_x is open in X. Take an open neighborhood U of x such that $\overline{U} \subset N_x$. By Lemma 2.13, $\overline{U}$ is Fréchet. Assume that L_x contains infinitely many non-isolated points in X. Then $x \in \overline{E^2(\{x\}) \setminus E^1(\{x\})}$, so we can take a sequence $\{z_n\}$ in $E^2(\{x\}) \setminus E^1(\{x\})$ converging to x. This is a contradiction, because $L_x = E^1(\{x\})$ is a sequential neighborhood at x. Take a $k \in \mathbb{N}$ such that x_n is isolated in X for all $n \geq k$. Using sequentiality again, we can conclude that $L_x \setminus \{x_n : n < k\}$ is open in X. Hence, X is a locally convergent sequence space. □

3. Sequentially-Quotient Mappings and Pseudo-Sequence-Covering Mappings

This section discusses sequentially-quotient mappings and pseudo-sequence-covering mappings, which are weaker than sequence-covering mappings.

Definition 3.1. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a *sequentially-quotient mapping* [4] if for each subset U of Y, $f^{-1}(U)$ is a sequential open subset of X if and only if U is a sequential open subset of Y.

(2) f is called a *pseudo-sequence-covering mapping* [9, 10] if whenever S is a convergent sequence in Y containing its limit point, there is a compact subset K of X such that $f(K) = S$.

It is clear that sequence-covering mappings are pseudo-sequence-covering mappings. The following lemma shows that sequence-covering mappings are sequentially-quotient mappings.

Lemma 3.2 ([4]). *A mapping $f : X \longrightarrow Y$ is a sequentially-quotient mapping if and only if whenever S is a convergent sequence in Y, there is a convergent sequence L in X such that $f(L)$ is a subsequence of S.*

Further, around sequence-covering mappings in Theorem 2.11, M. Sakai proved the following theorem.

Lemma 3.3 ([19]). *A space X is a locally convergent sequence space if and only if each pseudo-sequence-covering mapping onto X is a bi-quotient mapping.*

By Theorem 2.11 and Lemma 3.3, some questions related sequence-covering mappings arise. R. Shen and the first author of this paper [20] characterize a space X, where each sequentially-quotient mapping onto X is a bi-quotient mapping (see the following Corollary 3.8), which answers F. Siwiec's Question 1.2. However, the following question is open.

Question 3.4. *How characterize the following space X?*

(1) Each sequentially-quotient mapping onto X is a pseudo-sequence-covering mapping.

(2) Each pseudo-sequence-covering mapping onto X is a sequence-covering mapping.

Theorem 3.5. *Let X be a regular space. Then each pseudo-sequence-covering mapping onto X is a sequence-covering mapping if and only if X has no non-trivial convergent sequence.*

Proof. Sufficiency is clear. Suppose that each pseudo-sequence-covering mapping onto X is sequence-covering. Since X is regular, there exists a regular extremally disconnected space EX and a perfect mapping k_X from EX onto X [24, Theorem 2.1]. Since EX is extremally disconnected[1], EX has no non-trivial convergent sequence. In fact, suppose that a non-trivial sequence $\{x_n\}$ converges to a point x in EX. Since EX is Hausdorff, there is a sequence $\{V_n\}$ of open subsets in EX such that each $x_n \in V_n$ and $V_n \cap V_m = \emptyset$ for each $n \neq m \in \mathbb{N}$. Let

[1]A space is said to be *extremally disconnected* if the closure of every open set is open.

$V = \bigcup\{V_{2n+1} : n \in \mathbb{N}\}$. Then $x \in \overline{V}$ and $\overline{V}$ is open in EX, thus $x_{2m} \in \overline{V}$ for some $m \in \mathbb{N}$, hence $V_{2m} \cap V_{2n+1} \neq \emptyset$ for some $n \in \mathbb{N}$, a contradiction.

Since k_X is perfect[2], it is pseudo-sequence-covering, thus it is sequence-covering, so X has no non-trivial convergent sequence. □

Lemma 3.6. *Let X be an sn-locally convergent sequence space. If each sequentially-quotient mapping onto X is a pseudo-sequence-covering mapping, then X has not any nontrivial convergent sequence.*

Proof. Assume that (X, τ) has a nontrivial convergent sequence $\{x_n\}$ converging to $x \in X$. Since X is an sn-locally convergent sequence space, without loss of generality, we assume that $K = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$ is a sequential neighborhood of x. Let $\mathscr{A}$ be a maximal almost mutually disjoint family of subsets of $\{x_n : n \in \mathbb{N}\}$.[3] For each $z \in X - \{x\}$, Put $\mathscr{B}_z = \{B \in \tau : z \in B, B \bigcap K \subset \{z\}\}$. Further, set $Z = \mathscr{A} \bigcup (X - \{x\})$, and endow a topology on Z as follows: $\mathscr{B}_z$ is a neighborhood base of z in Z for $z \in X - \{x\}$, and

$$\{\{A\} \bigcup (\bigcup \{B_z : z \in A'\}) : B_z \in \mathscr{B}_z,\ A' \subset A \text{ and } A - A' \text{ is finite}\}$$

is a neighborhood base of A in Z for $A \in \mathscr{A}$. Define $f : Z \longrightarrow X$ as follows: $f(z) = x$ for $z \in \mathscr{A}$; $f(z) = z$ for $z \in X - \{x\}$. We prove that f is a sequentially-quotient mapping and is not a pseudo-sequence-covering mapping as follows, which results in a contradiction.

(i) f is a continuous mapping.

It is clear that f is continuous at z for each $z \in X - \{x\}$. Let $A \in \mathscr{A}$ and U be an open neighborhood of x in X. Then there is $m \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq m$. Also, there is an open neighborhood $B_n \subset U$ of x_n, such that $B_n \bigcap K \subset \{x_n\}$ (X is a T_2-space), i.e., $B_n \in \mathcal{B}_{x_n}$. Put $V = \{A\} \bigcup (\bigcup \{B_n : n \geq m, x_n \in A\})$, then V is an open neighborhood of A in Z and $f(V) \subset U$.

(ii) f is a sequentially-quotient mapping.

Let L be a nontrivial convergent sequence in X. Without loss of generality, we assume that L converges to x and $x \in L \subset K$. Since $\mathscr{A}$ is a maximal almost mutually disjoint family of subsets of $\{x_n : n \in \mathbb{N}\}$, there is $A \in \mathscr{A}$ such that $A \bigcap L$ is infinite. Thus, $A \cap L$ is a sequence in Z converging to $A \in \mathscr{A}$ and $f(A \bigcap L)$ is a subsequence of L. By Lemma 3.2, f is a sequentially-quotient mapping.

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[2]A closed mapping $f : X \longrightarrow Y$ is called a *perfect mapping* if $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

[3]Let D be an infinite set. The family $\mathscr{A}$ consisting of countable, infinite subsets of D is called almost mutually disjoint if $A_1 \bigcap A_2$ is finite for all $A_1, A_2 \in \mathscr{A}$ and $A_1 \neq A_2$.

(iii) f is not a pseudo-sequence-covering mapping.

If f is a pseudo-sequence-covering mapping, then there is a compact subset H in Z such that $f(H) = K$. Since $\mathscr{A}$ is closed discrete in Z, $H \bigcap \mathscr{A}$ is finite. Pick $A_0 \in \mathscr{A} - H$, by $\mathscr{A}$ being almost mutually disjoint, $A_0 - \bigcup\{A : A \in H \bigcap \mathscr{A}\}$ is infinite, hence $A_0 \in H$. This is a contradiction. □

Theorem 3.7. *A space X has not any nontrivial convergent sequence if and only if each sequentially-quotient mapping onto X is an 1-sequence-covering mapping.*

Proof. Necessity is clear. Sufficiency is obtained from Theorem 2.7 and Lemma 3.6. □

Corollary 3.8. *Let X be a space. Then the following are equivalent.*

(1) X is a discrete space.

(2) Each sequentially-quotient mapping onto X is an 1-sequence-covering mapping.

(3) Each sequentially-quotient mapping onto X is a bi-quotient mapping [20].

(4) Each pseudo-sequence-covering mapping onto X is an almost open mapping.

Proof. (1) $\Longrightarrow$ (2), (3) and (4). They are clear.

(2) $\Longrightarrow$ (1). Assume that (2) holds. Then each sequence-covering mapping onto X is a quotient mapping. By Lemma 2.8, X is a sequential space. It follows that X is a discrete space from Theorem 3.7.

(3) $\Longrightarrow$ (1). Assume that (3) holds. By Theorem 2.11, X is a locally convergent sequence space. If X is not a discrete space, then X has a non-isolated point x, and so there is a convergent sequence $S = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$, which is an open neighborhood of x in X. It follows that $X = S \oplus (X - S)$. Let $\Psi(\mathbb{N}) = \mathscr{A} \bigcup \mathbb{N}$ be the Isbell-Mrówka space [5, Example 4.4], where $\mathscr{A}$ is a maximal almost mutually disjoint family of subsets of $\mathbb{N}$. Put $Z = \Psi(\mathbb{N}) \oplus (X - S)$. Define $f : Z \longrightarrow X$ as follows.

$$f(z) = \begin{cases} x, & z \in \mathcal{A} \\ x_n, & z = n \in \mathbb{N} \\ z, & z \in X - S. \end{cases}$$

By a similar way as in the proof of Lemma 3.6, $f : Z \longrightarrow X$ is a sequentially-quotient mapping and not a bi-quotient mapping. This is a contradiction.

(4) $\Longrightarrow$ (1). Assume that (4) holds. If X is not a discrete space, $X = S \oplus (X - S)$, where S is described in the proof of the above (3) $\Longrightarrow$ (1).

Put $Z = (\{-1,1\} \bigcup \{-1+\frac{1}{2n+1}, 1+\frac{1}{2n} : n \in \mathbb{N}\}) \oplus (X-S)$. Define $f : Z \longrightarrow X$ as follows.

$$f(z) = \begin{cases} x, & z = -1, 1 \\ x_n, & z = (-1)^{n+1} + \frac{1}{n+1}, n \in \mathbb{N} \\ z, & z \in X - S. \end{cases}$$

Then f is a pseudo-sequence-covering mapping and not an almost open mapping. This is a contradiction. □

4. sn-Bi-Quotient Mappings and Almost sn-Open Mappings

In order to give some mapping theorems for $\aleph$-spaces, M. Sakai [18] introduced property ω: a mapping $f : X \longrightarrow Y$ is called to have *property* ω if for each $y \in Y$ and for each increasing open cover $\{U_n : n \in \mathbb{N}\}$ of X, $f(U_n)$ is a sequential neighborhood of y for some $n \in \mathbb{N}$. Having gained some enlightenment from the above, we introduce the following definition.

Definition 4.1. A mapping $f : X \longrightarrow Y$ is called an *sn-bi-quotient mapping* if whenever $y \in Y$ and $\mathscr{U}$ is an open family in X covering $f^{-1}(y)$, there is a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $f(\bigcup \mathscr{U}')$ is a sequential neighborhood of y.

In investigations for sequence-covering mappings, Y. Ge [8] introduced almost sn-open mappings, which is equivalent to the following definition.

Definition 4.2. A mapping $f : X \longrightarrow Y$ is called an *almost sn-open mapping* if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that for every neighborhood U of x in X, $f(U)$ is a sequential neighborhood of y in Y.

Remark 4.3. It is easy to check the following.

(1) Bi-quotient mappings $\Longrightarrow$ sn-bi-quotient mappings.

(2) 1-sequence-covering mappings $\Longrightarrow$ almost sn-open mappings $\Longrightarrow$ sn-bi-quotient mappings $\Longrightarrow$ property ω.

Theorem 4.4. *Let X be a space. Then the following are equivalent.*

(1) X is an sn-locally convergent sequence space.

(2) Each pseudo-sequence-covering mapping onto X is an sn-bi-quotient mapping.

(3) Each sequence-covering mapping onto X is an sn-bi-quotient mapping.

(4) X is an image of a locally convergent sequence (metric) space under an sn-bi-quotient mapping.

Proof. (1) $\Longrightarrow$ (2). Let $f : Z \longrightarrow X$ be a pseudo-sequence-covering mapping from a space Z onto an sn-locally convergent sequence space X. For each $x \in X$

and each family $\mathscr{U}$ consisting of open subsets of Z covering $f^{-1}(x)$, Choose a sequential neighborhood S of x such that S is a sequence converging to x. Without loss of generality, we assume that S is infinite. Since f is a pseudo-sequence-covering mapping, there is a compact subset K of Z such that $f(K) = S$. By the compactness of K, there is a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $K \bigcap f^{-1}(x) \subset U'$, where $U' = \bigcup \mathscr{U}'$, i.e., $(K - U') \bigcap f^{-1}(x) = \emptyset$, and hence $x \notin f(K - U')$. Since $f(K - U')$ is a closed subset of S, $f(K - U') = S - f(U')$ is finite by the T_2-separation of X, so $f(U')$ is still a sequential neighborhood of x. It follows that f is an sn-bi-quotient mapping.

(2) $\Longrightarrow$ (3). It is clear.

(3) $\Longrightarrow$ (4). It holds from Lemma 2.6.

(4) $\Longrightarrow$ (1). Let $f : Z \longrightarrow X$ be an sn-bi-quotient mapping from a locally convergent sequence space Z onto X. Let $x \in X$. For each $z \in f^{-1}(x)$, there is an open neighborhood U_z of z such that U_z is a sequence converging to z. Since f is an sn-bi-quotient mapping, there is a finite subset F of $f^{-1}(x)$ such that $V = f(\bigcup\{U_z : z \in F\})$ is a sequential neighborhood of x. Note that $f(U_z)$ is a sequence converging to x for each $z \in f^{-1}(x)$, so V is still a sequence converging to x. It follows that X is an sn-locally convergent sequence space. □

The following lemma is a parallel result for Lemma 2.8.

Lemma 4.5 ([9]). *A space X is a sequential space if and only if each pseudo-sequence-covering mapping onto X is a quotient mapping.*

Corollary 4.6. *Let X be a space. Then the following are equivalent.*

(1) X is a weak-locally convergent sequence space.

(2) Each pseudo-sequence-covering mapping onto X is an sn-bi-quotient, quotient mapping.

(3) Each sequence-covering mapping onto X is an sn-bi-quotient, quotient mapping.

(4) X is an image of a locally convergent sequence (metric) space under an sn-bi-quotient (or almost sn-open), quotient mapping.

Proof. (1) $\Longrightarrow$ (2). Note that each weak-locally convergent sequence space is sequential space. So (1) $\Longrightarrow$ (2) from Theorem 4.4 and Lemma 4.5.

(2) $\Longrightarrow$ (3). It is clear.

(3) $\Longrightarrow$ (1). Assume that (3) holds. By Theorem 4.4, X is an sn-locally convergent sequence space. Moreover, X is a sequential space from Lemma 2.8. It follows that X is a weak-locally convergent sequence space by Lemma 2.3.

(1) $\Longrightarrow$ (4). It holds from Theorem 2.9 and Remark 4.3.

(4) $\Longrightarrow$ (1). It holds from Theorem 4.4 and Lemma 2.3. $\square$

Theorem 4.7. *Let X be a space. Then the following are equivalent.*

(1) X has not any nontrivial convergent sequence.

(2) Each mapping onto X is an 1-sequence-covering mapping.

(3) Each sequentially-quotient mapping onto X is an sn-bi-quotient mapping.

(4) Each pseudo-sequence-covering mapping onto X is an almost sn-open mapping.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (4). They are clear.

(3) $\Longrightarrow$ (1). Suppose that each sequentially-quotient mapping onto X is an sn-bi-quotient mapping. By Theorem 4.4, X is an sn-locally convergent sequence space. In order to complete the proof, we use the same way as in the proof of Lemma 3.6. Assume that X has a nontrivial convergent sequence. Let $f : Z \longrightarrow X$ be a mapping, which is described in the proof of Lemma 3.6. Then f is a sequentially-quotient mapping. It suffices to prove that f is not an sn-bi-quotient mapping. In the following proof, we use some notations described in the proof of Lemma 3.6. For each $z \in X - \{x\}$, choose $B_z \in \mathscr{B}_z$. For each $A \in \mathscr{A}$, put $U_A = \{A\} \cup (\cup\{B_z : z \in A\})$, then U_A is an open neighborhood of A in Z and $f(U_A) \bigcap K = \{y\} \bigcup A$. Let $\mathscr{A}'$ be a finite subfamily of $\mathscr{A}$. Since $f^{-1}(y) = \mathscr{A} \subset \bigcap\{U_A : A \in \mathscr{A}\}$, choose $A_0 \in \mathscr{A} - \mathscr{A}'$, then $A_0 \bigcap(\bigcup \mathscr{A}')$ is finite, and hence $\{y\} \cup (\cup\mathscr{A}')$ is not a sequential neighborhood of y. However, $K \bigcap(\bigcup\{f(U_A) : A \in \mathscr{A}'\}) = \{y\} \bigcup(\bigcup \mathscr{A}')$, so $\bigcup\{f(U_A) : A \in \mathscr{A}'\}$ is not a sequential neighborhood of y. Consequently, f is not an sn-bi-quotient mapping.

(4) $\Longrightarrow$ (1). Assume that (4) holds. By Theorem 4.4 and Remark 4.3, X is an sn-locally convergent sequence space. Suppose that X has a nontrivial convergent sequence $\{x_n\}$ converging to x. It suffices to give a contradiction between the above supposition and (4). Without loss of generality, we assume that $K = \{x\} \bigcup\{x_n : n \in \mathbb{N}\}$ is a sequential neighborhood of x. For each $z \in X - \{x\}$, put $\mathscr{B}_z = \{B \in \tau : z \in B \text{ and } B \bigcap K \subset \{z\}\}$, where τ is a topology on X. Put $Z = \{a_1, a_2\} \bigcup(X - \{x\})$, where $a_1, a_2 \notin X$. Put $A_1 = \{x_{2n+1} : n \in \mathbb{N}\}$ and $A_2 = \{x_{2n} : n \in \mathbb{N}\}$. Endow a topology on Z as follows: if $z \in X - \{x\}$, B_z is a neighborhood base at z in Z; if $i = 1, 2$,

$$\{\{a_i\} \bigcup(\bigcup_{z \in A'} B_z) : B_z \in \mathscr{B}_z,\ A' \subset A_i \text{ and } A_i - A' \text{ is finite}\}$$

is a neighborhood base at a_i in Z. Define $f : Z \longrightarrow X$ as follows: $f(z) = x$ for $z \in \{a_1, a_2\}$ and $f(z) = z$ for all $z \in X - \{x\}$. It is easy to check that f is continuous. Thus, we only need to prove the following claims, which contradicts with (4).

(i) f is a pseudo-sequence-covering mapping.

Let L is a nontrivial convergent sequence in X. Without loss of generality, we assume that L converges to x and $x \in L \subset K$. Put $H = \{a_1, a_2\} \bigcup (L - \{x\})$, then H is a compact subset of Z and $f(H) = L$. So f is a pseudo-sequence-covering mapping.

(ii) f is not an almost sn-open mapping.

For each $z \in X - \{x\}$, choose $B_z \in \mathscr{B}_z$. Put $U_i = \{a_i\} \bigcup (\bigcup \{B_z : z \in A_i\})$ for $i = 1, 2$, then U_i is an open neighborhood of a_i in Z. However, $f(U_i)$ is not a sequential neighborhood of x. So f is not an almost sn-open mapping. □

Remark 4.8. (1) $\Longleftrightarrow$ (3) in Theorem 4.7 generalizes Theorem 3.7.

The following is a further result of Corollary 3.8.

Corollary 4.9. *Let X be a space. Then the following are equivalent.*

(1) X is a discrete space.

(2) Each pseudo-sequence-covering mapping onto X is an almost sn-open, quotient mapping.

(3) Each sequentially-quotient mapping onto X is an sn-bi-quotient, quotient mapping.

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3). They are clear.

(2) $\Longrightarrow$ (1). Assume that (2) holds. By Theorem 4.7, X has not any nontrivial convergent sequence. X is a sequential space from Lemma 4.5. It follows that X is a discrete space.

(3) $\Longrightarrow$ (1). The proof is similar to the proof of (2) $\Longrightarrow$ (1). □

5. Almost Weak-Open Mappings

Recently, almost weak-open mappings attract the attention of some topological scholars. In this section, we give some characterizations of a space X by some results in the previous sections, where each sequence-covering mapping onto X is an almost weak-open mapping.

Definition 5.1 ([8, 25])**.** A mapping $f : X \longrightarrow Y$ is called an almost weak-open mapping[4], if there is a weak base $\mathscr{P} = \bigcup_{y \in Y} \mathscr{P}_y$ of Y satisfying that for each $y \in Y$, there is $x \in f^{-1}(y)$ such that for every neighborhood U of x in X, $P \subset f(U)$ for some $P \in \mathscr{P}_y$.

[4]At the earliest, the mapping was called a weak-open mapping by S. Xia [25]. Later, the mapping was renamed an almost weak-open mapping and weak-open mappings were introduced in a natural way by Y. Ge [8].

It is clear that almost open mappings $\Longrightarrow$ almost weak-open mappings $\Longrightarrow$ almost *sn*-open mappings.

The definition of almost weak-open mappings is a little complicated. However, almost weak-open mappings are closely connected to 1-sequence-covering mappings. In investigations for images of metric spaces, there are similar results between almost weak-open mappings and 1-sequence-covering, quotient mappings. The following lemma can be used to realize the transition between these two class of mappings.

Lemma 5.2. *The following hold.*

(1) Each almost weak-open mapping is a quotient mapping [25]*.*

(2) Each almost sn-open mapping from a first countable space is an 1-sequence-covering mapping [8]*.*

(3) Each 1-sequence-covering mapping onto a sequential space is an almost weak-open mapping [25]*.*

Remark 5.3. An open mapping onto a first countable space need not to be a pseudo-sequence-covering mapping or a sequentially-quotient mapping [12, Example 3.10].

Corollary 5.4 ([17])**.** *Let X be a space. Then the following are equivalent.*

(1) X is a weak-locally convergent sequence space.

(2) X is an image of a locally convergent sequence (metric) space under an almost weak-open mapping.

(3) Each sequence-covering mapping onto X is an almost weak-open mapping.

Proof. (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3). They hold from Theorem 2.9 and Lemma 5.2.

(3) $\Longrightarrow$ (2). It holds from Lemma 2.6.. □

Corollary 5.5 ([17])**.** *Let X be a space. Then the following are equivalent.*

(1) X is a discrete space.

(2) Each pseudo-sequence-covering mapping onto X is an almost weak-open mapping.

(3) Each sequentially-quotient mapping onto X is an almost weak-open mapping.

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3). They are clear.

(2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1). They hold from Corollary 4.9 and Lemma 5.2. □

6. Conclusions

So far we have discussed some relations among the class of sequence-covering mappings and some classes of locally convergent sequence spaces completely. However, the following question is unavoidable.

Question 6.1. *What classes of mappings preserve locally convergent sequence spaces?*

By Theorem 2.7, 1-sequence-covering mappings preserve sn-locally convergent sequence spaces. By Theorem 2.9, 1-sequence-covering, quotient mappings preserve weak-locally convergent sequence spaces. By Theorem 2.11, bi-quotient mappings preserve locally convergent sequence spaces.

It is known that the sequential fan S_ω is a perfect image of the Arens space S_2 [13], and S_2 is a weak-locally convergent sequence space, but S_ω is not an sn-locally convergent sequence space. So perfect mappings (hence, bi-quotient mappings) preserve neither weak-locally convergent sequence spaces nor sn-locally convergent sequence spaces.

M. Sakai [16] showed that an open mapping need not to preserve a space in which each point has a countable weak neighborhood base. However, the following question is open.

Question 6.2. *(1) Do open mappings preserve weak-locally convergent sequence spaces or sn-locally convergent sequence spaces?*

(2) How characterize a space X? Here, each quotient mapping onto X is an almost open mapping.

Based on ideas of mappings characterizing spaces, this paper full discusses some conditions such that sequence-covering mappings, pseudo-sequence-covering mappings and sequentially-quotient mappings are almost open mappings, bi-quotient mappings, 1-sequence-covering mappings, almost weak-open mappings, almost sn-open mappings, sn-bi-quotient mappings. In the end, we summarize the main results of this paper by the following table.

$\longrightarrow$	AO	B-Q	1-SC	AWO	Asn-O	sn-B-Q
SC	LCS	LCS	sn-LCS	w-LCS	sn-LCS	sn-LCS
SQ	DS	DS	sn-DS	DS	sn-DS	sn-DS
PSC	DS	LCS	sn-DS	DS	sn-DS	sn-LCS

We give some explanations for the above table.

(1) AO, B-Q, 1-SC, SC, SQ, PSC, AWO, Asn-O and sn-B-Q denote almost open, bi-quotient, 1-sequence-covering, sequence-covering, sequentially-quotient, pseudo-sequence-covering, almost weak open, almost sn-open and sn-bi-quotient mappings, respectively; LCS, sn-LCS and w-LCS denote locally convergent sequence, sn-locally convergent sequence and weak-locally convergent sequence spaces, respectively; DS and sn-DS denote discrete space and space having not any nontrivial convergent sequence, respectively.

(2) Let A-mapping lie in the first column and B-mapping lie in the first row, P-space lie in the cross of the row labeled by A and the column labeled by B. Then a space X is a P-space if and only if each A-mapping onto X is a B-mapping.

Acknowledgment

The authors would like to thank the referee for reviewing our paper and offering many valuable comments. In particular, the authors would like to thank that the referee gives Theorem 2.14 and Theorem 3.5 in this revised version, which answer a question posed in the original version and Question 3.4(2), respectively.

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Received January 30, 2012
Revised version received February 19, 2012
Second revision received February 21, 2012

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