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THE BAIRE PROPERTY IN THE REMAINDERS OF SEMITOPOLOGICAL GROUPS

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Abstract

It is proved that every remainder of a nonlocally compact semitopological group G is a Baire space if and only if G is not Čech-complete, which improves a dichotomy theorem of topological groups by Arhangel'skiĭ ['The Baire property in remainders of topological groups and other results', *Comment. Math. Univ. Carolin.* **50**(2) (2009), 273–279], and also gives a positive answer to a question of Lin and Lin ['About remainders in compactifications of paratopological groups', ArXiv: 1106.3836v1 [Math. GN] 20 June 2011]. We also show that for a nonlocally compact rectifiable space G every remainder of G is either Baire, or meagre and Lindelöf.

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1. Introduction

'A space' in this paper stands for a Tychonoff topological space. A remainder of a space X is the space $bX \setminus X$, where bX is a Hausdorff compactification of X.

An important question in the study of Hausdorff compactifications is when a Tychonoff space X has a Hausdorff compactification with the remainder belonging to a given class of spaces. A famous classical result in this study is the following theorem of Henriksen and Isbell [15].

THEOREM 1.1 [15, Theorem 3.6]. *A space X is of countable type if and only if the remainder in any (or some) compactification of X is Lindelöf.*

Recall that a space X is of *countable type* [12] if every compact subspace of X is contained in a compact subspace $F \subset X$ that has a countable base of open neighbourhoods in X.

A *semitopological group* G is a group G with a topology such that the product map of $G \times G$ onto G associating xy with arbitrary $x, y \in G$ is separately continuous.

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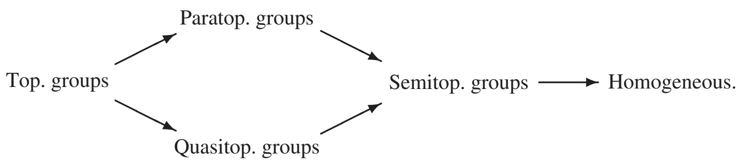
If G is a semitopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *quasitopological group*. A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ onto G associating xy with arbitrary $x, y \in G$ is jointly continuous. If G is a paratopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *topological group*. A topological space X is said to be *homogeneous* if for each $x \in X$ and each $y \in X$ there exists a homeomorphism f of X onto itself such that $f(x) = y$.

A topological space G is said to be a *rectifiable space* [14] provided that there exist a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. Every rectifiable space is homogeneous. It is well known that rectifiable spaces and paratopological groups are all good generalisations of topological groups [8].

It is obvious that



Recently, many topologists have studied how properties of the remainders of topological spaces with algebraic structures are related to their topological properties. They have shown that, in general, remainders of topological groups are much more sensitive to the topological properties of groups than remainders of topological spaces are sensitive to the topological properties of spaces. Some dichotomy theorems, among them the following, have been obtained.

Theorem 1.2 [3, Theorem 2.4]. *Every remainder of a topological group is either pseudocompact or Lindelöf.*

Theorem 1.3 [4, Theorem 1.1]. *Let G be a nonlocally compact topological group. Then either every remainder of G is Baire or every remainder of G is σ-compact.*

Theorem 1.4 [9, Theorem 3.2]. *The remainder of a homogeneous space is either Baire, or meagre and realcompact.*

Lin and Lin established in 2011 a dichotomy theorem for paratopological groups and posed a related question on semitopological groups or quasitopological groups as follows.

Theorem 1.5 [16, Theorem 3.1]. *Every remainder of a nonlocally compact paratopological group is either Baire, or meagre and Lindelöf.*

Question 1.6 [16, Question 3.1]. Let G be a nonlocally compact semitopological group or quasitopological group and bG be a compactification of G. Is the remainder $bG \setminus G$ either Baire, or meagre and Lindelöf?

In this paper, we mainly prove that either every remainder of a nonlocally compact semitopological group G has the Baire property, or every remainder of G is a σ-compact space (Theorem 2.4), which gives a positive answer to Question 1.6. A dichotomy theorem for rectifiable spaces is established, that is, every remainder of a rectifiable space G either has the Baire property, or is meagre and Lindelöf (Theorem 2.7).

2. Main results

In this section some dichotomy theorems about semitopological groups and rectifiable spaces are given. The following lemma is obvious.

Lemma 2.1. *Let $\mathscr{U}$ be a pairwise disjoint family of open subsets in a space X. Then $Z = \bigcup_{U\in\mathscr{U}} Z_U$ is a G_δ-set in X, where each Z_U is a G_δ-set in U for each $U \in \mathscr{U}$.*

A space X has the *Baire property* if the intersection of an arbitrary countable family of dense open subsets of X is dense in X.

Theorem 2.2. *If some remainder of a homogeneous space G is not Baire, then:*

(1) *G has a dense Čech-complete subspace;*

(2) *G has a compact subset which has a countable base of open neighbourhoods in G.*

Proof. Assume that a remainder $Y = bG \setminus G$ of G does not have the Baire property. Since every remainder of a locally compact space is compact, G is not locally compact. Thus G is nowhere locally compact by the homogeneity of G. Therefore Y is dense in bG, that is, bG is also a compactification of Y.

Since Y does not have the Baire property, one can find a countable family $\{V_n\}_{n\in\omega}$ of open dense subsets of Y such that $\bigcap_{n\in\omega} V_n$ is not dense in Y. For each $n \in \omega$, take an open subset U_n of bG such that $V_n = Y \cap U_n$. From Y being dense in bG it follows that U_n is an open dense subset of bG. Put $H = \bigcap_{n\in\omega} U_n$. Then H is a dense G_δ-set of bG, since bG is compact, which implies that bG has the Baire property. Thus, from $H \cap Y = \bigcap_{n\in\omega} V_n$ not being dense in Y it follows that there exists an open subset W of bG such that $W \cap H \cap Y = \emptyset$ and $W \cap H \cap G \neq \emptyset$. Since W is locally compact, W has the Baire property. Then $H \cap G$ is Čech-complete and dense in the open subset $W \cap G$ of G.

Since G is a homogeneous space, every nonempty open subset V of G contains a nonempty open subset with a dense Čech-complete subspace. Thus, by Zorn's lemma, we can take a maximal disjoint family $\mathscr{V}$ of nonempty open subsets of G such that each element of $\mathscr{V}$ contains a dense Čech-complete subspace. Put $F = \bigcup \mathscr{V}$. Clearly, F is dense in G. For each $V \in \mathscr{V}$, fix a dense Čech-complete subspace Z_V of V, and put $Z = \bigcup_{V\in\mathscr{V}} Z_V$.

(1) Clearly, Z is also dense in G. Now let us show that Z is a dense Čech-complete subspace of G. For each $V \in \mathscr{V}$, one can find an open subset U_V of bG such that $V = U_V \cap G$. In addition, Z_V is a dense Čech-complete subspace of V, so

is in U_V. Since a Čech-complete subspace is a G_δ-set in all its compactifications [12, Theorem 3.9.1], Z_V is also a G_δ-set in U_V. Since G is dense in bG and the family $\mathscr{V}$ of open subsets of G is pairwise disjoint, the family $\{U_V : V \in \mathscr{V}\}$ is also pairwise disjoint in bG. Thus, from Lemma 2.1 it follows that $Z = \bigcup_{V\in\mathscr{V}} Z_V$ is a G_δ-set in bG as well. Then Z is a dense Čech-complete subspace of G.

(2) Since Z is a G_δ-set in bG, one can take a sequence $\{Z_n\}_{n\in\omega}$ of open subsets in bG such that $Z = \bigcap_{n\in\omega} Z_n$. Fix a point $x \in Z$. One can easily find a sequence $\{W_n\}_{n\in\omega}$ of open subsets in bG such that $x \in \overline{W_{n+1}} \subset W_n \subset Z_n$ for each $n \in \omega$. Put $K = \bigcap_{n\in\omega} W_n$. Clearly, K is a compact G_δ-set in bG with $K \subset G$, and it has a countable base of open neighbourhoods in G by the compactness of bG. □

The following lemma is an important part of our techniques.

Lemma 2.3 [7, Corollary 5.4]. *Let G be a semitopological group. If G has a dense Čech-complete subspace, then G is a Čech-complete topological group.*

Theorem 2.4. *Let G be a semitopological group. Then either every remainder of G has the Baire property, or every remainder of G is σ-compact.*

Proof. If some remainder of G does not have the Baire property, then G has a dense Čech-complete subspace by Theorem 2.2. From Lemma 2.3 it follows that G is a Čech-complete space, which implies that G is a G_δ-set in each of its compactifications [12, Theorem 3.9.1], and therefore, every remainder of G is σ-compact. □

Remark 2.5. (1) Theorem 2.4 holds if G is a quasitopological (respectively, paratopological) group.

(2) A remainder $Y = bG\setminus G$ of a nonlocally compact semitopological group G cannot be a σ-compact Baire space. Indeed, otherwise the interior of Y in bG is not empty and clearly Y must be dense in the compactification bG. Therefore, Y has to intersect its complement G, since G is also dense in bG, a contradiction.

(3) A remainder of a nonlocally compact semitopological group G is a meagre space when it is σ-compact by Theorems 1.4 and 2.4. Thus Theorem 2.4 gives a positive answer to Question 1.6 and improves Theorems 1.3 and 1.5 as well.

(4) Let G be a nonlocally compact semitopological group. Then the following statements are equivalent. Every remainder (or some remainder) of G has the Baire property; every remainder is not σ-compact; G is not Čech-complete.

(5) Let X be a neither Baire nor σ-compact space. Then X cannot be a remainder of any semitopological group. Thus semitopological groups can be used to produce nontrivial topological spaces with the Baire properties.

Next, a dichotomy theorem for rectifiable spaces is established. That is, Question 1.6 is positive for rectifiable spaces as well.

Lemma 2.6 [6, Corollary 2.8]. *Let G be a rectifiable space. Then G is of countable type if there exists a nonempty compact subset with a countable base of open neighbourhoods in G.*

THEOREM 2.7. *Let G be a rectifiable space. Then every remainder of G is either Baire, or meagre and Lindelöf.*

PROOF. Let bG be a compactification of G. Assume that the remainder $Y = bG \setminus G$ is not Baire. Then Y is meagre by Theorem 1.4, and G has a nonempty compact subset which has a countable base of open neighbourhoods in G by Theorem 2.2. Thus G is of countable type by Lemma 2.6, and Y is Lindelöf by Theorem 1.1. □

REMARK 2.8. (1) Basile and Bella [9, Example 3.3] have shown that there exists a homogeneous space X such that the remainder $bX \setminus X$ in some compactification bX is neither Baire nor Lindelöf. Hence, Theorems 2.4 and 2.7 cannot be generalised to the case of homogeneous spaces.

(2) If G is nonlocally compact, then the remainder $bG \setminus G$ cannot be a Baire, Lindelöf and meagre space. Indeed, it is easy to see that the failure of the Baire property is equivalent to the existence of some nonempty open meagre subset.

(3) Let X be neither Baire nor meagre. Then X cannot be a remainder of any rectifiable space.

(4) Let X be neither Baire nor Lindelöf. Then X cannot be a remainder of any rectifiable space.

3. Applications

A space X is called a *p-space* if there exists a sequence $\{\mathscr{U}_n\}_{n\in\omega}$ of families of open subsets of the Stone–Čech compactification βX of X such that $x \in \bigcap_{n\in\omega} \mathrm{st}(x, \mathscr{U}_n) \subset X$ for each $x \in X$, where each $\mathrm{st}(x, \mathscr{U}_n) = \bigcup\{U \in \mathscr{U}_n : x \in U\}$.

THEOREM 3.1. *Let G be a semitopological group with a countable Souslin number. Then either every remainder of G is Baire, or every remainder of G is a σ-compact p-space.*

PROOF. Assume that the first alternative does not hold, that is, there exists some compactification bG of G such that the remainder $bG \setminus G$ is not Baire. Hence, from Theorems 2.2 and 2.4 and Lemma 2.3 it follows that G is a Čech-complete topological group, which implies that G is a G_δ-set in each compactification [12, Theorem 3.9.1]. Thus, every remainder of G is σ-compact and G is a paracompact p-space [8, Corollary 4.3.21]. Since the Souslin number of G is countable, it follows that G is Lindelöf. Thus, G is a Lindelöf p-space. From [2, Theorem 2.1] it follows that every remainder of G is Lindelöf p-space. Hence, every remainder of G is a σ-compact p-space. □

A *π-base* of a space X at a point $x \in X$ is a family $\mathscr{B}$ of nonempty open subsets of X such that every open neighbourhood of x contains at least one element of $\mathscr{B}$. Put $\pi_\chi(x, X) = \min\{|\mathscr{B}| : \mathscr{B} \text{ is a } \pi\text{-base at } x\} + \omega$. Then $\pi_\chi(X) = \sup\{\pi_\chi(x, X) : x \in X\}$ is called the *π-character* of X.

Corollary 3.2. *Let $Y = bG\backslash G$ be a non-Baire remainder of a semitopological group G with countable Souslin number. If Y is of countable pseudocharacter, then Y is first-countable, and G is a separable and metrisable topological group.*

Proof. Since Y is not Baire, from Theorem 3.1 it follows that Y is a σ-compact p-space and from the proof of Theorem 3.1 it follows that G is a Lindelöf topological group. Since every p-space with countable pseudocharacter is first-countable, Y is first-countable. Hence, it is enough to show that G is metrisable. Indeed, since Y is not Baire, it is not countably compact. Thus, there exists an infinite countable set $A = \{a_n : n \in \omega\}$ such that A is closed discrete in Y. Since bG is compact, A has an accumulation point c in G. For each $n \in \omega$, choose $\{V(n, k) : k \in \omega\}$ as a countable local base at a_n in Y. For each $n, k \in \omega$, take an open subset $U(n, k)$ in bG with $V(n, k) = U(n, k) \cap Y$, and let $W(n, k) = U(n, k) \cap G$. It is not difficult to see that $\{W(n, k) : n, k \in \omega\}$ is a countable π-base at c in G. Since G is a nonlocally compact topological group, Y is dense in the compactification bG. This implies that G is metrisable, since every topological group with countable π-character is metrisable [8, Theorem 3.3.12 and Proposition 5.2.6]. □

Recall that a *neighbourhood assignment* for a space X is a function ψ from X to the topology of X such that $x \in \psi(x)$ for each $x \in X$. A space X is a *D-space* [11], if for any neighbourhood assignment ψ for X there is a closed discrete subset D of X such that $X = \bigcup_{d\in D} \psi(d)$. Theorem 3.4 was proved by Arhangel'skiĭ [5, Theorem 3.5] for topological groups.

Lemma 3.3 [13, Theorem 2.1]. *If a countably compact space X is the union of a countable family of D-spaces, then X is compact.*

Theorem 3.4. *Let G be a nonlocally compact rectifiable space, and some remainder $Y = bG \setminus G$ be the union of countably many hereditarily D-spaces each of which is of countable π-character at a dense set of points such that Y has the Baire property. Then G is metrisable.*

Proof. Since Y has the Baire property and is the union of countably many hereditarily D-spaces, one of them, call it M, is not nowhere dense in Y, that is, there exists a nonempty open subset V in Y such that $M \cap V$ is dense in V. Let U be an open subset of bG such that $U \cap Y = V$, and let Z_M be dense in M such that M is of countable π-character at Z_M. Since G is nonlocally compact, it is nowhere locally compact by the homogeneity of G. Therefore Y is dense in bG. Then the subset $A = U \cap Z_M$ is dense in U. Thus, each point of A has a countable π-character in U, and therefore, each point of A has a countable π-character in bG as well.

Put $B = \bigcup\{\overline{S}^{bG} : S \subset A \text{ and } |S| \leq \omega\}$. Then $B \not\subset Y$. Indeed, assume that $B \subset Y$. It is easy to see that B is a countably compact subspace, and it is the union of countably many D-spaces. It follows that B is compact from Lemma 3.3. This implies that $U \subset \overline{A}^{bG} = B$. Since G is dense in bG, $U \cap G \neq \emptyset$. Hence, we have $B \not\subset Y$. Thus this contradicts $B \subset Y$.

So $B \cap G \neq \emptyset$ and we may consequently fix a point $x \in G$ and a countable subset $C \subset A$ such that $x \in \overline{C}^{bG}$. The fact that bG has a countable π-character at each point of C ensures that x has a countable π-base in bG and therefore also in G, since G is dense in bG. As G is homogeneous we have that $\pi_\chi(G) = \omega$, which implies that G is metrisable by [14, Theorems 3.2 and 3.3]. □

Since all metrisable spaces, and, more generally, all semimetric spaces, are hereditarily D-spaces [10], and a space with a point-countable base is also a hereditarily D-space [1, Theorem 3.6], Theorem 3.4 yields the following corollary.

Corollary 3.5. *Let G be a nonlocally compact rectifiable space, such that some remainder $Y = bG \setminus G = \bigcup_{n\in\omega} Y_n$ has the Baire property. If one of the following conditions holds, then G is metrisable:*

(1) *Y_i is a semimetric space for each $i \in \omega$;*
(2) *Y_i has a point-countable base for each $i \in \omega$.*

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