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Additive k-Metric Spaces are γ-spaces

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Abstract In this paper, we prove that additive k-metric spaces are γ-spaces. So if an additive k-metric space is a quasi-Nagata space or a paracompact β-space, it is a metrizable space.

Key Words γ-space; β-space; Additive k-metric Space; Quasi-Nagata Space; Paracompact Space

Ščepin[1] introducted the notion of k-metrizability as a generalization of metric spaces. Drani\v{s}nikov[2] defined the additivity of k-metric space and asserted that if an additive k-metric space is compact, it is metrizable. Isiwata[3] discussed the metrization problem of additive k-metric spaces and proved that if an additive k-metric space is a wM-space or a paracompact β-space, it is a metrizable space. One can ask the following question: is an additive k-metric space any kind of generalized metric space?

Let $\mathrm{Rc}(X)$ be the collection of regular closed subsets of a topological space X. (X, d) is said to be an additive k-metric space if there exists a mapping $d: X\times \mathrm{Rc}(X)\to[0, \infty)$ satisfying the following conditions:

(i) $d(x, C)=0\Leftrightarrow x\in C$,

(ii) $C\subset D\Rightarrow d(x,C)\geqslant d(x, D)$ for each $x\in X$,

(iii) $d(x, C)$ is continuous with respect to x for every $C\in \mathrm{Rc}(X)$,

(iv) $d(x, \mathrm{cl}(\bigcup\{C_a: a\in A\}))=\inf\{d(x, C_a): a\in A\}$.

d is called an additive k-metric on X.

Throughout this paper, we mean by a space a Tychonoff space. We denote by N the set of positive integers, and by (x_n) the sequence whose n-th element is x_n. For a given closed set F of an additive k-metric space (X, d) we put $\mathrm{Rci}(F)=\{C\in \mathrm{Rc}(X): F\subset \mathrm{int}C\}$, $\mathrm{Rci}(x)=\mathrm{Rci}(\{x\})$, and define

$$\begin{cases} z(x, F) = \sup\{d(x, C): C \in \mathrm{Rci}(F)\}, \\ z(H, F) = \inf\{z(x, F): x \in H\}. \end{cases} \tag{1}$$

Then for each $x\in H$ and $y\in F$, $z(H, F)\leqslant z(x, y)$ (see [3]). A space (X, T) is a γ-space (see [4]) if there exists a function $g: N\times X\to T$ such that $x_n\to p$ whenever $y_n\in g(n, p)$ and $x_n\in g(n, y_n)$.

Theorem Additive k-metric spaces are γ-spaces.

Proof Suppose (X, d) is an additive k-metric space. For each $n\in N$, $x\in X$, let

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$$g(n, x) = \text{int}\{y: z(x, y) < 1/n\}.$$

If $z(x, y) \geqslant 1/n$, then there exists $C(y) \in \text{Rci}(y)$ with $d(x, C(y)) > 1/2n$. Put $C = \text{cl}(\bigcup \{C(y): z(x, y) \geqslant 1/n\})$. Then $d(x, C) \geqslant 1/2n$ by (iv). Hence

$$x \in X - C = \text{int}(X - C) \subset \{y: z(x, y) < 1/n\},$$

so $x \in g(n, x)$. Suppose $y_n \in g(n, p)$ and $x_n \in g(n, y_n)$. If p is not a cluster point of the sequence (x_n), without loss of generality, we can assume that $p \notin \{x_n: n \in N\}$ and hence $p \notin \text{cl}(\{x_n: n \in N\}) = F$. Therefore $z(p, F) = 2a > 0$ by (1) and (i). Take $C \in \text{Rci}(F)$ with $d(p, C) > a$. By (iii) there exists $H \in \text{Rci}(p)$ with $d(h, C) > a$ for each $h \in H$, and hence $z(h, F) > a$ for each $h \in H$, so $z(H, F) > 0$. But, on the other hand, if $\{h: z(p, h) < 1/n\} \not\subset H$ for each $n \in N$, then there exists a sequence (h_n) such that $z(p, h_n) < 1/n$ and $h_n \in X - H$. Put $F_1 = X - \text{int}H$. Then $p \notin F_1$ and $\{h_n: n \in N\} \subset F_1$. By (1), (i) and (iii), $z(H_1, F_1) > 0$ for some $H_1 \in \text{Rci}(p)$. This contradicts $1/n > z(p, h_n) \geqslant z(H_1, F_1)$. So there exists $m \in N$ such that $\{h: z(p, h) < 1/m\} \subset H$. Therefore for each $n \geqslant m$, $y_n \in g(n, p) \subset g(m, p) \subset H$, and $z(H, F) \leqslant z(y_n, x_n) < 1/n$, a contradiction. Hence p is a cluster point of the sequence (x_n). If the sequence (x_n) does not converge to p, then there exists a neighborhood W of p and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \notin W$ for each $k \in N$. Now $y_{n_k} \in g(n_k, p) \subset g(k, p)$, $x_{n_k} \in g(n_k, y_{n_k}) \subset g(k, y_{n_k})$, and so p is a cluster point of the sequence (x_{n_k}). But this is impossible, and so we conclude that (x_n) converges to p and X is a γ-space.

A space (X, T) is a quasi-Nagata space (see [5]) if there exists a function $g: N \times X \to T$ such that $x \in g(n, x)$, and (y_n) has a cluster point whenever $x_n \in g(n, y_n)$ and (x_n) is a convergent sequence, wM-spaces are wN-spaces (see [4]), and wN-spaces are quasi-Nagata spaces (see [5]).

Corollary If an additive k-metric space is a quasi-Nagata space or a paracompact β-space, it is a metrizable space.

Proof A quasi-Nagata, γ-space is a metrizable space (see [5]), and a γ, β-space is a developable space (see [4]).

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