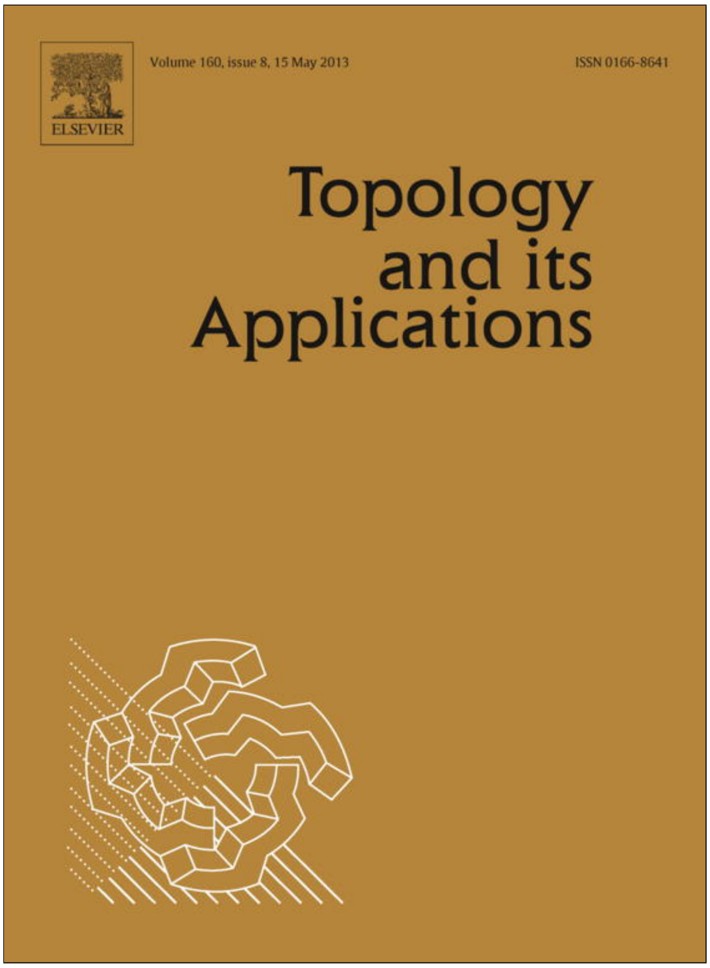
Volume 160, issue 8, 15 May 2013
ISSN 0166-8641
ELSEVIER
Topology
and its
Applications

Topology and its Applications 160 (2013) 979–990

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Cardinal invariants and $\mathbb{R}$-factorizability in paratopological groups ☆

Li-Hong Xie [a], Shou Lin [b,c,*]

[a] School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, PR China
[b] Department of Mathematics, Zhangzhou Normal University, Zhangzhou 363000, PR China
[c] Institute of Mathematics, Ningde Normal University, Ningde 352100, PR China

ARTICLE INFO

Article history:
Received 10 April 2012
Received in revised form 19 November 2012
Accepted 26 March 2013

MSC:
22A05
54A25
54B10
54C05
54C10
54H11

Keywords:
Paratopological group
Cardinal invariant
$\mathbb{R}$-factorizability
ω-Quasi-uniform continuity
Property ω-QU
τ-Narrow

ABSTRACT

In this paper, cardinal invariants and $\mathbb{R}$-factorizability in paratopological groups are studied. The main results are that (1) $w(G) = ib(G^*) \times \chi(G)$ holds for every paratopological group G; (2) every paratopological group G satisfies $|G| \leqslant 2^{ib(G^*)\psi(G)}$; (3) $nw(G) = Nag(G) \times \psi(G)$ is valid for every completely regular paratopological group G; (4) a completely regular paratopological group G is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) if and only if it is a totally ω-narrow paratopological group with property ω-QU and $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$); (5) if G is a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open homomorphism onto a paratopological group K such that $p^{-1}(e)$ is countably compact, then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable), which gives a partial answer to the question posed by M. Sanchis and M.G. Tkachenko (2010) [17].

1. Introduction

A semigroup H which is also a topological space is said to be a topological semigroup provided that the operation in H is jointly continuous. Following Bourbaki [5], a topological semigroup which is algebraically a group is called a *paratopological* group. The importance of the latter concept was clarified in the articles of Banach [3], Numakura [12], Wallace [22], and in the papers [6,7] by Ellis. Most recently, paratopological groups have become a field of intensive research. Among other sources, the readers interested in this topic can consult [4,9–11,13,15].

In the theory of general topology, cardinal functions are very useful. Roughly speaking, cardinal functions extend such important topological properties as countable base, separable, and first-countable to higher cardinality. Cardinal functions then allow one to formulate, generalize, and prove results of the type just discussed in a systematic and elegant manner. Thus, cardinal invariants in topological groups have been extensively investigated by many topologists. Many very useful results were established. For example, it is well known that $w(G) = d(G) \times \chi(G)$ and $w(G) = \pi_w(G)$ hold for every topological

☆ The project is supported by the NSFC (Nos. 10971185, 11171162, 11201414).

* Corresponding author at: Institute of Mathematics, Ningde Normal University, Ningde 352100, PR China.
E-mail addresses: xielihong2011@yahoo.cn (L.-H. Xie), shoulin60@yahoo.cn, shoulin60@163.com (S. Lin).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.03.013

group G [2]. Recently, cardinal invariants in paratopological groups have been studied in [1,13,14,21]. One of aims in this paper is to establish certain new connections between cardinal invariants in paratopological groups.

In the theory of topological groups, M.G. Tkachenko introduced the class of $\mathbb{R}$-factorizable groups as the topological groups satisfying the condition that every real-valued continuous function is factorized through a continuous homomorphism onto a second-countable topological group. The class of $\mathbb{R}$-factorizable groups is unexpectedly wide. For example, it contains all Lindelöf groups and arbitrary (not necessarily closed) subgroups of Lindelöf Σ-groups. In particular, all precompact groups and arbitrary subgroups of σ-compact groups are $\mathbb{R}$-factorizable (see for instance [18,19]).

This motivated M. Sanchis and M.G. Tkachenko to introduce the class of $\mathbb{R}$-factorizable paratopological groups as the generalization of the class of $\mathbb{R}$-factorizable groups in [17]. Recall that a paratopological group G is called *$\mathbb{R}_i$-factorizable*, for $i = 1, 2, 3, 3.5$, if G is a T_i-space and for every continuous real-valued function f on G, one can find a continuous homomorphism $p : G \to K$ onto a paratopological group K of countable weight satisfying the T_i-separation axiom and a continuous real-valued function g on K such that $f = g \circ p$. Similarly, the classes of $\mathbb{R}_i$-factorizable paratopological groups are unexpectedly wide.

In [17], a question on the open homomorphic images of $\mathbb{R}$-factorizable paratopological groups was posed. To solve this problem, the concept of property ω-QU in paratopological groups is introduced in this paper. Recently, a decomposition theory of $\mathbb{R}$-factorizable topological groups was obtained by the concept of property ω-U in [23]. They established that a topological group G is $\mathbb{R}$-factorizable if and only if it is ω-narrow with property ω-U. Being analogous to the case of $\mathbb{R}$-factorizable topological groups, the characterizations of the classes of $\mathbb{R}_i$-factorizable paratopological groups for $i = 2, 3$ are established in this paper.

The paper is organized as follows. The aim of Section 2 is to establish certain new connections between cardinal invariants in paratopological groups. We show, among other things, that every totally τ-narrow paratopological group G has $inv(G) \leqslant \tau$, that $w(G) = ib(G^*) \times \chi(G)$ holds for every paratopological group G and that $nw(G) = Nag(G) \times \psi(G)$ is valid for every completely regular paratopological group G, which is a generalization of an earlier result in [2] saying that $nw(G) = Nag(G) \times \psi(G)$ holds for every topological group G. In Section 3 we present a result saying that dense subgroups of a topological product of regular paratopological groups which are Lindelöf Σ-spaces are $\mathbb{R}_3$-factorizable. Section 4 contains a decomposition theorem of $\mathbb{R}_i$-factorizable paratopological groups, which says that a completely regular paratopological group G is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) if and only if it is a totally ω-narrow paratopological group with property ω-QU and $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$). In Section 5, we establish that if G is a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open and continuous homomorphism onto a paratopological group K with identity e such that $p^{-1}(e)$ is countably compact, then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable), which gives a partial answer to the question posed by M. Sanchis and M.G. Tkachenko in [17]. In Section 6, some question are posed.

All spaces are assumed to be T_1-separation axiom. T_3 and $T_{3.5}$ mean regularity and complete regularity, respectively.

2. Cardinal invariants in paratopological groups

In this section, we establish certain new connections between cardinal invariants in paratopological groups. Below $c(X)$, $d(X)$, $w(X)$, $nw(X)$, $l(X)$, and $k(X)$ denote the cellularity, density, weight, network weight, Lindelöf degree, and compact-covering number of a space X defined, respectively, as follows.

Cellularity: $c(X) = \sup\{|\mathcal{U}|\colon \mathcal{U}$ is a disjoint family of open subsets of $X\} + \omega$.

Density: $d(X) = \min\{|S|\colon S \subset X$ and $\overline{S} = X\} + \omega$.

Weight: $w(X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a base for $X\} + \omega$.

Network weight: $nw(X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a network for $X\} + \omega$.

Lindelöf degree: $l(X) = \min\{\lambda \in \mathrm{Card}\colon$ for every open covering $\mathcal{V}$ of X there is a subfamily $\mathcal{U} \subset \mathcal{V}$ such that $|\mathcal{U}| \leqslant \lambda$ and $\bigcup \mathcal{U} = X\} + \omega$.

Compact-covering number: $k(X) = \min\{\lambda \in \mathrm{Card}\colon \mathcal{U}$ is a family of compact subsets of X such that $|\mathcal{U}| \leqslant \lambda$ and $\bigcup \mathcal{U} = X\} + \omega$.

For a paratopological group G with identity e we will consider the following cardinal functions:

Character: $\chi(G) = \min\{|\mathcal{B}|\colon \mathcal{B}$ is a neighborhood base at e of $G\} + \omega$.

Pseudocharacter: $\psi(G) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a family of open subsets of G such that $\bigcap \mathcal{U} = \{e\}\} + \omega$.

For a paratopological group G with a topology τ, one defines the *conjugate topology* τ^{-1} on G by $\tau^{-1} = \{U^{-1} \mid U \in \tau\}$. Then $G' = (G, \tau^{-1})$ is also a paratopological group, and the inversion $x \to x^{-1}$ is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G, and we call $G^* = (G, \tau^*)$ the topological group *associated to* G. Those notations are used throughout the article. Clearly, the associated topological group G^* is Hausdorff for every paratopological group G, the identity mapping $i : G^* \to G$ is continuous, and $H = H^*$ holds for every topological group H.

Let $\mathcal{P}$ be a (topological) property. Recall that a paratopological group G is called *totally* $\mathcal{P}$ [16, Definition 3.1] if the associated topological group G^* has the property $\mathcal{P}$.

The following is a general form of [16, Corollary 3.3] and is quite simple, but is of much importance, since it relates the properties of a paratopological group G to those of the associated topological group G^*.

Proposition 2.1. *Let $\mathcal{P}$ be a topological property which is finitely productive and hereditary with respect to closed subspaces. If a paratopological group G has the property $\mathcal{P}$, so does G^*.*

Proof. Define a function $f : G \times G' \to G$ as follows: $f(x, y) \mapsto xy^{-1}$. One can easily verify that the function f is continuous. Since G is a T_1-paratopological group, the diagonal $\Delta = \{(x, x)\colon x \in G\} = f^{-1}(e)$ is closed in $G \times G'$, where e is the identity in G. By [17, Lemma 3.2], Δ is a Hausdorff topological group topologically isomorphic to the topological group G^*. Therefore, the statement is obvious, since the property $\mathcal{P}$ is finitely productive and hereditary with respect to closed subspaces. □

According to Proposition 2.1 one can easily obtain the following.

Corollary 2.2. *Let $\mathcal{P}$ be a topological property which is finitely productive, hereditary with respect to closed subspaces and preserved by continuous mappings. Then a paratopological group G has the property $\mathcal{P}$ if and only if so does G^*.*

Let G be a paratopological group. We say that the *invariance number* of G is less than or equal to τ, in symbols, $inv(G) \leqslant \tau$ if for every open neighborhood U of identity e in G, there exists a family γ of open neighborhoods of e in G with $|\gamma| \leqslant \tau$ such that for each $x \in G$ one can find $V \in \gamma$ satisfying $xVx^{-1} \subset U$. A paratopological group G is called *τ-balanced* if $inv(G) \leqslant \tau$.

Recall that for a Hausdorff paratopological group G with identity e the *Hausdorff number* [20] of G, denoted by $Hs(G)$, is the minimum cardinal number κ such that for every open neighborhood U of e in G, there exists a family γ of open neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subset U$ and $|\gamma| \leqslant \kappa$. Similarly, the *index of regularity* [20] $Ir(G)$ of a regular paratopological group G with identity e, is the minimum cardinal number κ such that for every open neighborhood U of e in G, one can find a open neighborhood V of e and a family γ of open neighborhoods of e in G such that $\bigcap_{W\in\gamma} VW^{-1} \subset U$ and $|\gamma| \leqslant \kappa$. Clearly, every topological group G has $Hs(G) = 1$ and $Ir(G) = 1$, and every first-countable Hausdorff (resp. regular) paratopological group G has $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$).

From the proofs of [20, Theorems 2.7 and 3.6] one can obtain the following which plays an important role as well in our paper.

Lemma 2.3. *Let G be an ω-balanced paratopological group with $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$). Then for every open neighborhood U of the identity in G, there exists a continuous homomorphism π of G onto a Hausdorff (resp. regular) first-countable paratopological group H such that $\pi^{-1}(V) \subset U$ for some open neighborhood V of the identity in H.*

Let τ be an infinite cardinal number and G a paratopological group. G is called *τ-narrow* if, for every neighborhood U of the identity in G, there exists a subset $K \subset G$ with $|K| \leqslant \tau$ such that $KU = UK = G$. An important cardinal function is the *index of narrowness* of G denoted by $ib(G)$. $ib(G)$ is defined as the minimal cardinal number $\tau \geqslant \omega$ such that G is τ-narrow.

Theorem 2.4. *Let G be an ω-balanced paratopological group with $Hs(G) \leqslant \omega$ and suppose that every continuous homomorphic image H of G with $\chi(H) \leqslant \omega$ is τ-narrow. Then G is τ-narrow.*

Proof. Let U be an open neighborhood of the identity in G. Take an open neighborhood V of the identity in G such that $V^2 \subset U$. Then, by Lemma 2.3, there exists a continuous homomorphism π of G onto a Hausdorff first-countable paratopological group H such that $\pi^{-1}(W) \subset V$ for some open neighborhood W of identity e in H. Thus H is τ-narrow by the hypothesis. For the open neighborhood W, choose a set $K \subset H$ such that $|K| \leqslant \tau$ and $KW = WK = H$. Let F be any subset of G such that $|F| \leqslant \tau$ and $\pi(F) = K$. We claim that $UF = FU = G$. Indeed, take an arbitrary element $x \in G$. Then $\pi(x) \in bW$ for some $b \in K$. Choose an element $a \in F$ such that $\pi(a) = b$. Clearly, $\pi(x) \in bW = \pi(a)W \subset \pi(aV)$, whence it follows that

$$x \in \pi^{-1}\big(\pi(aV)\big) = aV\pi^{-1}(e) \subset aV\pi^{-1}(W) \subset aVV \subset aU \subset FU.$$

This implies that $FU = G$. Similarly, one can easily prove that $UF = G$, so G is τ-narrow. □

Clearly, every Abelian topological group is ω-balanced. Thus we have the following.

Corollary 2.5. *([2, Proposition 5.1.13]) Let G be an Abelian topological group and suppose that every continuous homomorphic image H of G with $\chi(H) \leqslant \omega$ is τ-narrow. Then the topological group G is τ-narrow.*

Theorem 2.6. *Every totally τ-narrow paratopological group is τ-balanced.*

Proof. Let G be a totally τ-narrow paratopological group with identity e and U an open neighborhood of e in G. Choose an open neighborhood V of e in G such that $V^3 \subset U$. Since the topological group G^* associated to G is τ-narrow, and the set $O = V \cap V^{-1}$ is an open neighborhood of e in G^*, there exists a subset $C \subset G^*$ such that $CO = OC = G^*$ and $|C| \leqslant \tau$. Clearly, $O \subset V$ and $O \subset V^{-1}$. Since the multiplication in G is continuous, we can find, for every $x \in C$, an open neighborhood W_x of e in G such that $xW_xx^{-1} \subset V$. Then the family $\gamma = \{W_x\colon x \in C\}$ is subordinated to U.

Indeed, for an arbitrary $y \in G$, there exists $x \in C$ such that $y \in Ox$. We have, therefore, that

$$yW_xy^{-1} \subset OxW_xx^{-1}O^{-1} \subset V\left(xW_xx^{-1}\right)V \subset V^3 \subset U.$$

This proves that G has $inv(G) \leqslant \tau$, that is, G is τ-balanced. □

Corollary 2.7. *([16, Proposition 3.8]) Every totally ω-narrow paratopological group G is ω-balanced, that is, $inv(G) \leqslant \omega$.*

A subset B of a paratopological group G is called *τ-narrow* in G if, for every open neighborhood U of the identity in G, there exists a subset $F \subset G$ with $|F| \leqslant \tau$ such that $B \subset FU \cap UF$. Clearly, G is τ-narrow iff G is τ-narrow in itself, and every subset of a τ-narrow paratopological group is τ-narrow in this group.

Theorem 2.8. *A subset B of a paratopological group G is τ-narrow in each of the following cases*:

(1) $l(B) \leqslant \tau$;
(2) $c(B^*) \leqslant \tau$, *where the space B^* is considered subset B with the topology induced from G^*.*

Proof. For case (1), the result is almost obvious. Indeed, if U is an open neighborhood of the identity in G, then $\{xU\colon x \in G\}$ and $\{Ux\colon x \in G\}$ are two open coverings of G. Since $l(B) \leqslant \tau$, there are two subsets C_1, C_2 of G such that $|C_i| \leqslant \tau$ $(i = 1, 2)$ and both the families $\{xU\colon x \in C_1\}$ and $\{Ux\colon x \in C_2\}$ cover B or, equivalently, $B \subset C_1U \cap UC_2$. Hence, B is τ-narrow.

For case (2), take an open set U of the identity in G. Then U is also an open set in G^*. Since $c(B^*) \leqslant \tau$, there exits a subset $C \subset G^*$ such that $B \subset CU \cap UC$ and $|C| \leqslant \tau$ by [2, Proposition 5.1.3], which implies that B is also τ-narrow. □

Corollary 2.9. *Every paratopological group G satisfies the inequalities $ib(G) \leqslant l(G)$ and $ib(G) \leqslant c(G^*)$.*

In what follows we focus in new connections between cardinal invariants in paratopological groups.

Theorem 2.10. *$w(G) = ib(G^*) \times \chi(G)$ holds for every paratopological group G.*

Proof. Clearly, $\chi(G^*) \leqslant \chi(G) \times \chi(G) = \chi(G)$ by Proposition 2.1. Since $w(G^*) = ib(G^*) \times \chi(G^*)$ [2, Proposition 5.2.3], we have $nw(G) \leqslant w(G^*) \leqslant ib(G^*) \times \chi(G)$ because of G as a continuous image of G^*. Suppose that $nw(G) = \delta$, $ib(G^*) = \kappa$, and $\chi(G) = \gamma$. Take a base $\mathcal{V} = \{V_\alpha\colon \alpha \in \gamma\}$ at the identity of G and a network $\mathcal{U} = \{U_\beta\colon \beta \in \delta\}$ for G. We claim that the family $\{U_\beta V_\alpha\colon (\beta, \alpha) \in \delta \times \gamma\}$ is a base of G. It will imply that $w(G) \leqslant \delta \times \gamma = nw(G) \times \chi(G) \leqslant ib(G^*) \times \chi(G) \times \chi(G) = ib(G^*) \times \chi(G)$.

In fact, take any open set $U \subset G$ and any point $x \in U$. By the joint continuity of G, one can find two open sets V_α and W in G such that $V_\alpha \in \mathcal{V}$, $x \in W$ and $WV_\alpha \subset U$. Since $\mathcal{U}$ is a network, there exists $U_\beta \in \mathcal{U}$ such that $x \in U_\beta \subset W$. Hence, $x \in U_\beta V_\alpha \subset WV_\alpha \subset U$.

Now we shall prove $ib(G^*) \times \chi(G) \leqslant w(G)$. Clearly, $\chi(G) \leqslant w(G)$. It is enough to show that $ib(G^*) \leqslant w(G)$. Indeed, by Proposition 2.1 and Corollary 2.9, we have $ib(G^*) \leqslant l(G^*) \leqslant w(G^*) \leqslant w(G) \times w(G) = w(G)$. □

Remark 2.11. The "$ib(G^*)$" cannot be replaced by "$ib(G)$" in Theorem 2.10, that is, $w(G) = ib(G) \times \chi(G)$ need not hold for every paratopological group G. Indeed, the Sorgenfrey line S is a first-countable and Lindelöf paratopological group, thus, by Corollary 2.9, $ib(S) \leqslant \omega$ and $\chi(S) \leqslant \omega$. Clearly, $w(S) > \omega = \omega \times \omega \geqslant ib(S) \times \chi(S)$.

According to Theorem 2.10, one can easily obtain Corollaries 2.12 and 2.13.

Corollary 2.12. *$w(G) = \chi(G)$ holds for every totally ω-narrow paratopological group G, i.e. $ib(G^*) \leqslant \omega$.*

Corollary 2.13. *([16, Proposition 3.5]) Every first-countable totally ω-narrow paratopological group has a countable base.*

Corollary 2.14. *$w(G) = nw(G) \times \chi(G)$ holds for every paratopological group G.*

Proof. Clearly, $nw(G) \times \chi(G) \leqslant w(G)$. From Proposition 2.1 and Corollary 2.9 it follows that $ib(G^*) \leqslant l(G^*) \leqslant nw(G^*) \leqslant nw(G)$. Thus $w(G) = ib(G^*) \times \chi(G) \leqslant nw(G) \times \chi(G)$ by Theorem 2.10. □

Corollary 2.15. *([1, Proposition 2.13]) Every first-countable paratopological group with a countable network has a countable base.*

Proof. The statement directly follows from Corollary 2.14. □

Since every completely regular $\mathbb{R}_1$-factorizable paratopological group is totally ω-narrow [17, Proposition 3.5], we have the following.

Corollary 2.16. *([17, Proposition 3.4]) Every $\mathbb{R}_1$-factorizable and completely regular paratopological group G satisfies the equality $w(G) = \chi(G)$.*

Corollary 2.17. *Every paratopological group G satisfies*:

(1) $w(G) = d(G^*) \times \chi(G)$;
(2) $w(G) = c(G^*) \times \chi(G)$;
(3) $w(G) = l(G^*) \times \chi(G)$;
(4) $w(G) = w(G^*) \times \chi(G)$;
(5) $w(G) \leqslant k(G^*) \times \chi(G) \leqslant k(G) \times \chi(G)$.

Proof. (1) It is well known that $ib(H) \leqslant c(H) \leqslant d(H)$ holds for every topological group H. Thus, from Theorem 2.10 it follows that $w(G) = ib(G^*) \times \chi(G) \leqslant d(G^*) \times \chi(G)$ for the paratopological group G. Clearly, $\chi(G) \leqslant w(G)$. According to Proposition 2.1, one can easily obtain that $w(G^*) \leqslant w(G) \times w(G) = w(G)$, which implies that $d(G^*) \leqslant w(G^*) \leqslant w(G)$. Thus, $d(G^*) \times \chi(G) \leqslant w(G) \times w(G) = w(G)$.

(2) By Corollary 2.9, $ib(G^*) \leqslant c(G^*)$, and from Theorem 2.10 it follows that $w(G) = ib(G^*) \times \chi(G) \leqslant c(G^*) \times \chi(G)$. Clearly, $\chi(G) \leqslant w(G)$ and $c(G^*) \leqslant w(G^*) \leqslant w(G) \times w(G) = w(G)$ by Proposition 2.1, thus $c(G^*) \times \chi(G) \leqslant w(G) \times w(G) = w(G)$.

(3) Note that $ib(G) \leqslant l(G)$ according to Corollary 2.9. From Theorem 2.10 it follows that $w(G) = ib(G^*) \times \chi(G) \leqslant l(G^*) \times \chi(G)$. Clearly, $l(G^*) \leqslant w(G^*) \leqslant w(G) \times w(G) = w(G)$ by Proposition 2.1 and $\chi(G) \leqslant w(G)$. Thus, $l(G^*) \times \chi(G) \leqslant w(G) \times w(G) = w(G)$.

(4) Note that $w(G^*) \leqslant w(G) \times w(G) = w(G)$ by Proposition 2.1. It follows that $w(G^*) \times \chi(G) \leqslant w(G) \times w(G) = w(G)$. Clearly, $ib(G^*) \leqslant w(G^*)$. From Theorem 2.10 it follows that $w(G) = ib(G^*) \times \chi(G) \leqslant w(G^*) \times \chi(G)$.

(5) According to Proposition 2.1 one can easily obtain that $k(G^*) \leqslant k(G) \times k(G) = k(G)$. Since $ib(H) \leqslant l(H) \leqslant k(H)$ holds for any topological group H, we have $ib(G^*) \leqslant k(G^*) \leqslant k(G)$. Thus, $w(G) = ib(G^*) \times \chi(G) \leqslant k(G^*) \times \chi(G) \leqslant k(G) \times \chi(G)$ by Theorem 2.10. □

Remark 2.18. The $d(G^*)$, $c(G^*)$, and $l(G^*)$ in Corollary 2.17 cannot be replaced by $d(G)$, $c(G)$ and $l(G)$, respectively. Indeed, let S be the Sorgenfrey line which is a paratopological group. Clearly, $c(S) = d(S) = l(G) = \omega$ and $\chi(S) = \omega$. However, $w(S) > \omega = d(S) \times \chi(S) = l(S) \times \chi(S) = c(S) \times \chi(S)$.

Corollary 2.19. *$w(G) = \chi(G)$ holds for every σ-compact paratopological group G.*

Proof. Clearly, $w(G) \geqslant \chi(G)$. The statement directly follows from (5) of Corollary 2.17 because of $k(G) \leqslant \omega$. □

Theorem 2.20. *Every paratopological group G satisfies $|G| \leqslant 2^{ib(G^*)\psi(G)}$. Therefore, $|G| \leqslant 2^{l(G^*)\psi(G)}$, and $|G| \leqslant 2^{c(G^*)\psi(G)}$.*

Proof. According to Proposition 2.1, one can easily obtain $\psi(G^*) \leqslant \psi(G) \times \psi(G) = \psi(G)$. From [2, Theorem 5.2.15], $|G^*| \leqslant 2^{ib(G^*)\psi(G^*)}$, it follows that $|G| = |G^*| \leqslant 2^{ib(G^*)\psi(G^*)} \leqslant 2^{ib(G^*)\psi(G)}$. By Corollary 2.9, the rest of the theorem is immediate. □

Corollary 2.21. *Every totally ω-narrow paratopological group G satisfies $|G| \leqslant 2^{\psi(G)}$.*

Theorem 2.22. *Let G be a paratopological group. Then $nw(G) \leqslant k(G) \times \psi(G)$.*

Proof. According to Proposition 2.1 one can easily obtain $k(G^*) \leqslant k(G) \times k(G) = k(G)$ and $\psi(G^*) \leqslant \psi(G) \times \psi(G) = \psi(G)$. From [2, Proposition 5.2.17], $nw(G^*) \leqslant k(G^*) \times \psi(G^*)$, it follows that $nw(G) \leqslant nw(G^*) \leqslant k(G^*) \times \psi(G^*) \leqslant k(G) \times \psi(G)$ because of G as a continuous image of G^*. □

Corollary 2.23. *$nw(G) = \psi(G)$ holds for every Hausdorff σ-compact paratopological group G.*

Proof. According to Theorem 2.22, it is enough to show $\psi(G) \leqslant nw(G)$. Take a network $\{V_\alpha \mid \alpha \in \gamma\}$ in G such that $\gamma = nw(G)$. Since G is Hausdorff, for any point $y \in G \setminus \{e\}$ where e is the identity in G, one can find $\alpha \in \gamma$ such that $y \in V_\alpha \subset \overline{V_\alpha} \subset G \setminus \{e\}$. It implies that $\psi(G) \leqslant nw(G)$ by the homogeneity of G. □

Suppose that X is a subset of Y and that γ is a family of subsets of Y. We say that γ *separates X from $Y \setminus X$* if for every $x \in X$ and every $y \in Y \setminus X$, there exists $F \in \gamma$ such that $x \in F$ and $y \notin F$.

Let βX be the Čech–Stone compactification of a Tychonoff space X and $\mathscr{F}$ the family of all closed subsets of βX. We recall that the *Nagami number* [2], denoted by $Nag(X)$, of X as follows:

$$Nag(X) = \min\bigl\{|\mathscr{P}|: \ \mathscr{P} \subset \mathscr{F} \text{ and } \mathscr{P} \text{ separates } X \text{ from } \beta X \setminus X\bigr\} + \omega.$$

Theorem 2.24. *Let G be a completely regular paratopological group. Then $nw(G) = Nag(G) \times \psi(G)$.*

Proof. It is well known that $Nag(X) \leqslant nw(X)$ [2, Proposition 5.3.3] and $\psi(X) \leqslant nw(X)$ for any completely regular space X. Thus, $nw(G) = nw(G) \times nw(G) \geqslant Nag(G) \times \psi(G)$. It remains to show $nw(G) \leqslant Nag(G) \times \psi(G)$. Indeed, it is well known that $Nag(Y) \leqslant Nag(X)$ holds for each closed subspace Y of a completely regular space X [2, Corollary 5.3.2]. Thus, by [2, Proposition 5.3.9], $Nag(G^*) \leqslant Nag(G) \times Nag(G) = Nag(G)$. According to Proposition 2.1 one can easily obtain that $\psi(G^*) \leqslant \psi(G) \times \psi(G) = \psi(G)$. Thus, by [2, Corollary 5.3.25], $nw(G^*) = Nag(G^*) \times \psi(G^*)$, so $nw(G) \leqslant nw(G^*) = Nag(G^*) \times \psi(G^*) \leqslant Nag(G) \times \psi(G)$ because of G as a continuous image of G^*. $\square$

Since every regular paratopological group G which is a Lindelöf Σ-space has $Nag(G) \leqslant \omega$ [2], Theorem 2.24 implies the following.

Corollary 2.25. *$nw(G) = \psi(G)$ holds for every regular paratopological group G which is a Lindelöf Σ-space.*

3. Lindelöf Σ-paratopological groups

In this section we establish that every dense subgroup of a topological product of regular paratopological groups which are Lindelöf Σ-spaces are $\mathbb{R}_3$-factorizable. Firstly, let us give some auxiliary facts.

Proposition 3.1. *Let G be a paratopological group satisfying T_i-separation axiom with the property that for every continuous function $f : G \to \mathbb{R}$, there exist a continuous homomorphism $\pi : G \to H$ onto an $\mathbb{R}_i$-factorizable paratopological group H and a continuous function $h : H \to \mathbb{R}$ such that $f = h \circ \pi$. Then G is $\mathbb{R}_i$-factorizable $(i = 1, 2, 3, 3.5)$.*

Proof. Take any continuous function $f : G \to \mathbb{R}$. By the assumptions, we can find a continuous homomorphism $\pi : G \to H$ onto an $\mathbb{R}_i$-factorizable paratopological group H and a continuous function $h : H \to \mathbb{R}$ such that $f = h \circ \pi$. By the $\mathbb{R}_i$-factorizability of H, there are a continuous homomorphism $p : H \to K$ onto a second-countable paratopological group K with T_i-separation axiom and a continuous real-valued function g on K such that $h = g \circ p$. Clearly, the continuous homomorphism $\varphi = p \circ \pi$ of G onto K satisfies $f = h \circ \varphi$. $\square$

Recall that a space X is called *ω-cellular* [2] if every family γ consisting of G_δ-sets in X contains a subfamily γ_0 such that $\overline{\bigcup \gamma} = \overline{\bigcup \gamma_0}$ and $|\gamma_0| \leqslant \omega$.

Lemma 3.2. *([17, Theorem 3.13]) Let G be a regular paratopological group such that G is a Lindelöf Σ-space. Then every subgroup of G is $\mathbb{R}_3$-factorizable.*

Lemma 3.3. *([2, Corollary 5.3.31]) The product of any family of Lindelöf Σ-groups is an ω-cellular space.*

Proposition 3.4. *The topological product of any family of paratopological groups which are Lindelöf Σ-spaces and satisfy T_3-separation axiom is an ω-cellular space.*

Proof. Let $G = \prod_{\alpha \in \Lambda} G_\alpha$ be the topological product of a family of paratopological groups $\{G_\alpha\colon \alpha \in \Lambda\}$, where the paratopological group G_α with T_3-separation axiom is a Lindelöf Σ-space for each $\alpha \in \Lambda$. According to Proposition 2.1, we can obtain that the topological group G_α^* is a Lindelöf Σ-space for each $\alpha \in \Lambda$. Thus, by Lemma 3.3, $G^* = \prod_{\alpha \in \Lambda} G_\alpha^*$ is an ω-cellular space. Consider the identity mapping $i : G^* = \prod_{\alpha \in \Lambda} G_\alpha^* \to G$. Clearly, i is a continuous mapping. Thus G is an ω-cellular space as a continuous image of the ω-cellular space G^*. $\square$

Lemma 3.5. *([2, Theorem 1.7.7]) Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ be an ω-cellular space and suppose that S is a dense subset of X. Then, for every continuous mapping $f : S \to Y$ to a regular first-countable space Y, there exist a countable set $K \subset \Lambda$ and a continuous mapping $h : p_K(S) \to Y$ such that $f = h \circ p_K \restriction S$, where $p_K : X \to \prod_{\alpha \in K} X_\alpha$ is the projection.*

Theorem 3.6. *Dense subgroups of a topological product of regular paratopological groups which are Lindelöf Σ-spaces are $\mathbb{R}_3$-factorizable.*

Proof. Let $G = \prod_{i \in I} G_i$ be the topological product of regular paratopological groups $\{G_i\colon i \in I\}$ which are Lindelöf Σ-spaces, H a dense subgroup of G and $f : H \to \mathbb{R}$ a continuous function. It follows from Proposition 3.4 that the space G is ω-cellular. By Lemma 3.5, we can find a countable set $K \subset I$ and a continuous function $h : p_K(H) \to Y$ such that $f = h \circ p_K \restriction H$, where $p_K : G \to G_K = \prod_{\alpha \in K} G_\alpha$ is the projection. Since Lindelöf Σ-spaces are countably productive, $H_K = p_K(H)$ is a subgroup of the paratopological group G_K which is a regular Lindelöf Σ-space. Thus, H_K is $\mathbb{R}_3$-factorizable by Lemma 3.2. Hence Proposition 3.1 implies that H is also $\mathbb{R}_3$-factorizable. $\square$

It is well known that every regular space with a countable network is a Lindelöf Σ-space. Thus we have the following by Theorem 3.6.

Corollary 3.7. *Dense subgroups of a topological product of regular paratopological groups which have a countable network are $\mathbb{R}_3$-factorizable.*

4. ω-Quasi-uniform continuity in paratopological groups

In this section, some decomposition theorems of $\mathbb{R}$-factorizable paratopological groups are obtained. Firstly, some simple properties of ω-quasi-uniformly continuous real-valued functions on paratopological groups are discussed. Those results will be used in Section 5.

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ is a reflexive relation on X, and (b) if $U \in \mathcal{U}$ then $V \circ V \subset U$ for some $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*.

Let $(X, \mathcal{U})$ be a quasi-uniform space. The topology $\tau(\mathcal{U}) = \{H \subset X\colon$ for each $x \in H$ there is $U \in \mathcal{U}$ with $U(x) \subset H\}$ is called the *topology induced* by $\mathcal{U}$. (Here $U(x) = \{y \in X\colon (x, y) \in U\}$.) A topological space (X, τ) *admits* $\mathcal{U}$ provided that τ is the topology induced by $\mathcal{U}$.

Let (G, τ) be a paratopological group. Denote by $\mathscr{N}(G)$ the family of all open neighborhoods at the identity in G throughout this section. Following [8], there are two natural quasi-uniformities on G. For each $U \in \mathscr{N}(G)$ put $U_L = \{(x, y) \in G \times G\colon x^{-1}y \in U\}$. It follows that $\{U_L\colon U \in \mathscr{N}(G)\}$ is a base for the *left quasi-uniformity* $\mathcal{U}_L$ on G. Moreover for each $U \in \mathscr{N}(G)$ put $U_R = \{(x, y) \in G \times G\colon yx^{-1} \in U\}$. Then $\{U_R\colon U \in \mathscr{N}(G)\}$ is also a base for the *right quasi-uniformity* $\mathcal{U}_R$ on G. Clearly, the paratopological group (G, τ) admits $\mathcal{U}_L$ and $\mathcal{U}_R$.

A real-valued function f on a paratopological group G is called *left quasi-uniformly continuous* (resp. *right quasi-uniformly continuous*) if for any $\varepsilon > 0$, there exists $V \in \mathscr{N}(G)$ such that $|f(y) - f(x)| < \varepsilon$ whenever $x^{-1}y \in V$ (resp. $yx^{-1} \in V$) for all $x \in G$. We say that a real-valued function f on G is *quasi-uniformly continuous* if f is both left quasi-uniformly continuous and right quasi-uniformly continuous.

In [23], as the generalization of uniformly continuous functions on topological groups, ω-uniformly continuous functions on topological groups were defined and discussed. Therefore, the notion of ω-quasi-uniform continuity of mappings is naturally defined in the classes of paratopological groups what follows.

Definition 4.1. A real-valued function f on a paratopological group G is *left* (resp. *right*) *ω-quasi-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U} \subset \mathscr{N}(G)$ such that for every $x \in G$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$ (resp. whenever $yx^{-1} \in U$).

Definition 4.2. A real-valued function f on a paratopological group is *ω-quasi-uniformly continuous* if f is both left and right ω-quasi-uniformly continuous.

Remark 4.3. (1) According to Definition 4.1, one can easily obtain that every continuous real-valued function defined on a first-countable paratopological group is ω-quasi-uniformly continuous.

(2) It is clear that every left (resp. right) quasi-uniformly continuous function defined on a paratopological group is left (resp. right) ω-quasi-uniformly continuous;

(3) It is clear that quasi-uniformly continuous $\Rightarrow$ ω-quasi-uniformly continuous, but the converse does not hold. In fact, the Sorgenfrey line S is a first-countable paratopological group, so every continuous real-valued f on S is ω-quasi-uniformly continuous according to (1). However, there exists some continuous real-valued function f on S which is not quasi-uniformly continuous.

The following proposition gives a characterization of left or right ω-quasi-uniformly continuous functions on a paratopological group. The proof is easy, so we omit it.

Proposition 4.4. *Let f be a real-valued function defined on a paratopological group G. The following are equivalent.*

(1) *f is left (resp. right) ω-quasi-uniformly continuous;*
(2) *there exists a countable family $\mathcal{U} \subset \mathscr{N}(G)$ satisfying that for every point $x \in G$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$ (resp. $yx^{-1} \in U$).*

Proposition 4.5. *A real-valued function defined on an ω-balanced paratopological group is left ω-quasi-uniformly continuous if and only if it is right ω-quasi-uniformly continuous.*

Proof. Let G be an ω-balanced paratopological group.

Necessity. Suppose that a real-valued function f on G is left ω-quasi-uniformly continuous. According to Definition 4.1, for every $\varepsilon > 0$, there exists a countable family $\gamma \subset \mathscr{N}(G)$ such that for every $x \in G$, there exists $U \in \gamma$ such that

$|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$. Since G is ω-balanced, for every $U \in \gamma$ one can find a countable family $\delta_U \subset \mathscr{N}(G)$ such that for each $x \in G$, there exists $V \in \delta_U$ with $xVx^{-1} \subset U$. Put $\delta = \bigcup_{U\in\gamma} \delta_U$. Take any $x \in G$. According to the property of δ_U, there exists $V \in \delta_U \subset \delta$ such that $x^{-1}Vx \subset U$. Thus $|f(x) - f(vx)| = |f(x) - f(xu)| < \varepsilon$ for all $v \in V$, which implies that f is right ω-quasi-uniformly continuous.

Sufficiency. Similar to the necessity. □

Corollary 4.6. *Every continuous (resp. bounded and continuous) real-valued function defined on an ω-balanced paratopological group is left ω-quasi-uniformly continuous if and only if every continuous (resp. bounded and continuous) real-valued function on it is right ω-quasi-uniformly continuous.*

Similar to the concepts of property ω-U and property $B\omega$-U [23] in topological groups, we have the following definition in paratopological groups, which plays an important role in our decomposition theorems.

Definition 4.7. A paratopological group G has *property ω-QU* (resp. *property $B\omega$-QU*) if each continuous (resp. bounded and continuous) real-valued function on G is ω-quasi-uniformly continuous.

Remark 4.8. (1) The (1) of Remark 4.3 implies that every first-countable paratopological group has property ω-QU.

(2) It is well known that every totally precompact paratopological group is a precompact topological group [1, Theorem 1.8] and that every precompact topological group has property ω-U [23, Corollary 4.12], thus, every totally precompact paratopological group has property ω-QU.

It is well known that a topological group has property $B\omega$-U if and only if it has property ω-U [23, Theorem 4.3]. In the same way in the proof of [23, Theorem 4.3] one can easily obtain the following.

Theorem 4.9. *A paratopological group has property $B\omega$-QU if and only if it has property ω-QU.*

The following theorem was proved for topological groups in [23].

Theorem 4.10. *Every Lindelöf paratopological group has property ω-QU.*

Proof. Let G be a Lindelöf paratopological group and $f : G \to \mathbb{R}$ a continuous function. Take any $\varepsilon > 0$. Then one can find two families $\mathcal{V} = \{V_i\colon i \in \omega\}$ and $\mathcal{U} = \{U_j\colon j \in \omega\}$, and two subsets $A = \{x_i\colon i \in \omega\}$ and $B = \{y_j\colon j \in \omega\}$ of G such that:

(a) V_i and U_j are open neighborhoods at the identity of G for each $i, j \in \omega$;
(b) $f(x_iV_i^2) \subset (f(x_i) - \frac{\varepsilon}{2}, f(x_i) + \frac{\varepsilon}{2})$ and $f(U_j^2y_j) \subset (f(y_j) - \frac{\varepsilon}{2}, f(y_j) + \frac{\varepsilon}{2})$ for each $i, j \in \omega$;
(c) $G = \bigcup_{i\in\omega} x_iV_i = \bigcup_{j\in\omega} U_jy_j$.

Indeed, since G is a paratopological group, for each $x \in G$, one can find an open neighborhood V_x at the identity of G such that $f(xV_x^2) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. Clearly, $G = \bigcup_{x\in G} xV_x$. Since G is Lindelöf, one can find a countable subset $A \subset G$ such that $G = \bigcup_{x\in A} xV_x$. Put $\mathcal{V} = \{V_x\colon x \in A\}$. It is obvious that A and $\mathcal{V}$ satisfy (a)–(c). Similarly, one can find B and $\mathcal{U}$.

Put $\mathcal{W} = \mathcal{V} \cup \mathcal{U}$. We claim that for each $x \in G$, there exists $V \in \mathcal{W}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in V$. It implies that the function f is left ω-quasi-uniformly continuous by Proposition 4.4. In fact, by (c), there exists $i \in \omega$ such that $x \in x_iV_i$. Thus, $f(xV_i) \subset f(x_iV_i^2) \subset (f(x_i) - \frac{\varepsilon}{2}, f(x_i) + \frac{\varepsilon}{2})$, which implies that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in V_i$.

Similarly, one can prove that f is right ω-quasi-uniformly continuous by Proposition 4.4. Thus, the Lindelöf paratopological group G has property ω-QU. □

Corollary 4.11. *Every subgroup of a paratopological group with a countable network has property ω-QU, in particular, so does every subgroup of a second-countable paratopological group.*

Since it is easy to prove that the continuous homomorphic image of a totally ω-narrow paratopological group is totally ω-narrow, from Lemma 2.3 and Corollary 2.13 it follows the following result which plays an important role in the proof of Lemma 4.13.

Lemma 4.12. *Let G be a totally ω-narrow paratopological group with $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$). Then for every open neighborhood U of the identity in G, there exists a continuous homomorphism π of G onto a Hausdorff (resp. regular) second-countable paratopological group H such that $\pi^{-1}(V) \subset U$ for some open neighborhood V of the identity in H.*

Lemma 4.13. *Let G be a totally ω-narrow paratopological group with $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$) and $f : G \to \mathbb{R}$ either left or right ω-quasi-uniformly continuous. Then there exist a continuous homomorphism $\pi : G \to K$ onto a Hausdorff (resp. regular) second-countable paratopological group K and a continuous function $p : K \to \mathbb{R}$ such that $f = p \circ \pi$.*

Proof. Suppose that $f : G \to \mathbb{R}$ is left ω-quasi-uniformly continuous. Thus, by Proposition 4.4, there exists a countable family $\mathcal{U}$ of open neighborhoods at the identity in G such that for any $\varepsilon > 0$ and any point $x \in G$, there exists $U \in \mathcal{U}$ satisfying $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$. According to Lemma 4.12, one can find a continuous homomorphism $\pi_U : G \to H_U$ onto a Hausdorff (resp. regular) second-countable paratopological group H_U such that $\pi_U^{-1}(V) \subset U$ for some open neighborhood V of the identity in H_U. Define $\pi = \Delta_{U\in\mathcal{U}}\pi_U$ as the diagonal product of the family $\{\pi_U\colon U \in \mathcal{U}\}$.

It is obvious that $\pi(G)$ is a Hausdorff (resp. regular) second-countable paratopological group, since $\prod_{U\in\mathcal{U}} H_U$ is Hausdorff (resp. regular) and second-countable.

Claim. *$f(g_1) = f(g_2)$ for all $g_1, g_2 \in G$ satisfying $\pi(g_1) = \pi(g_2)$.*

Indeed, assume to the contrary, and choose $g_1, g_2 \in G$ and $\varepsilon > 0$ such that

$$\pi(g_1) = \pi(g_2) \quad \text{and} \quad f(g_1) \notin \big(f(g_2) - \varepsilon, f(g_2) + \varepsilon\big).$$

By the property of $\mathcal{U}$, for g_2 and ε there exists $U \in \mathcal{U}$ such that $|f(g_2) - f(g_2u)| < \varepsilon$ for all $u \in U$, which is equivalent to $f(g_2U) \subset (f(g_2) - \varepsilon, f(g_2) + \varepsilon)$. Therefore, there exists an open neighborhood V of the identity in H_U such that $\pi_U^{-1}(V) \subset U$. Take an open neighborhood W of the identity in H_U such that $W^2 \subset V$. Put $g = \pi_U(g_1)$, then $g = \pi_U(g_2)$ by $\pi(g_1) = \pi(g_2)$ and

$$\begin{aligned} g_1 \in \pi_U^{-1}(gW) &= \pi_U^{-1}(g)\pi_U^{-1}(W) \\ &= g_2\pi_U^{-1}(e)\pi_U^{-1}(W) \subset g_2\pi_U^{-1}(W)\pi_U^{-1}(W) \\ &= g_2\pi_U^{-1}\big(W^2\big) \subset g_2\pi^{-1}(V) \subset g_2U, \end{aligned}$$

which implies that

$$f(g_1) \in f(g_2U) \subset \big(f(g_2) - \varepsilon, f(g_2) + \varepsilon\big).$$

This contradiction completes the proof of Claim.

From Claim it follows that there is a function $p : \pi(G) \to \mathbb{R}$ such that $f = p \circ \pi$. It remains to prove that p is continuous.

Take any $\varepsilon > 0$, $h \in \pi(G)$ and choose a point $g \in G$ such that $h = \pi(g)$. According to $f = p \circ \pi$ and the property of $\mathcal{U}$ there exists $U \in \mathcal{U}$ such that

$$f(gU) \subset \big(f(g) - \varepsilon, f(g) + \varepsilon\big) = \big(p(h) - \varepsilon, p(h) + \varepsilon\big).$$

By the property of π_U above, there is an open neighborhood V containing the identity in H_U such that $\pi_U^{-1}(V) \subset U$. Choose an open neighborhood W of the identity in H_U such that $W^2 \subset V$. Put

$$O = \pi(G) \cap \Bigg(W \times \prod_{U'\in\mathcal{U}\setminus\{U\}} H_{U'}\Bigg).$$

We claim that $p(hO) \subset (p(h) - \varepsilon, p(h) + \varepsilon)$, which implies that p is continuous.

In fact, since $h_U = \pi_U(g)$,

$$\begin{aligned} p(hO) &\subset f\big(\pi^{-1}(hO)\big) \\ &= f\Bigg(\pi^{-1}\Bigg(\pi(G) \cap \Bigg(h_UW \times \prod_{U'\in\mathcal{U}\setminus\{U\}} H_{U'}\Bigg)\Bigg)\Bigg) \\ &= f\big(\pi_U^{-1}(h_UW)\big) \subset f\big(g\pi_U^{-1}(V)\big) \subset f(gU) \\ &\subset \big(f(g) - \varepsilon, f(g) + \varepsilon\big) = \big(p(h) - \varepsilon, p(h) + \varepsilon\big). \end{aligned}$$

This completes the proof when f is left ω-quasi-uniformly continuous.

Similarly, one can prove the result when f is right ω-quasi-uniformly continuous. $\square$

Lemma 4.14. *$Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$) holds for every completely regular and $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group G.*

Proof. Let $\mu = \{f_\alpha\colon \alpha \in \gamma\}$ be all continuous real-valued functions on G. Define $f = \Delta_{\alpha\in\gamma} f_\alpha$ as the diagonal product of the family μ. Since G is completely regular, $f : G \to \prod_{\alpha\in\gamma} \mathbb{R}_\alpha$ is an embedding mapping. By the $\mathbb{R}_2$-factorizability (resp. $\mathbb{R}_3$-factorizability) of G, there exist a continuous homomorphism $p_\alpha : G \to K_\alpha$ onto a Hausdorff (resp. regular) second-countable paratopological group K_α and a continuous function $h_\alpha : K_\alpha \to \mathbb{R}_\alpha$ such that $f_\alpha = h_\alpha \circ p_\alpha$ for each $\alpha \in \gamma$. Define $p = \Delta_{\alpha\in\gamma} p_\alpha$ as the diagonal product of the family $\{p_\alpha\colon \alpha \in \gamma\}$ and $h = \Delta_{\alpha\in\gamma} h_\alpha$ as the diagonal product of the family

$\{h_\alpha\colon \alpha \in \gamma\}$. Clearly, $f = h \circ p$. Since f is an embedding mapping and h, p are continuous, $p : G \to \prod_{\alpha\in\gamma} K_\alpha$ is also an embedding mapping. By [20, Propositions 2.1, 2.2 and 2.3] (resp. [20, Propositions 3.1, 3.2 and 3.3]), one can easily obtain that $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$). □

The following is one of main results in this section.

Theorem 4.15. *A completely regular paratopological group G is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) if and only if it is a totally ω-narrow paratopological group with property ω-QU and $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$).*

Proof. The sufficiency is obtained by Lemma 4.13. Conversely, suppose that G is a completely regular and $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group. Then $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$) holds by Lemma 4.14. Since every completely regular $\mathbb{R}_1$-factorizable paratopological group is totally ω-narrow [17, Proposition 3.5], so is G (because of $\mathbb{R}_i$-factorizable $\Rightarrow$ $\mathbb{R}_j$-factorizable for $1 \leqslant j < i \leqslant 3.5$). Thus, it is enough to show that G has property ω-QU.

Let $f : G \to \mathbb{R}$ be a continuous function. Since G is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable), there exist a continuous homomorphism $\pi : G \to K$ onto a Hausdorff (resp. regular) second-countable paratopological group K and a continuous function $p : K \to \mathbb{R}$ such that $f = p \circ \pi$. Let $\mathscr{B}$ be a countable local base of the identity in K. Put $\mathscr{U}_f = \{\pi^{-1}(U)\colon U \in \mathscr{B}\}$. By (1) of Remark 4.8, one can easily verify that $\mathscr{U}_f$ is a countable family of open neighborhoods of the identity in G and satisfies that for every point $x \in G$ and $\varepsilon > 0$, there exist $U_1, U_2 \in \mathcal{U}_f$ such that $|f(x) - f(y)| < \varepsilon$ and $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U_1$ and $yx^{-1} \in U_2$, respectively. It implies that the function f is ω-quasi-uniformly continuous by Proposition 4.4. □

Remark 4.16. From the proof of Theorem 4.15 it is not difficult to obtain that every $\mathbb{R}_i$-factorizable paratopological group has property ω-QU (for $i = 1, 2, 3, 3.5$).

Since $Ir(G) \leqslant \omega$ holds for every regular Lindelöf paratopological group G [20, Proposition 3.5], from Theorems 4.10 and 4.15 it follows Corollary 4.17.

Corollary 4.17. *([17, Theorem 3.6]) Every Lindelöf totally ω-narrow regular paratopological group is $\mathbb{R}_3$-factorizable.*

Since every subgroup of a Hausdorff σ-compact paratopological group is $\mathbb{R}_2$-factorizable [17, Proposition 3.16], from Lemma 3.2, Theorem 3.6 and Remark 4.16 it follows Corollaries 4.18 and 4.19.

Corollary 4.18. *Every subgroup of a paratopological group G satisfying one of the following conditions has property ω-QU.*

(1) *G is a Hausdorff σ-compact space*;
(2) *G is a regular Lindelöf Σ-space.*

Corollary 4.19. *Every dense subgroup of a topological product of regular paratopological groups which are Lindelöf Σ-spaces has property ω-QU.*

Theorem 4.15 implies Theorem 4.20.

Theorem 4.20. *A totally ω-narrow Hausdorff (resp. regular) paratopological group G with $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$) and with property ω-QU is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

Corollary 4.21. *([17, Theorem 3.7]) Let G be a Hausdorff totally ω-narrow paratopological group. If G is Lindelöf, then it is $\mathbb{R}_2$-factorizable.*

Proof. Since every Hausdorff Lindelöf paratopological group H has $Hs(H) \leqslant \omega$ [20], the statement directly follows from Theorems 4.10 and 4.20. □

5. Open homomorphic images of $\mathbb{R}$-factorizable paratopological groups

In this section, we answer partially to the following question posed in [17].

Question 5.1. *([17, Problem 5.2]) Let G be an $\mathbb{R}_i$-factorizable paratopological group, for some $i \in \{1, 2, 3, 3.5\}$. Is every open continuous homomorphic image H of G an $\mathbb{R}_i$-factorizable paratopological group, provided that H satisfies the T_i-separation axiom?*

Lemma 5.2. *Let G be a paratopological group with property ω-QU (resp. property $B\omega$-QU). If N is a closed normal subgroup of G, then the quotient group G/N has property ω-QU (resp. property $B\omega$-QU).*

Proof. Let $p : G \to G/N$ be a quotient homomorphism. Then p is an open continuous homomorphism [2, Theorem 1.5.1]. Take any continuous (resp. bounded and continuous) real-valued function f on G/N. Then $f \circ p$ is a continuous (resp. bounded and continuous) real-valued function on G. Since G has property ω-QU (resp. $B\omega$-QU), $f \circ p$ is ω-quasi-uniformly continuous by Definition 4.2. According to Proposition 4.4, there exists a countable family $\mathcal{U}_{f\circ p} \subset \mathscr{N}(G)$ satisfying that for every $x \in G$ and $\varepsilon > 0$, there exists $U_{x,\varepsilon} \in \mathcal{U}_{f\circ p}$ such that $|f \circ p(x) - f \circ p(y)| < \varepsilon$ whenever $x^{-1}y \in U_{x,\varepsilon}$. Put $\mathscr{U}_f = \{p(U): U \in \mathscr{U}_{f\circ p}\}$. Since p is an open homomorphism, one can easily verify that $\mathscr{U}_f$ satisfies the condition (2) in Proposition 4.4, which implies that f is left ω-quasi-uniformly continuous. Similarly, one can easily prove that f is right ω-quasi-uniformly continuous. Thus G/N has property ω-QU (resp. property $B\omega$-QU). $\square$

Theorem 5.3. *Let G be a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open and continuous homomorphism onto a Hausdorff (resp. regular) paratopological group K. If $Hs(K) \leqslant \omega$ (resp. $Ir(K) \leqslant \omega$) holds, then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

Proof. To show that K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable), by Theorem 4.20, it is enough to show that K is a totally ω-narrow paratopological group with property ω-QU. Indeed, Remark 4.16 implies that G has property ω-QU. From Lemma 5.2 it follows that K has property ω-QU as well. Thus, it remains to prove that K is totally ω-narrow.

It is well known that every completely regular $\mathbb{R}_1$-factorizable paratopological group is totally ω-narrow [17, Proposition 3.5]. Thus G is totally ω-narrow, that is, G^* is an ω-narrow topological group. Clearly, $p : G^* \to K^*$ is a continuous homomorphism. Since the ω-narrowness is preserved by continuous homomorphisms for topological groups [2, Proposition 3.4.2], K^* is ω-narrow, that is, K is totally ω-narrow. $\square$

Corollary 5.4. *Let G be a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open and continuous homomorphism onto a Hausdorff (resp. regular) Lindelöf paratopological group K. Then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

Proof. Since every Hausdorff (resp. regular) Lindelöf paratopological group K has $Hs(K) \leqslant \omega$ (resp. $Ir(K) \leqslant \omega$) [20], the statement directly follows from Theorem 5.3. $\square$

Corollary 5.5. *Let G be a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open and continuous homomorphism onto a Hausdorff (resp. regular) first-countable paratopological group K. Then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

Proof. It is well known that a Hausdorff (resp. regular) first-countable paratopological group K has $Hs(K) \leqslant \omega$ (resp. $Ir(K) \leqslant \omega$) [20]. Thus, the statement directly follows from Theorem 5.3. $\square$

Theorem 5.6. *Let G be a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ an open and continuous homomorphism onto a Hausdorff (resp. regular) paratopological group K with identity e such that $p^{-1}(e)$ is countably compact. Then K is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

Proof. By Theorem 5.3, it is enough to prove that $Hs(K) \leqslant \omega$ (resp. $Ir(K) \leqslant \omega$).

Take any open set U of the identity in K and put $W = p^{-1}(U)$. By Lemma 4.14, $Hs(G) \leqslant \omega$, thus, one can easily find a countable family $\{V_i \mid i \in \omega\}$ of open neighborhoods at the identity in G such that $\bigcap_{i\in\omega} V_i V_i^{-1} \subset W$ and $V_{i+1}^2 \subset V_i$ for each $i \in \omega$. We claim that the family $\{p(V_i) \mid i \in \omega\}$ satisfies $\bigcap_{i\in\omega} p(V_i)p(V_i)^{-1} \subset U$, which implies that $Hs(K) \leqslant \omega$.

To prove that $\bigcap_{i\in\omega} p(V_i)p(V_i)^{-1} \subset U$, it is enough to show that for each $x \notin U$, there exists $i_0 \in \omega$ such that $V_{i_0} V_{i_0}^{-1} \cap p^{-1}(x) = \emptyset$.

Since $\overline{V_{i+1}V_{i+1}^{-1}} \subset V_{i+1}V_{i+1}^{-1}V_{i+1}^{-1} \subset V_{i+1}V_i^{-1} \subset V_i V_i^{-1}$ for each $i \in \omega$, we have $\bigcap_{i\in\omega} V_i V_i^{-1} = \bigcap_{i\in\omega} \overline{V_i V_i^{-1}}$. Thus, $\bigcap_{i\in\omega} \overline{V_i V_i^{-1}} \cap p^{-1}(x) = \emptyset$ by $\bigcap_{i\in\omega} V_i V_i^{-1} \subset W = p^{-1}(U)$. According to the countable compactness of $p^{-1}(x)$, one can easily obtain that there exists $i_0 \in \omega$ such that $V_{i_0} V_{i_0}^{-1} \cap p^{-1}(x) = \emptyset$.

Similarly, one can obtain that $Ir(K) \leqslant \omega$. $\square$

Corollary 5.7. *Let G be a completely regular $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable) paratopological group and N a compact normal subgroup in G. Then the quotient group G/N is $\mathbb{R}_2$-factorizable (resp. $\mathbb{R}_3$-factorizable).*

6. Open problems

It is well known that every subgroup of a Hausdorff σ-compact paratopological group is $\mathbb{R}_2$-factorizable [17, Proposition 3.16]. Thus, according to Theorem 3.6, the following question arises in a natural way.

Question 6.1. *Are dense subgroups of a topological product of Hausdorff σ-compact paratopological groups $\mathbb{R}_2$-factorizable?*

In [23], it shows that every real-valued function on a topological group G is left ω-uniformly continuous if and only if every real-valued function on it is right ω-uniformly continuous, so we pose the following.

Question 6.2. *Can the condition "ω-balanced" be dropped in Corollary 4.6?*

It is known that every $\mathbb{R}_1$-factorizable paratopological group is totally ω-narrow [17, Proposition 3.5], however, the proof of it seems to use the condition "completely regular". Thus, we have the following.

Question 6.3. *Is every regular $\mathbb{R}_1$-factorizable paratopological group totally ω-narrow?*

If the answer to Question 6.3 is positive, then the sufficiency of Theorem 4.20 is also true. We do not know if the condition "completely regular" can be dropped in Lemma 4.14.

Question 6.4. *Does $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$) hold for every Hausdorff $\mathbb{R}_2$-factorizable (resp. regular $\mathbb{R}_3$-factorizable) paratopological group G?*

If the answers to Questions 6.3 and 6.4 are affirmative, then the condition "completely regular" can be dropped in Theorem 4.15. Thus the following question is posed.

Question 6.5. *Let G be a Hausdorff (resp. regular) paratopological group with $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$). Does every open homomorphic image H of G have $Hs(G) \leqslant \omega$ (resp. $Ir(G) \leqslant \omega$), provided H is a T_2-space (resp. T_3-space)?*

Acknowledgements

We wish to thank the referee for the detailed list of corrections, suggestions, in particular quasi-uniformity on paratopological groups to the paper, and all her/his efforts in order to improve the paper.

References

[1] A.V. Arhangel'skiĭ, E.A. Reznichenko, Paratopological and semitopological groups versus topological groups, Topology Appl. 151 (2005) 107–119.
[2] A.V. Arhangel'skiĭ, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Press, World Sci., 2008.
[3] S. Banach, Remarques sur les groupes et les corps métriques, Studia Math. 10 (1948) 178–181.
[4] B. Bokalo, I. Guran, Sequentially compact Hausdorff cancellative semigroup is a topological group, Mat. Stud. 6 (1996) 39–40.
[5] N. Bourbaki, Elements of Mathematics. General Topology, Part 1, Hermann, Paris, 1966.
[6] R. Ellis, Locally compact transformation groups, Duke Math. J. 24 (1957) 119–125.
[7] R. Ellis, A note on the continuity of the inverse, Proc. Amer. Math. Soc. 8 (1957) 372–373.
[8] P. Fletcher, W.F. Lindgren, Some unsolved problems concerning countably compact spaces, Rocky Mountain J. Math. 5 (1975) 95–106.
[9] D.L. Grant, The Wallace problem and continuity of the inverse in pseudocompact groups, in: General Topology and Applications, Staten Island, NY, 1989, in: Lect. Notes Pure Appl. Math., vol. 134, Dekker, New York, 1991, pp. 93–114.
[10] D.L. Grant, Sequentially compact cancellative topological semigroups: Some progress on the Wallace problem, in: Papers on General Topology and Applications, Madison, WI, 1991, in: Ann. New York Acad. Sci., vol. 704, New York Acad. Sci., New York, 1993, pp. 150–154.
[11] C. Liu, A note on paratopological groups, Comment. Math. Univ. Carolin. 47 (2006) 633–640.
[12] K. Numakura, On bicompact semigroups, Math. J. Okayama Univ. 1 (1952) 99–108.
[13] O. Ravsky, Paratopological groups I, Mat. Stud. 16 (2001) 37–48.
[14] O. Ravsky, Paratopological groups II, Mat. Stud. 17 (2002) 93–101.
[15] E.A. Reznichenko, Extension of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups, Topology Appl. 59 (1994) 233–244.
[16] M. Sanchis, M.G. Tkachenko, Totally Lindelöf and totally ω-narrow paratopological groups, Topology Appl. 155 (4) (2007) 322–334.
[17] M. Sanchis, M.G. Tkachenko, $\mathbb{R}$-factorizable paratopological groups, Topology Appl. 157 (2010) 800–808.
[18] M.G. Tkachenko, Subgroups, quotient groups and products of $\mathbb{R}$-factorizable groups, Topology Proc. 16 (1991) 201–231.
[19] M.G. Tkachenko, Introduction to topological groups, Topology Appl. 86 (1998) 179–231.
[20] M.G. Tkachenko, Embedding paratopological groups into topological products, Topology Appl. 156 (2009) 1298–1305.
[21] M. Tkachenko, Paratopological and semitopological groups vs topological groups, in: J. van Mill, J. Vaughan (Eds.), Recent Progress in General Topology III, Elsevier, in press.
[22] A.D. Wallace, The structure of topological semigroups, Bull. Amer. Math. Soc. 61 (1955) 95–112.
[23] L.H. Xie, S. Lin, $\mathbb{R}$-factorizability and ω-uniform continuity in topological groups, Topology Appl. 159 (2012) 2711–2720.