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Compact-valued continuous relations on TVS-cone metric spaces

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Abstract. This paper uses cones of topological vector spaces in place of cones of Banach spaces to investigate relations on TVS-cone metric spaces. It is proved that if f is a compact-valued continuous relation on a TVS-cone metric space X, then f^n is a compact-valued continuous relation on X for each $n \in \mathbb{N}$. This result generalizes domains of compact-valued continuous relations from metric spaces to TVS-cone metric spaces, and improves a result for compact-valued continuous relations by omitting "locally compactness" of domains.

1. Introduction

Cone metric spaces were introduced and discussed by Huang and Zhang in [11]. In the past years, the following question arouses mathematical scholar interest and many interesting results have been obtained (see [2, 3, 8, 9, 11, 12, 15, 16, 18, 19], for example).

Question 1.1. *In relevant results about metric spaces, can metric spaces be generalized to cone metric spaces?*

As a positive answer for Question 1.1, Khani and Pourmahdian [12] proved that each cone metric space is metrizable, which shows that some generalizations from metric spaces to cone metric spaces are trivial. Moreover, Khani and Pourmahdian [12] pointed out: "However, considering certain topological groups in place of Banach spaces may result in the construction of new spaces which are not in general metrizable. This can serve as a topic for further studies". Also, Kadelburg, Radenović and Rakočević [13] noted that: "proper generalizations when passing from norm-valued cone metric spaces of [11] to TVS-valued cone metric spaces can be obtained only in the case of nonnormal cones", and discussed TVS-cone metric spaces to develop "further the theory of topological vector space valued cone metric spaces (with nonnormal cones)". In fact, the notion of a TVS-cone metric space was first used by Beg, Abbas and Arshad in [4], and it was proved that each TVS-cone metric space is metrizable by using the Minkowski functional under assumption that the topological vector space is locally convex and Hausdorff (see [14, 17], for example). In addition, many interesting results had been obtained for some nonnormal spaces in place of Banach spaces (see [1, 4, 7, 8, 13, 14, 17], for example). Having gained some enlightenment from the above, we

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consider TVS-cone metric spaces in place of cone metric spaces to investigate relations on TVS-cone metric spaces, and give a new result around compact-valued continuous relations on TVS-cone metric spaces. This paper deals with mainly some non-topological properties (for example, TVS-metric properties) and does not assume that the topological vector space is locally convex and Hausdorff, so some corresponding results from the metric setting can not be applied directly in our investigations.

Recall that a relation f on a space X is a set-valued mapping $f : X \longrightarrow \mathscr{P}_0(X)$, where $\mathscr{P}_0(X) = \{P \subseteq X : P \neq \emptyset\}$, i.e., $f(x)$ is a nonempty subset of X for each $x \in X$.

Chu and Park [6] obtained the following results for compact-valued continuous relations on metric spaces.

Proposition 1.2. ([6]) *Let f be a compact-valued continuous relation on a locally compact metric space X. Then for each compact subset K of X, $f(K)$ is a compact subset of X.*

Proposition 1.3. ([6]) *Let f be a compact-valued continuous relation on a locally compact metric space X. Then f^n is a compact-valued continuous relation on X for each $n \in \mathbb{N}$.*

In this paper, we improve Proposition 1.2 and Proposition 1.3 by weakening "metric space X" to "TVS-cone metric space X" and omitting "locally compact" for metric space X. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}^+$ denote the set of all natural numbers and the set of all positive real numbers, respectively.

2. TVS-cone metric spaces

Definition 2.1. ([4]) Let E be a topological vector space. A subset P of E is called a topological vector space cone (abbr. TVS-cone) if the following are satisfied.
(1) P is closed, $P \neq \emptyset$ and $P \neq \{0\}$.
(2) $\alpha, \beta \in P$ and $a, b \in \mathbb{R}^+ \Longrightarrow a\alpha + b\beta \in P$.
(3) $\alpha, -\alpha \in P \Longrightarrow \alpha = 0$.

For $r \in \mathbb{R}^+$, $\alpha \in E$ and $B \subseteq E$, rB and $\alpha + B$ denote $\{r\beta : \beta \in B\}$ and $\{\alpha + \beta : \beta \in B\}$, respectively.

Remark 2.2. Let P be a TVS-cone of a topological vector space E. Then $0 \in P - P^\circ$, where P° denotes the interior of P in E. In fact, pick $\alpha, \beta \in P$, then $\dfrac{\alpha+\beta}{n} \in P$ for each $n \in \mathbb{N}$ from Definition 2.1(2). Note that $\{\dfrac{\alpha+\beta}{n}\} \longrightarrow 0$. So $0 \in P$ because P is closed from Definition 2.1(1). On the other hand, pick $\gamma \in E - \{0\}$, then $\{\dfrac{\gamma}{n}\} \longrightarrow 0$ and $\{-\dfrac{\gamma}{n}\} \longrightarrow 0$. If $0 \in P^\circ$, then there is $n \in \mathbb{N}$, such that $\dfrac{\gamma}{n}, -\dfrac{\gamma}{n} \in P$. By Definition 2.1(3), $\dfrac{\gamma}{n} = 0$. This contradicts that $\gamma \neq 0$. So $0 \notin P^\circ$.

Definition 2.3. ([4]) Let P be a TVS-cone of a topological vector space E. Some partial orderings $\leq$, $<$ and $\ll$ on E with respect to P are defined as follows, respectively.
(1) $\alpha \leq \beta$ if $\beta - \alpha \in P$.
(2) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.
(3) $\alpha \ll \beta$ if $\beta - \alpha \in P^\circ$, where P° denotes the interior of P in E.

Remark 2.4. In this paper, for the sake of conveniences, we also use notations "$\geq$", "$>$" and "$\gg$" on E with respect to P. The meanings of these notations are clear and the following hold.
(1) $\alpha \geq 0$ if and only if $\alpha \in P$.
(2) $\alpha \gg 0$ if and only if $\alpha \in P^\circ$.
(3) $\alpha - \beta \gg 0$ if and only if $\alpha \gg \beta$.
(4) $\alpha - \beta \geq 0$ if and only if $\alpha \geq \beta$.
(5) If $\alpha \geq 0$ and $\beta \geq 0$, then $a\alpha + b\beta \geq 0$ for all $a, b \in \mathbb{R}^+$.
(6) $\alpha \gg \beta \Longrightarrow \alpha > \beta \Longrightarrow \alpha \geq \beta$.

Lemma 2.5. *Let P be a TVS-cone of a topological vector space E. Then the following hold.*
(1) If $\alpha \gg 0$, then $r\alpha \gg 0$ for each $r \in \mathbb{R}^+$.
(2) If $\alpha_1 \gg \beta_1$ and $\alpha_2 \geq \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.
(3) If $\alpha \gg 0$ and $\beta \gg 0$, then there is $\gamma \gg 0$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.

Proof. (1) Let $\alpha \gg 0$, i.e., $\alpha \in P^\circ$. Then there is an open neighborhood B_α of α in E such that $B_\alpha \subseteq P$. If $r \in \mathbb{R}^+$, then $rB_\alpha \subseteq P$ from Definition 2.1(2). Note that $r\alpha \in rB_\alpha$ and rB_α is an open subset of E. So $r\alpha \in P^\circ$, i.e. $r\alpha \gg 0$.

(2) Let $\alpha_1 \gg \beta_1$ and $\alpha_2 \geq \beta_2$. Then $\alpha_1 - \beta_1 \gg 0$ and $\alpha_2 - \beta_2 \geq 0$, i.e., $\alpha_1 - \beta_1 \in P^\circ$ and $\alpha_2 - \beta_2 \in P$. So there is an open neighborhood B of $\alpha_1 - \beta_1$ in E such that $B \subseteq P$. Note that $(\alpha_2 - \beta_2) + B$ is an open subset of E, and $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \in (\alpha_2 - \beta_2) + B \subseteq P$ from Definition 2.1(2). So $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \in P^\circ$, i.e., $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \gg 0$, hence $(\alpha_1 + \alpha_2) - (\beta_1 + \beta_2) \gg 0$. It follows that $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.

(3) Let $\alpha \gg 0$ and $\beta \gg 0$, i.e., $\alpha, \beta \in P^\circ$. Then there is $n_1, n_2 \in \mathbb{N}$ such that $\alpha - \dfrac{\alpha+\beta}{n} \in P^\circ$ for all $n \geq n_1$ and $\beta - \dfrac{\alpha+\beta}{n} \in P^\circ$ for all $n \geq n_2$. Put $\gamma = \dfrac{\alpha+\beta}{n_0}$, where $n_0 = \max\{n_1, n_2\}$. Then $\gamma \gg 0$ from the above (1) and (2). It is clear that $\alpha - \gamma \in P^\circ$ and $\beta - \gamma \in P^\circ$, i.e., $\alpha - \gamma \gg 0$ and $\beta - \gamma \gg 0$. So $\gamma \ll \alpha$ and $\gamma \ll \beta$. □

Definition 2.6. ([4]) Let X be a non-empty set and let E be a topological vector space equipped with partial order $\leq$ given by an order cone P. A vector-valued function $d : X \times X \longrightarrow E$ is called a TVS-cone metric on X, and (X, d) is called a TVS-cone metric space if the following are satisfied.
(1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark 2.7. By Definition 2.6(1), the TVS-cone metric $d : X \times X \longrightarrow E$ on X can be replaced by $d : X \times X \longrightarrow P$.

Proposition 2.8. *Let (X, d) be a TVS-cone metric space. Put $\mathscr{B} = \{B(x, \varepsilon) : x \in X$ and $\varepsilon \gg 0\}$, where $B(x, \varepsilon) = \{y \in X : d(x, y) \ll \varepsilon\}$ for each $x \in X$ and each $\varepsilon \gg 0$. Then $\mathscr{B}$ is a base for some topology on X.*

Proof. It is clear that $X = \bigcup \mathscr{B}$. Let $z \in B(x, \alpha) \bigcap B(y, \beta)$, where $B(x, \alpha)$, $B(y, \beta) \in \mathscr{B}$. Since $z \in B(x, \alpha)$, $d(x, z) \ll \alpha$. Put $\gamma_1 = \alpha - d(x, z)$, then $\gamma_1 \gg 0$. We claim that $B(z, \gamma_1) \subseteq B(x, \alpha)$. In fact, if $u \in B(z, \gamma_1)$, then $d(z, u) \ll \gamma_1$. It follows that $d(x, u) \leq d(x, z) + d(z, u) \ll d(x, z) + \gamma_1 = \alpha$, hence $u \in B(x, \alpha)$. By the same way, we can obtain that there is $\gamma_2 \gg 0$ such that $B(z, \gamma_2) \subseteq B(y, \beta)$. Thus, there is $\gamma \gg 0$ such that $\gamma \ll \gamma_1$ and $\gamma \ll \gamma_2$ from Lemma 2.5(3). Let $v \in B(z, \gamma)$, then $d(z, v) \ll \gamma \ll \gamma_1$ and $d(z, v) \ll \gamma \ll \gamma_2$, so $v \in B(z, \gamma_1) \subseteq B(x, \alpha)$ and $v \in B(z, \gamma_2) \subseteq B(y, \beta)$, and hence $v \in B(x, \alpha) \bigcap B(y, \beta)$. This has proved that $B(z, \gamma) \subseteq B(x, \alpha) \bigcap B(y, \beta)$. Note that $z \in B(z, \gamma) \in \mathscr{B}$. Consequently, $\mathscr{B}$ is a base for some topology on X. In fact, put $\mathscr{T} = \{U \subseteq X :$ there is $B' \subseteq \mathscr{B}$ such that $U = \bigcup \mathscr{B}'\}$, then $\mathscr{T}$ is a topology on X and $\mathscr{B}$ is a base for $\mathscr{T}$. □

In this paper, we always suppose that a cone P is a TVS-cone of a topological vector space E and a TVS-cone metric space (X, d) is a topological space with the topology $\mathscr{T}$ described in Proposition 2.8.

3. Relations on TVS-cone metric spaces

Throughout this section, we use the following brief notations

Notation 3.1. *Let (X, d) be a TVS-cone metric space, f be a relation on (X, d) and $D \subseteq X$.*
(1) $f(D) = \bigcup\{f(x) : x \in D\}$.
(2) $B(D, \varepsilon) = \bigcup\{B(x, \varepsilon) : x \in D\}$.
(3) $S_f(D) = \{x \in X : f(x) \subseteq D\}$.
(4) $W_f(D) = \{x \in X : f(x) \bigcap D \neq \emptyset\}$.

Lemma 3.2. *Let (X, d) be a TVS-cone metric space. If $K \subseteq U$ with K compact in X and U open in X, then there is $\varepsilon \gg 0$ such that $B(K, \varepsilon) \subseteq U$.*

Proof. By Proposition 2.8, for each $x \in K \subseteq U$, there is $\eta_x \gg 0$ such that $B(x, \eta_x) \subseteq U$. Put $\varepsilon_x = \frac{1}{2}\eta_x$, then $\varepsilon_x \gg 0$ from Lemma 2.5(1). Since $\{B(x, \varepsilon_x) : x \in K\}$ is an open cover of K and K is compact, there is a finite subset F of K such that $\{B(x, \varepsilon_x) : x \in F\}$ covers K. By Lemma 2.5(3), there is $\varepsilon \gg 0$ such that $\varepsilon \ll \varepsilon_x$ for each $x \in F$. We claim that $B(K, \varepsilon) \subseteq U$. In fact, let $u \in B(K, \varepsilon)$, then there is $y \in K$ such that $u \in B(y, \varepsilon)$, i.e., $d(u, y) \ll \varepsilon$. Furthermore, there is $z \in F$ such that $y \in B(z, \varepsilon_z)$, i.e., $d(y, z) \ll \varepsilon_z$. By Lemma 2.5(2), $d(u, z) \leq d(u, y) + d(y, z) \ll \varepsilon + \varepsilon_z \ll 2\varepsilon_z = \eta_z$. It follows that $u \in B(z, \eta_z) \subseteq U$. This has proved that $B(K, \varepsilon) \subseteq U$. □

The following two definitions refer to [5, 6, 10].

Definition 3.3. Let f be a relation on a TVS-cone metric space (X, d) and $x \in X$.

(1) f is called upper semicontinuous at x if for each $\varepsilon \gg 0$, there is $\delta \gg 0$ such that $y \in B(x, \delta) \implies f(y) \subseteq B(f(x), \varepsilon)$.

(2) f is called lower semicontinuous at x if for each $\varepsilon \gg 0$, there is $\delta \gg 0$ such that $y \in B(x, \delta) \implies f(x) \subseteq B(f(y), \varepsilon)$.

(3) f is called continuous at x if f is both upper semicontinuous and lower semicontinuous at x.

Definition 3.4. Let f be a relation on a TVS-cone metric space (X, d).

(1) f is called continuous (resp. upper semicontinuous, lower semicontinuous) , if f is continuous (resp. upper semicontinuous, lower semicontinuous) at each $x \in X$.

(2) f is called compact-valued (resp. closed-valued), if $f(x)$ is a compact (resp. closed) subset of X for each $x \in X$.

(3) f is called compact-set (resp. closed-set), if $f(F)$ is a compact (resp. closed) subset of X for each compact (resp. closed) subset F of X.

Theorem 3.5. *Let f be a compact-valued relation on a TVS-cone metric space (X, d). Then the following hold.*

(1) f is upper semicontinuous if and only if $S_f(U)$ is open in X for each open subset U of X.

(2) f is lower semicontinuous if and only if $W_f(U)$ is open in X for each open subset U of X.

Proof. (1) Necessity: Let f be upper semicontinuous. For each open subset U of X, let $x \in S_f(U)$, then $f(x) \subseteq U$. Since $f(x)$ is compact in X and U is open in X, there is $\varepsilon \gg 0$ such that $B(f(x), \varepsilon) \subseteq U$ by Lemma 3.2. Since f is upper semicontinuous at x, there is $\delta \gg 0$ such that $y \in B(x, \delta)$ implies $f(y) \subseteq B(f(x), \varepsilon) \subseteq U$, and hence $y \in B(x, \delta)$ implies $y \in S_f(U)$. It follows that $B(x, \delta) \subseteq S_f(U)$. This proves that $S_f(U)$ is open in X.

Sufficiency: Let $S_f(U)$ be open in X for each open subset U of X. For each $x \in X$ and each $\varepsilon \gg 0$, $B(f(x), \varepsilon)$ is open in X, so $S_f(B(f(x), \varepsilon))$ is open in X. Since $f(x) \subseteq B(f(x), \varepsilon)$, $x \in S_f(B(f(x), \varepsilon))$, and hence there is $\delta \gg 0$ such that $B(x, \delta) \subseteq S_f(B(f(x), \varepsilon))$. It is not difficult to check that $f(S_f(B(f(x), \varepsilon))) \subseteq B(f(x), \varepsilon)$. So, for each $y \in B(x, \delta)$, $f(y) \subseteq f(B(x, \delta)) \subseteq f(S_f(B(f(x), \varepsilon))) \subseteq B(f(x), \varepsilon)$. This proves that f is upper semicontinuous at x. Consequently, f is upper semicontinuous.

(2) Necessity: Let f be lower semicontinuous. For each open subset U of X, let $x \in W_f(U)$, then $f(x) \bigcap U \neq \emptyset$. Pick $b \in f(x) \bigcap U$. Since U is open neighborhood of b in X, there is $\varepsilon \gg 0$ such that $B(b, \varepsilon) \subseteq U$. f is lower semicontinuous at x, so there is $\delta \gg 0$ such that $y \in B(x, \delta)$ implies $f(x) \subseteq B(f(y), \varepsilon)$. Thus, for each $y \in B(x, \delta)$, $b \in f(x) \subseteq B(f(y), \varepsilon)$, so there is $c \in f(y)$ such that $b \in B(c, \varepsilon)$, hence $c \in B(b, \varepsilon) \subseteq U$. It follows that $c \in f(y) \bigcap U \neq \emptyset$, and so $y \in W_f(U)$. This proves that $B(x, \delta) \subseteq W_f(U)$. Consequently, $W_f(U)$ is open in X.

Sufficiency: Let $W_f(U)$ be open in X for each open subset U of X. For each $x \in X$ and each $\varepsilon \gg 0$, since $f(x)$ is compact in X, there is a finite subset F of $f(x)$ such that $f(x) \subseteq \bigcup\{B(a, \frac{1}{2}\varepsilon) : a \in F\}$. For each $a \in F$, $B(a, \frac{1}{2}\varepsilon)$ is open in X, so $W_f(B(a, \frac{1}{2}\varepsilon))$ is open in X. Since $a \in f(x) \bigcap B(a, \frac{1}{2}\varepsilon) \neq \emptyset$, $x \in W_f(B(a, \frac{1}{2}\varepsilon))$.

Put $W = \bigcap\{W_f(B(a, \frac{1}{2}\varepsilon)) : a \in F\}$, then W is an open neighborhood of x. So there is $\delta \gg 0$ such that $B(x, \delta) \subseteq W$. Let $y \in B(x, \delta)$. It suffices to prove that $f(x) \subseteq B(f(y), \varepsilon)$. Let $b \in f(x)$, then there is $a \in F$ such that $b \in B(a, \frac{1}{2}\varepsilon)$, and hence $d(b, a) \ll \frac{1}{2}\varepsilon$. Since $y \in B(x, \delta) \subseteq W \subseteq W_f(B(a, \frac{1}{2}\varepsilon))$, and so $f(y) \bigcap B(a, \frac{1}{2}\varepsilon) \neq \emptyset$. Pick $c \in f(y) \bigcap B(a, \frac{1}{2}\varepsilon)$, then $d(a, c) \ll \frac{1}{2}\varepsilon$. So $d(b, c) \leq d(b, a) + d(a, c) \ll \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. It follows that $b \in B(c, \varepsilon) \subseteq B(f(y), \varepsilon)$. This proves that $f(x) \subseteq B(f(y), \varepsilon)$. □

Remark 3.6. By the proof of Theorem 3.5, we can omit "compact-valued" in Sufficiency of Theorem 3.5(1) and Necessity of Theorem 3.5(2).

Corollary 3.7. *A compact-valued relation f on a TVS-cone metric space (X, d) is continuous if and only if both $S_f(U)$ and $W_f(U)$ are open in X for each open subset U of X.*

Theorem 3.8. *Let f be a compact-valued upper-semicontinuous relation on a TVS-cone metric space (X, d). If K is a compact subset of X, then so is $f(K)$.*

Proof. Let K be a compact subset of X and let $\mathscr{U}$ be a family consisting of open subsets of X covering $f(K)$. For each $x \in K$, $f(x)$ is compact in X, so there is a finite subfamily $\mathscr{U}_x$ of $\mathscr{U}$ such that $\mathscr{U}_x$ covers $f(x)$. Put $U_x = \bigcup\{U : U \in \mathscr{U}_x\}$, then $f(x) \subseteq U_x$, hence $x \in S_f(U_x)$. Since f is upper-semicontinuous, $S_f(U_x)$ is open in X from Theorem 3.5(1). Thus $\{S_f(U_x) : x \in K\}$ is a family consisting of open subsets of X covering K, so there is a finite subset K' of K such that $\{S_f(U_x) : x \in K'\}$ covers K. We claim that $\{U_x : x \in K'\}$ covers $f(K)$. In fact, let $y \in f(K)$, then there is $z \in K$ such that $y \in f(z)$. Furthermore, there is $x \in K'$ such that $z \in S_f(U_x)$. Thus $y \in f(z) \subseteq f(S_f(U_x)) \subseteq U_x$. So $\{U_x : x \in K'\}$ covers $f(K)$. Put $\mathscr{U}' = \{U : U \in \mathscr{U}_x \textit{ and } x \in K'\}$, then $\mathscr{U}'$ is a finite subfamily of $\mathscr{U}$ and $\mathscr{U}'$ covers $f(K)$. So $f(K)$ is a compact subset of X. □

Corollary 3.9. *Let f be an upper semicontinuous relation on a compact TVS-cone metric space (X, d). The the following are equivalent.*

(1) f is compact-valued.
(2) f is closed-valued.
(3) f is compact-set.
(4) f is closed-set.

Proof. Compact subsets and closed subsets are equivalent in a compact space, so (1) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (4). It is clear that (3) $\Longrightarrow$ (1). By Theorem 3.8, (1) $\Longrightarrow$ (3). □

Remark 3.10. Theorem 3.8 improves Proposition 1.2 by weakening "metric space" to "TVS-cone metric space", weakening "continuous" to "upper-semicontinuous", and omitting "locally compact" for the space X.

For two relations f and g on a TVS-cone metric space (X, d), the composition of f with g is denoted by gf, that is, $(gf)(x) = g(f(x))$ for each $x \in X$.

Lemma 3.11. *Let f and g be relations on a TVS-cone metric space (X, d), $x \in X$ and $D \subseteq X$. Then the following hold.*

(1) $gf(x) \subseteq D \Longleftrightarrow f(x) \subseteq S_g(D)$.
(2) $gf(x) \bigcap D \neq \emptyset \Longleftrightarrow f(x) \bigcap W_g(D) \neq \emptyset$.

Proof. (1) Suppose that $gf(x) \subseteq D$. Let $y \in f(x)$, then $g(y) \subseteq gf(x)$. Since $gf(x) \subseteq D$, $g(y) \subseteq D$, hence $y \in S_g(D)$. This proves that $f(x) \subseteq S_g(D)$. Conversely, suppose that $f(x) \subseteq S_g(D)$. Let $y \in gf(x)$, then there is $z \in f(x)$ such that $y \in g(z)$. Since $z \in f(x) \subseteq S_g(D)$, $g(z) \subseteq D$. It follows that $y \in g(z) \subseteq D$. This proves that $gf(x) \subseteq D$.

(2) Let $gf(x) \bigcap D \neq \emptyset$, then there is $y \in gf(x) \bigcap D$, and so there is $z \in f(x)$ such that $y \in g(z)$. Thus $y \in g(z) \bigcap D \neq \emptyset$. It follows that $z \in W_g(D)$. This proves that $z \in f(x) \bigcap W_g(D) \neq \emptyset$. Conversely, let $f(x) \bigcap W_g(D) \neq \emptyset$, then there is $y \in f(x) \bigcap W_g(D)$, and hence $g(y) \bigcap D \neq \emptyset$. It follows that $gf(x) \bigcap D \neq \emptyset$. □

Lemma 3.12. *Let f and g be relations on a TVS-cone metric space (X, d), and let $D \subseteq X$. Then the following hold.*
(1) $S_{gf}(D) = S_f(S_g(D))$.
(2) $W_{gf}(D) = W_f(W_g(D))$.

Proof. (1) By Lemma 3.11(1), $S_{gf}(D) = \{x \in X : gf(x) \subseteq D\} = \{x \in X : f(x) \subseteq S_g(D)\} = S_f(S_g(D))$.
(2) By Lemma 3.11(2), $W_{gf}(D) = \{x \in X : gf(x) \bigcap D \neq \emptyset\} = \{x \in X : f(x) \bigcap W_g(D) \neq \emptyset\} = W_f(W_g(D))$. □

Theorem 3.13. *Let f and g be compact-valued continuous relations on a TVS-cone metric space (X, d). Then gf is compact-valued continuous.*

Proof. For each $x \in X$, $f(x)$ is a compact subset of X. Since g is compact-valued upper semicontinuous, $(gf)(x) = g(f(x))$ is a compact subset of X by Theorem 3.8. So gf is compact-valued. Let U be an open subset of X. Since g is compact-valued upper semicontinuous, $S_g(U)$ is an open subset of X by Theorem 3.5(1). furthermore, since f is compact-valued upper semicontinuous, $S_f(S_g(U))$ is an open subset of X. By Lemma 3.12(1), $S_{gf}(U) = S_f(S_g(D))$ is an open subset of X. By a similar way, we can prove that $W_{gf}(U)$ is an open subset of X by Theorem 3.5(2) and Lemma 3.12(2). Consequently, gf is continuous from Corollary 3.7. □

Corollary 3.14. *Let f be a compact-valued continuous relation on a TVS-cone metric space (X, d). Then f^n is compact-valued continuous for each $n \in \mathbb{N}$.*

Remark 3.15. Corollary 3.14 improves Proposition 1.3 by weakening "metric space" to "TVS-cone metric space" and omitting "locally compact" for the space X.

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