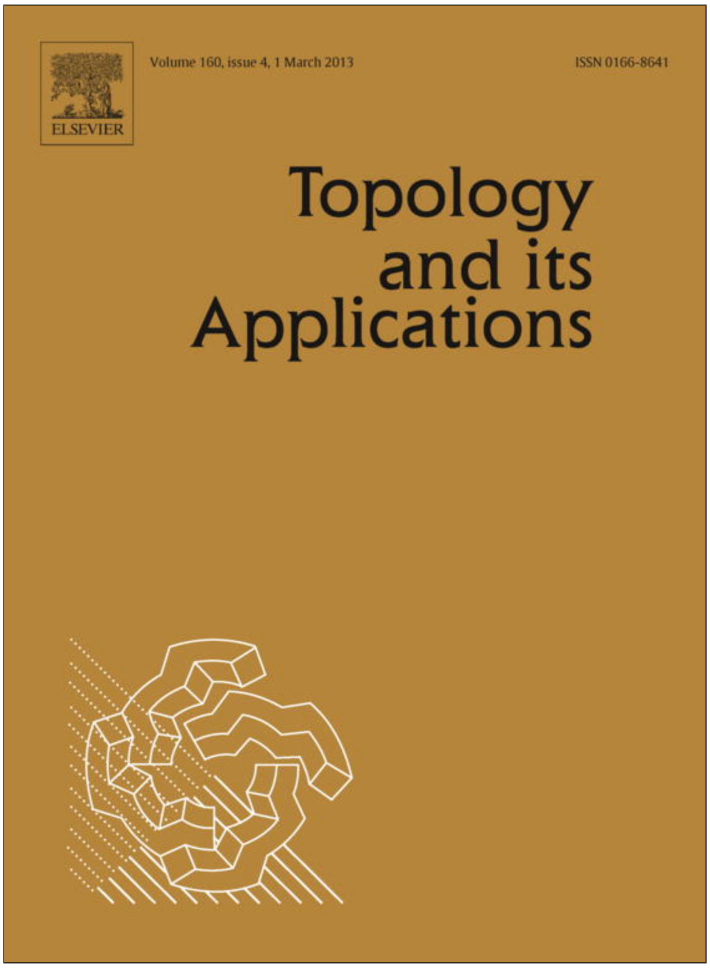
ELSEVIER
Volume 160, issue 4, 1 March 2013
ISSN 0166-8641
Topology and its Applications

Topology and its Applications 160 (2013) 648–655

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Remainders of topological and paratopological groups ☆

Li-Hong Xie [a], Shou Lin [b,*]

[a] School of Mathematics, Sichuan University, Chengdu 610065, PR China
[b] Institute of Mathematics, Ningde Normal University, Ningde 352100, PR China

ARTICLE INFO

Article history:
Received 19 October 2012
Received in revised form 14 January 2013
Accepted 17 January 2013

MSC:
54D20
54D40
54D45
54E18
54E35
54H11

Keywords:
Remainder
Topological group
Paratopological group
p-Space
Lindelöf p-space
Metrizability
G_δ-diagonal

ABSTRACT

In this paper, the remainders of topological and paratopological groups are discussed. Some conditions are given for a topological group G such that G or its compactification bG is separable and metrizable by the properties of the remainder $bG \setminus G$. Some properties are obtained for a paratopological group G such that the remainder $bG \setminus G$ is a p-space or a Lindelöf space. They improve some results of A.V. Arhangel'skiĭ (2005) [1] and F. Lin (2011) [14], respectively, and also give partial answers to some questions posed by F. Lin (2011) [14] and C. Liu (2009) [16].

1. Introduction

"A space" in this article stands for a Tychonoff topological space. A *remainder* of a space X is the space $bX \setminus X$, where bX is a Hausdorff compactification of X [10].

An important question in the study of Hausdorff compactifications is when a Tychonoff space X has a Hausdorff compactification with the remainder belonging to a given class of spaces. A famous classical result in this study is the following theorem [12]:

M. Henriksen and J. Isbell's theorem. *A space X is of countable type if and only if the remainder in any* (*in some*) *compactification of X is Lindelöf.*

A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ onto G associating $x \cdot y$ with arbitrary $x, y \in G$ is jointly continuous. If G is a paratopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *topological group*.

☆ The project is supported by the NSFC (Nos. 10971185, 11171162, 11201414).
* Corresponding author.
E-mail addresses: xielihong2011@yahoo.cn (L.-H. Xie), shoulin60@yahoo.cn, shoulin60@163.com (S. Lin).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.01.017

Recently, A.V. Arhangel'skiĭ [3] proved that if, for a non-locally compact topological group X, the remainder $Y = bX \setminus X$ of X has a G_δ-diagonal or a point-countable base, then both X and Y are separable metrizable. Chuan Liu [16] systematically investigated how the generalized metrizability properties of the remainder affect the metrizability of the group. Recall that a space X has *locally a property-Φ* if for each point $x \in X$ there exists an open neighborhood $U(x)$ of x such that $U(x)$ has a property-Φ. In 2011, F. Lin [14] systematically studied which locally a property-Φ is needed to make the following statement true:

Let G be a non-locally compact topological group and the remainder $bG \setminus G$ have locally a property-Φ, then bG and G are separable and metrizable.

In this paper, the following questions are considered and answered partially.

Question 1.1. ([14]) Let G be a non-locally compact topological group. If $Y = bG \setminus G$ satisfies the following conditions (1) and (2), are G and bG separable and metrizable?

(1) For each point $y \in Y$, there exists an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is a metrizable G_δ-subset of $U(y)$;
(2) π-character[1] of Y is countable.

Question 1.2. ([16]) Let G be a non-locally compact topological group, and $bG \setminus G$ have a base of countable order. Are G and bG separable and metrizable?

Question 1.3. ([16]) Let G be a non-locally compact topological group, and $bG \setminus G$ have a σ-locally countable network. Are G and bG separable and metrizable?

Question 1.4. ([16]) Let G be a non-locally compact topological group, and $bG \setminus G$ be locally symmetrizable. Are G and bG separable and metrizable?

The paper is organized as follows. In Section 2, the remainders of topological groups are discussed. It is proved that G is separable and metrizable, and Y is a first-countable, Lindelöf p-space if G is a non-locally compact topological group and the remainder $Y = bG \setminus G$ satisfies the condition (1) in Question 1.1. Thus the condition (1) implies the condition (2) in Question 1.1. Many results on remainders of topological groups can be obtained easily by it, and partial solutions of Questions 1.2, 1.3 and 1.4 are given.

In Section 3, the remainders of paratopological groups are investigated. A series of results on remainders of paratopological groups have been obtained. They show that the remainders of paratopological groups are much more sensitive to the topological properties of paratopological groups than the remainders of topological spaces are in general.

Recall that a *Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping.[2] The following theorem was proved in [1].

Lemma 1.5. *([1]) If X is a Lindelöf p-space, then every remainder of X is a Lindelöf p-space.*

Remark 1.6. If G is a non-locally compact paratopological group, G is nowhere locally compact. Then $Y = bG \setminus G$ is dense in bG, i.e., bG is also a compactification of Y. By Henriksen and Isbell's theorem and Lemma 1.5, the following statements hold:

(1) Y is of countable type $\Leftrightarrow$ G is Lindelöf;
(2) Y is a Lindelöf p-space $\Leftrightarrow$ G is a Lindelöf p-space.

2. Remainders of topological groups

First, we consider the separable metrizability for a topological group G.

Recall that a space X is of *countable type* [10] if every compact subset of X is contained in a compact subspace $F \subset X$ such that F has a countable base of open neighborhoods in X. A space X is of *subcountable type* [5] if every compact subset of X is contained in a compact G_δ-subset of X.

[1] A *π-base* of a space at a point x of X is a family γ of non-empty open subsets of X such that every open neighborhood of x contains at least one element of γ. Put $\pi_\chi(x, X) = \min\{|\gamma|\colon \gamma \text{ is a } \pi\text{-base at } x\} + \omega$. Then the *$\pi$-character* of X is $\pi_\chi(X) = \sup\{\pi_\chi(x, X)\colon x \in X\}$.

[2] A continuous mapping f of X onto Y is perfect if f is closed mapping such that $f^{-1}(y)$ is compact in X for each $y \in Y$.

Clearly, every k-perfect space[3] is of subcountable type. In [5], A.V. Arhangel'skiĭ proved that every non-locally compact topological group G with a Lindelöf k-perfect remainder Y in a compactification bG is separable metrizable, and Y is separable and first-countable, while following from Lemma 2.1 we have Theorem 2.2, which gives a partial answer to Question 1.1.

Lemma 2.1. *([5]) Suppose that G is a non-locally compact topological group with a remainder Y in some compactification bG of G satisfying the following conditions*:

(1) *Every closed countably compact subspace of Y is locally metrizable*; *and*
(2) *Y is of subcountable type.*

Then G is separable and metrizable.

Theorem 2.2. *Let G be a non-locally compact topological group. If $Y = bG \setminus G$ satisfies the condition* (1) *in Question 1.1, i.e., for each point $y \in Y$, there exists an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is a metrizable G_δ-subset of $U(y)$, then G is separable metrizable and Y is a first-countable, Lindelöf p-space.*

Proof. Clearly, Y is locally k-perfect, so one can easily obtain that Y is k-perfect. Thus Y is of subcountable type, and by Lemma 2.1 G is separable and metrizable. Hence, by Remark 1.6 Y is a Lindelöf p-space, and Y is first-countable. □

Next, we discuss the separable metrizability of bG for a topological group G. Let Φ be a topological property satisfying (L) as follows:

(L_1) every Lindelöf p-space X with property-Φ is metrizable;
(L_2) every countably compact subset of X with property-Φ is a metrizable G_δ-subset in X;
(L_3) property-Φ is hereditary with respect to closed subspaces.

Since every countably compact metrizable space is compact, and every compact space is a Lindelöf p-space, the condition (L_2) in (L) above can be replaced by the following condition (L_2'):

(L_2') every countably compact subset of X with property-Φ is a compact G_δ-subset in X.

Theorem 2.3. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ have locally a property-Φ satisfying (L). Then bG is separable metrizable.*

Proof. Since Y has locally a property-Φ satisfying (L), Y satisfies the condition (1) in Question 1.1, i.e., for each point $y \in Y$, there exists an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is a metrizable G_δ-subset of $U(y)$. Thus G is separable metrizable and Y is a Lindelöf p-space by Theorem 2.2.

Let $y \in Y$. There exists an open neighborhood $U(y)$ of y such that $\overline{U(y)}^Y$ has the property-Φ by locally a property-Φ satisfying (L).[4] $\overline{U(y)}^Y$ is a Lindelöf p-space as a closed subset of Y, thus $\overline{U(y)}^Y$ is separable and metrizable by (L_1), which implies that Y is locally separable and locally metrizable.

Since Y is Lindelöf and locally metrizable, Y has a countable network.[5] Clearly, G has a countable network as well, so $bG = G \cup Y$ has a countable network. Therefore, bG is separable metrizable by bG being compact. □

Since G is a subspace of bG, it is obvious that G is also separable metrizable in Theorem 2.3. From Theorem 2.3 we can obtain many results on remainders of topological groups. First, we recall some definitions.

A collection $\mathscr{A}$ of subsets of a space X is a *p-meta-base* [15] for X if for distinct points $x, y \in X$, there exists a finite subfamily $\mathscr{F} \subset \mathscr{A}$ such that $x \in (\bigcup \mathscr{F})^\circ \subset \bigcup \mathscr{F} \subset X \setminus \{y\}$.

A space X has a *quasi-G_δ-diagonal* [13] if there exists a sequence $\{\mathscr{G}_n\}_{n\in\omega}$ of families of open subsets of X such that $\bigcap_{n\in\omega}\{\mathrm{st}(x, \mathscr{G}_n)\colon \mathrm{st}(x, \mathscr{G}_n) \neq \emptyset\} = \{x\}$ for each $x \in X$.

Recall that a collection $\mathscr{B} = \bigcup_{n\in\omega} \mathscr{B}_n$ of open subsets of a space X is a *$\delta\theta$-base* [11] if whenever $x \in U$ with U open, there exist an $n \in \omega$ and a $B \in \mathscr{B}_n$ such that (i) $x \in B \subset U$; (ii) $\mathrm{ord}(x; \mathscr{B}_n) \leqslant \omega$.

Let X be a topological space. X is called a *c-semistratifiable space* (abbr. CSS) [18] if for each compact subset K of X and each $n \in \omega$ there is an open set $G(n, K)$ in X such that: (i) $\bigcap_{n\in\omega} G(n, K) = K$; (ii) $G(n + 1, K) \subset G(n, K)$ for each $n \in \omega$; (iii) if for any compact subsets K, L of X with $K \subset L$, then $G(n, K) \subset G(n, L)$ for each $n \in \omega$.

The results (1) and (4) of Corollary 2.4 improve [14, Theorems 2.5 and 2.13], respectively.

---

[3] A space X is *k-perfect* [5] if every compact subset of X is a G_δ-set.
[4] The closure of a subset A in a space Y is denoted by $\overline{A}^Y$.
[5] A family $\mathscr{P}$ of subsets of a space Y is a *network* of Y if $y \in U$ with U open in Y there exists $P \in \mathscr{P}$ such that $y \in P \subset U$.

Corollary 2.4. *Let G be a non-locally compact topological group. Then bG is separable metrizable if the remainder $bG \setminus G$ of G satisfies one of the following conditions.*

(1) *locally a point-countable p-meta-base*;
(2) *locally a $\delta\theta$-base* [14];
(3) *locally a quasi-G_δ-diagonal* [14];
(4) *locally a CSS-space.*

Proof. Suppose that the conditions $(1')$–$(4')$ are defined by dropping the condition "locally" in (1)–(4), respectively. It is obvious that the topological properties in $(1')$–$(4')$ are hereditary with respect to closed subspaces. Thus, according to Theorem 2.3, it is enough to show the following two facts.

Fact 1. Every Lindelöf p-space satisfying one of the conditions $(1')$–$(4')$ is metrizable.

Fact 2. Every countably compact subset in a space X satisfying one of the conditions $(1')$–$(4')$ is a metrizable G_δ-subset in X.

Indeed, Fact 1 was proved in [15, Theorem 3.1.8], [11, Corollary 8.3], [13, Corollary 3.6] and [8, Proposition 3.8], respectively. Fact 2 was proved in [14, Lemma 2.3], [8, Proposition 2.1], [8, Proposition 2.3] and [8, Proposition 3.8], respectively. □

The following corollary gives partial answers to Questions 1.2 and 1.3, respectively.

Lemma 2.5. *Let X have a σ-locally countable network. Then each compact subset of X is a metrizable G_δ-set.*

Proof. Let $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$ be a network for X, where each $\mathcal{P}_n$ is locally countable in X. Since X is regular, we may assume that each element of $\mathcal{P}$ is closed. Take an arbitrary compact subset F of X. It is clear that F is metrizable, since F has a countable network. Fix $n \in \omega$, for each $x \in X$, let $V_n(x)$ be an open neighborhood of x such that $V_n(x)$ meets at most countably many elements of $\mathcal{P}_n$. Put $\gamma_n = \{V_n(x)\colon x \in X\}$. One can find a finite subcover $\gamma'_n \subset \gamma_n$ of F, and let $V_n = \bigcup \gamma'_n$. Put $\{P \in \mathcal{P}_n\colon P \cap V_n \neq \emptyset, P \cap F = \emptyset\} = \{P_n(i)\colon i \in \omega\}$.

Claim. $F = \bigcap_{n,i\in\omega} (X \setminus P_n(i)) \cap V_n$.

Let $Q = \bigcap_{n,i\in\omega} (X \setminus P_n(i)) \cap V_n$. Clearly, $F \subset Q$. Suppose $x \in Q \setminus F$. Since $\mathcal{P}$ is a network and F is compact, there exists a $P \in \mathcal{P}_n$ for some $n \in \omega$ such that $x \in P \subset X \setminus F$. Then $P = P_n(i)$ for some $i \in \omega$. $x \notin X \setminus P_n(i)$, hence $x \notin Q$. This is a contradiction. Therefore, F is a G_δ-set in X. □

Lemma 2.6. *([8]) Every compact subset of a space with a BCO*[6] *is a metrizable G_δ-subset.*

Corollary 2.7. *Let G be a non-locally compact topological group and $Y = bG \setminus G$. Then bG is separable metrizable if and only if every countably compact subset of Y is compact, and Y satisfies one of the following conditions.*

(1) *locally a σ-locally countable network*;
(2) *locally a BCO.*

Proof. If bG is separable metrizable, then Y is metrizable, and it is obvious that every countably compact subset of Y is compact.

Conversely, suppose every countably compact subset of Y is compact. And the conditions $(1')$ and $(2')$ are denoted by dropping the condition "locally" in (1) and (2), respectively. It is obvious that the topological properties in $(1')$ and $(2')$ are hereditary with respect to closed subspaces. Thus, according to Theorem 2.3, it is enough to show the following two facts.

Fact 1. Every Lindelöf p-space satisfying one of the conditions $(1')$ and $(2')$ is metrizable.

Fact 2. Every compact subset in a space satisfying one of the conditions $(1')$ and $(2')$ is a metrizable G_δ-subset.

[6] A space X is said to have a *base of countable order* (abbr. BCO) if there is a sequence $\{\mathcal{B}_n\}$ of bases for X such that whenever $x \in B_n \in \mathcal{B}_n$ and $\{B_n\}$ is decreasing (by set inclusion), then $\{B_n\colon n \in \omega\}$ is a base at x in X.

Indeed, Fact 2 is obtained by Lemmas 2.5 and 2.6. As for Fact 1, if X is a Lindelöf p-space with a σ-locally countable network, then X has a countable network, thus X is metrizable by [11, Corollaries 3.20, 4.7]. On the other hand, every Lindelöf p-space with a BCO is metrizable by [11, Theorems 1.2 and 6.6]. □

Finally, we consider Question 1.4. Let X be a set and all non-negative real numbers be denoted by $\mathbb{R}^+$. A function $d : X \times X \to \mathbb{R}^+$ is a *symmetric* on the set X if, for each $x, y \in X$, (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$. A space X is said to be *symmetrizable* if there is a symmetric d on X satisfying the following condition: $U \subset X$ is open if and only if for each $x \in U$ there exists $\varepsilon > 0$ with $B(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\} \subset U$.

Lemma 2.8. *([4]) Suppose that G is a topological group with a remainder Y of countable pseudocharacter. Then Y is bisequential, and either G is metrizable and $|Y| \leqslant 2^\omega$, or Y is countably compact and first-countable.*

Lemma 2.9. *([16]) Let G be a non-locally compact topological group, and $Y = bG \setminus G$ be locally symmetrizable. Then bG is separable and metrizable if π-character of Y is countable.*

Theorem 2.10. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ be locally symmetrizable. Then bG is separable and metrizable if each singleton of Y is a G_δ-subset in Y.*

Proof. By Lemma 2.8, Y is bisequential. Hence, Y is Fréchet-Urysohn. Since Y is locally symmetrizable, it follows that Y is first-countable. Therefore, by Lemma 2.9, bG is separable and metrizable. □

Corollary 2.11. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ be locally symmetrizable. Then bG is separable and metrizable if Y satisfies one of the following conditions.*

(1) *Y is locally perfect;*
(2) *Y is locally Lindelöf;*
(3) *Y is locally ω_1-compact.*

Lemma 2.12. *([6]) Let G be a paratopological group. Then G is of countable type if and only if there exists a non-empty compact subset F of G such that F has a countable base of open neighborhoods in G.*

Lemma 2.13. *([11, Lemma 9.12]) Every ω_1-compact[7] symmetric space is hereditary Lindelöf.*

Corollary 2.14. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ be locally symmetrizable. Then bG is separable and metrizable if there exists an isolated point in Y.*

Proof. Let y be an isolated point in Y. Then there exists a G_δ-subset P in bG such that $\{y\} = P \cap Y$ and $P \cap G \neq \emptyset$. Take a sequence $\{U_n\}$ of open subsets in bG with $P = \bigcap_{n\in\omega} U_n$. Fix a point $g \in P \setminus \{y\}$. There is an open subset V_n in bG such that $y \notin \overline{V_n}^{bG}$, and $g \in V_{n+1} \subset \overline{V_{n+1}}^{bG} \subset V_n \cap U_{n+1}$ for each $n \in \omega$. Put $F = \bigcap_{n\in\omega} V_n$. Clearly, F is a non-empty closed G_δ-subset in bG with $F \subset G$. One can easily obtain that F has a countable base of open neighborhoods in bG by the compactness of bG. Therefore F has a countable base of open neighborhoods in G as well. It is obvious that F is a compact subset of G. Then G is of countable type by Lemma 2.12.

By Henriksen and Isbell's theorem, Y is Lindelöf, thus Y is ω_1-compact. Then Y is locally a hereditarily Lindelöf by Lemma 2.13. Thus the statement directly follows from Corollary 2.11 □

3. Remainders of paratopological groups

In this section, we consider the remainders of paratopological groups and improve some results of A.V. Arhangel'skiĭ in [1]. Recall that a space X is *Ohio complete* [1] if in each compactification bX of X there is a G_δ-subset Z such that $X \subset Z$ and each point $y \in Z \setminus X$ is separated from X by a G_δ-subset of Z. A space X is a *p-space* if there exists a sequence $\{\mathscr{U}_n\}$ of families of open subsets of the Stone–Čech compactification βX such that

(1) each $\mathscr{U}_n$ covers X for each $n \in \omega$;
(2) $\bigcap_{n\in\omega} \mathrm{st}(x, \mathscr{U}_n) \subset X$ for each $x \in X$, where $\mathrm{st}(x, \mathscr{U}_n) = \bigcup\{U \in \mathscr{U}_n\colon x \in U\}$.

A space X has a *G_δ-diagonal* if the set $\Delta = \{(x, x) \in X \times X\colon x \in X\}$ is a G_δ-subset in $X \times X$.

[7] A space X is *ω_1-compact* if every closed discrete subset of X has cardinality less than ω_1.

Lemma 3.1. *([1]) Every Lindelöf space, every space with a G_δ-diagonal or every p-space is Ohio complete.*

A.V. Arhangel'skiĭ [6] characterized the spaces such that the remainders of paratopological groups are Ohio complete as follows.

Lemma 3.2. *([6]) Let G be a paratopological group and $Y = bG \setminus G$. Then Y is Ohio complete if and only if G is σ-compact or a space of countable type.*

The Lindelöf remainder was characterized by Henriksen and Isbell's theorem. We have other equivalent conditions for paratopological groups as follows.

Theorem 3.3. *Let G be a non-locally compact paratopological group and $Y = bG \setminus G$. Then the following are equivalent*[8]:

(1) *Y is σ-metacompact and Ohio complete*;
(2) *Y is metacompact and Ohio complete*;
(3) *Y is paracompact and Ohio complete*;
(4) *Y is Lindelöf*;
(5) *G is of countable type.*

Proof. (5) $\Rightarrow$ (4) by Henriksen and Isbell's theorem. (4) $\Rightarrow$ (3) by Lemma 3.1. It is obvious that (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5). Suppose that Y is σ-metacompact and Ohio complete. We may assume that G is σ-compact by Lemma 3.2. Thus the Souslin number $c(G)$ of G is countable [7, Corollary 5.7.12] and Y is Čech-complete.[9] Since G is dense in bG, the Souslin number $c(bG)$ of bG is countable. Since G is a non-locally compact paratopological group, Y is dense in bG. Then the Souslin number $c(Y)$ of Y is countable. In addition, every point-finite open collection in a Čech-complete space X with $c(X) \leqslant \omega$ is countable [9, Proposition 8.3]. Thus, one can obtain that Y is Lindelöf by Y being σ-metacompact. Therefore from Henriksen and Isbell's theorem it follows that G is of countable type. □

Remark 3.4. Why do we consider the non-local compactness for G? If G is locally compact, then G is open in bG. Thus $Y = bG \setminus G$ is compact. Thus, "non-locally compact" in Theorem 3.3 can be dropped.

The following corollary is obtained by Remark 1.6, Lemma 3.1 and Theorem 3.3, which improves a result of A.V. Arhangel'skiĭ in [1, Theorem 4.9] as follows: if G is a non-locally compact topological group, then the remainder $bG \setminus G$ is a paracompact p-space if and only if G is a Lindelöf p-space.

Corollary 3.5. *Let G be a non-locally compact paratopological group and $Y = bG \setminus G$. Then G is a Lindelöf p-space if and only if Y is a Lindelöf (or σ-metacompact) p-space.*

The following corollary is proved by A.V. Arhangel'skiĭ in [1, Theorem 4.8] for topological groups.

Corollary 3.6. *Suppose that G is a paratopological group such that the remainder $bG \setminus G$ is a paracompact p-space. Then G is also a paracompact p-space.*

Proof. If G is locally compact, then G is a locally compact topological group [7, Theorem 2.3.12], which implies that G is a paracompact p-space [7, Theorem 4.3.35]. If G is non-locally compact, G is a paracompact p-space by Corollary 3.5. □

Corollary 3.7. *Let G be a non-locally compact paratopological group. Then G is separable metrizable if and only if $bG \setminus G$ is a σ-metacompact p-space and G has a G_δ-diagonal.*

Proof. If G is separable metrizable, it is obvious that G has a G_δ-diagonal, and $bG \setminus G$ is a σ-metacompact p-space by Corollary 3.5. Conversely, if $bG \setminus G$ is a σ-metacompact p-space and G has a G_δ-diagonal, G is a Lindelöf p-space by Corollary 3.5. Since every Lindelöf p-space with a G_δ-diagonal is metrizable by [11, Corollaries 3.8 and 3.20], G is separable metrizable. □

The following theorem is proved by A.V. Arhangel'skiĭ in [1, Theorem 4.5] for topological groups.

[8] A space X is *metacompact* (resp. *σ-metacompact*) [9] if every open cover of X has a point-finite (resp. σ-point-finite) open refinement.

[9] A completely regular space X is *Čech-complete* if X is a G_δ-subset in some compactification of X.

Theorem 3.8. *Suppose that G is a non-locally compact paratopological group. Then the remainder $Y = bG \setminus G$ is a p-space if and only if G is σ-compact or a Lindelöf p-space.*

Proof. Suppose Y is a p-space. By Lemmas 3.1 and 3.2, G is of countable type or σ-compact. If G is of countable type, Y is Lindelöf by Henriksen and Isbell's theorem, then G is a Lindelöf p-space by Remark 1.6.

Conversely, if G is σ-compact, then Y is Čech-complete and therefore, is a p-space. If G is a Lindelöf p-space, then Y is a Lindelöf p-space by Lemma 1.5. □

Remark 3.9. "Non-locally compact" in Corollary 3.7 cannot be dropped. To see this, we can take any discrete topological group G with uncountable cardinality. Let bG be the one-point compactification of G. It is clear that $bG \setminus G$ is a σ-metacompact p-space, but G is not separable. This example also shows that "non-locally compact" in Theorem 3.8 cannot be dropped and the topological group G in [1, Corollary 4.11] must be non-locally compact, which corrects a little mistake in [1].

Lemma 3.10. *([2]) If a paratopological group G has a countable π-base at some point in G, then G has a G_δ-diagonal.*

Theorem 3.11. *Let G be a non-locally compact paratopological group. Then G is separable and metrizable if the remainder $bG \setminus G$ is a σ-metacompact p-space with countable π-character.*

Proof. Since G is a non-locally compact paratopological group, $Y = bG \setminus G$ is not compact, thus Y is not countably compact. There exists an infinite countable set $A = \{a_n\colon n \in \omega\}$ such that A is closed discrete in Y. Since bG is compact, then A has an accumulation point c in G. For each $n \in \omega$, choose $\{V(n,k)\colon k \in \omega\}$ as a countable local π-base at a_n in Y. For each $n, k \in \omega$, take an open subset $U(n,k)$ in bG with $V(n,k) = U(n,k) \cap Y$, and let $W(n,k) = U(n,k) \cap G$. It is not difficult to see that $\{W(n,k)\colon n, k \in \omega\}$ is a countable π-base at c in G, then G has a G_δ-diagonal by Lemma 3.10. Then G is separable and metrizable by Corollary 3.7. □

Corollary 3.12. *([17]) Let G be a non-locally compact paratopological group. Then bG is separable and metrizable if the remainder $Y = bG \setminus G$ is metrizable.*

Proof. G is separable and metrizable by Theorem 3.11. Therefore Y is separable and metrizable by Henriksen and Isbell's theorem. Thus, the compact space $bG = G \cup Y$ is also separable and metrizable. □

In the final, we consider that the remainder of a paratopological group has a G_δ-diagonal.

Theorem 3.13. *Let G be a paratopological group and the remainder $bG \setminus G$ have a G_δ-diagonal. Then G either is of countable type or is a σ-compact space with a G_δ-diagonal.*

Proof. If G is locally compact, then it is obvious that G is of countable type. Thus we assume that G is non-locally compact.

Since the remainder $Y = bG \setminus G$ has a G_δ-diagonal, Y is Ohio complete by Lemma 3.1. Thus G either is of countable type or is σ-compact by Lemma 3.2. We also assume that G is not of countable type. Then it is enough to show that G has a G_δ-diagonal.

Claim 1. Y is first-countable.

If Y is not first-countable, there exists a point $y_0 \in Y$ such that $\{y_0\}$ is not a G_δ-subset of bG by the compactness of bG. Thus one can easily take a sequence $\{U_n\}$ of open subsets of bG such that $\{y_0\} = Y \cap \bigcap_{n\in\omega} U_n$ and $G \cap \bigcap_{n\in\omega} U_n \neq \emptyset$. Take a point $x_0 \in G \cap \bigcap_{n\in\omega} U_n$. There exists a decreasing sequence $\{V_n\}$ of open subsets of bG such that $\overline{V_{n+1}}^{bG} \subset V_n \subset U_n$ for each $n \in \omega$ and $y_0 \notin V_0$. Put $F = \bigcap_{n\in\omega} V_n$. Then F is compact and has a countable neighborhood base in G. Thus G is of countable type by Lemma 2.12, which is a contradiction.

Claim 2. Y is not countably compact.

Indeed, since every countably compact space with a G_δ-diagonal is metrizable [11], if Y is countably compact, then Y is compact. It is a contradiction with G being non-locally compact.

From Claims 1 and 2 it follows that G has a countable π-character by the same way of the proof of Theorem 3.11. Thus G has a G_δ-diagonal by Lemma 3.10. □

Remark 3.14. The converse of Theorem 3.13 does not hold. Let Z be the set of all integers endowed with the discrete topology. The topological group Z is of countable type and is a σ-compact space with a G_δ-diagonal, but the remainder $\beta Z \setminus Z$ has no a G_δ-diagonal.

Question 3.15. Find a sufficient and necessary condition for a paratopological group G such that the remainder $bG \setminus G$ has a G_δ-diagonal.

Acknowledgements

We wish to thank the referee for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper. Particularly, the referee simplifies the proofs of Theorems 2.2 and 2.10, and points out that Question 1.2 has been posed by A.V. Arhangel'skiĭ at some conference and at a seminar as well, Question 1.4 has not been answered even for symmetrizable remainders, which has been posed by A.V. Arhangel'skiĭ at a conference and at a seminar.

References

[1] A.V. Arhangel'skiĭ, Remainders in compactification and generalize metrizability properties, Topology Appl. 150 (2005) 79–90.
[2] A.V. Arhangel'skiĭ, Paratopological and semitopological groups versus topological groups, Topology Appl. 151 (2005) 107–119.
[3] A.V. Arhangel'skiĭ, More on remainders close to metrizable spaces, Topology Appl. 154 (2007) 1084–1088.
[4] A.V. Arhangel'skiĭ, G_δ-points in remainders of topological groups and some addition theorems in compacta, Topology Appl. 156 (2009) 2013–2018.
[5] A.V. Arhangel'skiĭ, A study of remainders of topological groups, Fund. Math. 203 (2009) 165–178.
[6] A.V. Arhangel'skiĭ, M.M. Choban, On remainders of rectifiable spaces, Topology Appl. 157 (2010) 789–799.
[7] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, World Sci., 2008.
[8] H. Bennett, R. Byerly, D. Lutzer, Compact G_δ-sets, Topology Appl. 153 (2006) 2169–2181.
[9] D.K. Burke, Covering properties, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, pp. 347–416.
[10] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[11] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, pp. 423–501.
[12] M. Henriksen, J. Isbell, Some properties of compactifications, Duke Math. J. 25 (1958) 83–106.
[13] R.E. Hodel, Metrizability of topological spaces, Pacific J. Math. 55 (1974) 441–459.
[14] F. Lin, Locall properties on the remainders of the topological groups, Kodai Math. J. 34 (2011) 505–518.
[15] S. Lin, Generalized Metrizable Spaces and Mappings, 2nd ed., China Science Press, Beijing, 2007 (in Chinese).
[16] C. Liu, Remainders in compactification of topological groups, Topology Appl. 156 (2009) 849–854.
[17] C. Liu, Metrizability of paratopological (semitopological) groups, Topology Appl. 159 (2012) 1415–1420.
[18] H. Martin, Metrizability of M-spaces, Canad. J. Math. 4 (1973) 840–841.