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Spaces with σ-point-discrete $\aleph_0$-weak bases

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Abstract. It is discussed in this paper the spaces with σ-point-discrete $\aleph_0$-weak bases. The main results are: (1) A space X has a σ-compact-finite $\aleph_0$-weak base if and only if X is a k-space with a σ-point-discrete $\aleph_0$-weak base; (2) Under (CH), every separable space with a σ-point-discrete $\aleph_0$-weak base has a countable $\aleph_0$-weak base.

§1 Introduction

In [34], Sirois-Dumais introduced the weakly quasi-first-countable spaces, which are natural generalizations of the well-known weakly first-countable spaces. Liu and Lin [23] introduced the notion of $\aleph_0$-weak bases, which revealed the elementary character of weakly quasi-first-countable spaces. It has been founded from the recent study that the notion of $\aleph_0$-weak bases plays an interesting role in the theory of generalized metric spaces and topological groups [23, 24, 30-33]. In [30], Shen gave a systemical discussion on the spaces with certain $\aleph_0$-weak bases, and revealed the relation between these spaces and the quotient, countable-to-one images of metric spaces. It has been proved in [30] that a regular space X has a σ-discrete $\aleph_0$-weak base if and only if it has a σ-locally finite $\aleph_0$-weak base. Also in [24], Liu, Lin and Li proved that a regular space X has a σ-locally finite $\aleph_0$-weak base if and only if it has a σ-hereditarily closure-preserving $\aleph_0$-weak base.

A family $\mathcal{B} = \{B_\alpha : \alpha \in H\}$ of subsets of a space X is called *hereditarily closure-preserving* [13]) if $\overline{\cup\{A_\alpha : \alpha \in H\}} = \cup\{\overline{A_\alpha} : \alpha \in H\}$ whenever $A_\alpha \subset B_\alpha$ for each $\alpha \in H$. $\mathcal{B}$ is called *point-discrete* (also called *weakly hereditarily closure-preserving* [4]) if $\{x_\alpha : \alpha \in H\}$ is closed discrete whenever $x_\alpha \in B_\alpha$ for each $\alpha \in H$. $\mathcal{B}$ is called *compact-finite* if every compact subset of X intersects at most finite members of $\mathcal{B}$. $\mathcal{B}$ is called *σ-point-discrete* (*σ-compact-finite*) if $\mathcal{B}$ is a countable union of point-discrete (compact-finite) families. It is easy to see that every locally finite family of subsets of a space is hereditarily closure-preserving and compact-finite, and every

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hereditarily closure-preserving family is point-discrete. Research on σ-point-discrete networks and σ-compact-finite networks is one of the important topics in the theory of generalized metric spaces. Burke, Engelking and Lutzer [4] discussed the spaces with σ-point-discrete bases. Boone [3] proved that every regular space with σ-compact-finite bases is metrizable. Liu and Tanaka [25], Lin and Yan [20] discussed the spaces with σ-point-discrete weak bases and the spaces with σ-compact-finite weak bases. Ge [9] characterized $\aleph_0$-spaces by σ-point-discrete strong *cs*-networks. Lin and Shen [15] gave a strict relationship between the spaces with σ-point-discrete *sn*-networks and the spaces with σ-compact-finite *sn*-networks. These works lead us to study the spaces with σ-point-discrete $\aleph_0$-weak bases. In this direction, we are interested in the following question:

Question 1.1. Does every k-space with a σ-point-discrete $\aleph_0$-weak base have a σ-compact-finite $\aleph_0$-weak base?

In Section 2, we shall give an affirmative answer to this question.

In Section 3, we discuss the separable spaces with σ-point-discrete $\aleph_0$-weak bases. We shall prove that under (CH), every separable space with a σ-point-discrete $\aleph_0$-weak base has a countable $\aleph_0$-weak base. As an application, each closed map on a space with a σ-point-discrete $\aleph_0$-weak base is compact-covering under (CH). It will be also pointed out that the assumption (CH) can be replaced by either of the following conditions: (1) X is $\aleph_1$-compact; (2) The sequential order of X is countable.

In this paper all spaces are regular. By $\mathbb{N}$ and ω_1, we denote the set of all natural numbers and the first uncountable ordinal, respectively. For a space X, $I(X)$ is the set of all isolated points of X. For a family $\mathcal{P}$ of subsets of X, $\cap\mathcal{P}$ and $\cup\mathcal{P}$ are respectively the intersection and union of all members of $\mathcal{P}$. $\mathcal{P}^{<\omega} = \{\cap\mathcal{P}' : \mathcal{P}'$ is a finite subfamily of $\mathcal{P}\}$. We recall some basic definitions.

Definition 1.1. [23] Let $\mathcal{B}$ be a family of subsets of a space X. $\mathcal{B}$ is said to be an $\aleph_0$-*weak base* for X if $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in \mathbb{N}\}$ satisfies

(1) For each $x \in X, n \in \mathbb{N}$, $\mathcal{B}_x(n)$ is closed under finite intersections and $x \in \cap\mathcal{B}_x(n)$;

(2) A subset U of X is open if and only if whenever $x \in U$ and $n \in \mathbb{N}$, there exists a $B_x(n) \in \mathcal{B}_x(n)$ such that $B_x(n) \subset U$.

X is called $\aleph_0$-*weakly first-countable* [36] or *weakly quasi-first-countable* in the sense of Sirois-Dumais [34] if $\mathcal{B}_x(n)$ is countable for each $x \in X, n \in \mathbb{N}$.

If $\mathcal{B}_x(n) = \mathcal{B}_x(1)$ for each $n \in \mathbb{N}$ in the definition of $\aleph_0$-weak bases, then $\mathcal{B}$ is called to be a *weak base* [2] for X. X is called *weakly first-countable* or *g-first-countable* in the sense of Arhangel'skiĭ [2] if $B_x(1)$ is countable for each $x \in X$.

Let X be a space. $P \subset X$ is called a *sequential neighborhood* [6] of x in X, if each sequence converging to $x \in X$ is eventually in P. A subset U of X is called *sequentially open* [6] if U is a sequential neighborhood of each of its points. X is called a *sequential space* [6] if each sequentially open subset of X is open. X is called a *k-space* [6] if every subset A of X is open whenever $A \cap K$ is open in K for each compact subset $K \subset X$. Note that every $\aleph_0$-weakly first-countable space is a sequential space [34], and every sequential space is a k-space [5].

Definition 1.2. Let $\mathcal{P}$ be a cover of a space X. Then

(1) $\mathcal{P}$ is called a *network* [1] for X if for any open set U and a point $x \in U$, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$;

(2) $\mathcal{P}$ is called a *k-network* [10] for X if for any compact set K and for any open set U such that $K \subset U$, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$;

(3) $\mathcal{P}$ is called a *cs-network* [12] for X if for any open set U and any sequence L converging to a point $x \in U$, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$ and $L - P$ is finite;

(4) $\mathcal{P}$ is called a *cs^*-network* [7] for X if for any open set U and any sequence L converging to a point $x \in U$, there exists a subsequence L' of L and a $P \in \mathcal{P}$ such that $L' \cup \{x\} \subset P \subset U$;

(5) $\mathcal{P}$ is called a *wcs^*-network* [19] if for any open set U and any sequence L converging to a point $x \in U$, there exists a subsequence L' of L and a $P \in \mathcal{P}$ such that $L' \subset P \subset U$;

(6) $\mathcal{P}$ is called an *sn-network* [8,17] for X if for any open set U and a point $x \in U$, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$ and P is a sequential neighborhood of x.

These notions have the following implications.

Remark 1.1. (1) weak bases $\rightarrow$ *sn*-networks $\rightarrow$ *cs*-networks $\rightarrow$ cs^*-networks $\rightarrow$ wcs^*-networks $\rightarrow$ networks;

(2) weak bases $\rightarrow$ $\aleph_0$-weak bases $\rightarrow$ cs^*-networks [30];

(3) k-networks $\rightarrow$ wcs^*-networks.

X is called an *$\aleph$-space* [28] if it has a σ-locally finite k-network. X is called an *$\aleph_0$-space* [27] if it has a countable k-network, which is equivalent to the spaces with a countable cs^*-network [7]. In [23], it is proved that a space X has a σ-locally finite $\aleph_0$-weak base (countable $\aleph_0$-weak base) if and only if it is an $\aleph_0$-weakly first-countable, $\aleph$-space ($\aleph_0$-space).

§2 Spaces with σ-compact-finite $\aleph_0$-weak bases and spaces with σ-point-discrete $\aleph_0$-weak bases

Lemma 2.1. *[30] Let X be a space. $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in \mathbb{N}\}$ is a family of subsets of X, here each $\mathcal{B}_x(n)$ is a network at x in X and $\mathcal{B}_x(n)$ is closed under finite intersections for each $x \in X, n \in \mathbb{N}$. Consider the following two conditions.*

(1) $\mathcal{B}$ is an $\aleph_0$-weak base for X.

(2) For any sequence L converging to x in X, there exist a subsequence L' of L and $n_0 \in \mathbb{N}$ such that L' is eventually in B for each $B \in \mathcal{B}_x(n_0)$.

We have (1)⇒(2). Moreover, if X is sequential, (2)⇒(1).

Lemma 2.2. *Let X be a sequential space with an $\aleph_0$-weak base $\mathcal{P} = \cup\{\mathcal{P}_x(n) : x \in X, n \in \mathbb{N}\}$. Then X has an $\aleph_0$-weak base $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in \mathbb{N}\}$ such that $\mathcal{B} \subset \mathcal{P}$, and for each $x \in X - I(X)$ and $n \in \mathbb{N}$, there is a non-trivial sequence L which converges to x and is eventually in each element of $\mathcal{B}_x(n)$.*

Proof. For each $x \in X$, if $x \in I(X)$, put $\mathcal{B}_x(n) = \mathcal{P}_x(n)$ for each $n \in \mathbb{N}$. If $x \in X - I(X)$, since X is a sequential space, there is a non-trivial sequence L_0 converging to x. By Lemma 2.1, there exist an $n_0 \in \mathbb{N}$ and a subsequence L_1 of L_0 such that L_1 is eventually in each element of $\mathcal{P}_x(n_0)$. For each $n \in \mathbb{N}$, if there is no non-trivial sequence L such that L converges to x and is eventually in each element of $\mathcal{P}_x(n)$, then we put $\mathcal{B}_x(n) = \mathcal{P}_x(n_0)$. Otherwise we put $\mathcal{B}_x(n) = \mathcal{P}_x(n)$. By Lemma 2.1, we can easily verify that $\mathcal{B} = \cup\{\mathcal{B}_x(n)\colon x \in X, n \in \mathbb{N}\} \subset \mathcal{P}$ is an $\aleph_0$-weak base for X which satisfies that for each $x \in X - I(X)$ and $n \in \mathbb{N}$, there is a non-trivial sequence L which converges to x and is eventually in each element of $\mathcal{B}_x(n)$. □

Lemma 2.3. *[20] Let $\mathcal{P}$ be a point-discrete family of a space X. Put $D = \{x \in X : \mathcal{P}$ is not point-finite at $x\}$. Then $\{P - D : P \in \mathcal{P}\} \cup \{\{x\} : x \in D\}$ is compact-finite.*

Theorem 2.1. *The following statements are equivalent for a space X.*

(1) X has a σ-compact-finite $\aleph_0$-weak base.

(2) X is an $\aleph_0$-weakly first-countable space with a σ-point-discrete $\aleph_0$-weak base.

(3) X is a k-space with a σ-point-discrete $\aleph_0$-weak base.

Proof. (1)⇒(2)⇒(3) are obvious. We now prove (3)⇒(1).

Let X be a k-space with a σ-point-discrete $\aleph_0$-weak base. First we prove that X is a sequential space. It is sufficient to show any compact subset of X is metrizable. By Lemma 2.3, X has a σ-compact-finite network. Thus any compact subset of X has a countable network. By [5,Theorem 3.1.19], any compact subset of X is metrizable.

Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\} = \cup\{\mathcal{P}_x(m) : x \in X, m \in \mathbb{N}\}$ be a σ-point-discrete $\aleph_0$-weak base for X, where each $\mathcal{P}_n$ is a point-discrete family and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. By Lemma 2.2, we can assume that $\mathcal{P}$ satisfies that for each $x \in X - I(X)$ and $m \in \mathbb{N}$, there is a non-trivial sequence $L_{x,m}$ which converges to x and is eventually in each element of $\mathcal{P}_x(m)$.

If $x \in I(X)$, then $\{x\}$ is open in X. Thus $\{x\} \in \mathcal{P}$. So $I(X)$ is a σ-closed discrete subspace of X. For $n, m \in \mathbb{N}$ and $P \in \mathcal{P}_n$, let

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\};$$
$$V_m(P) = \{x \in X - I(X) : P \in \mathcal{P}_x(m)\};$$
$$W_{n,m}(P) = (P - D_n) \cup V_m(P).$$

Then $W_{n,m}(P) \subset P$. Now we show $\{W_{n,m}(P) : P \in \mathcal{P}_n\}$ is compact-finite for each $n, m \in \mathbb{N}$. Since every point-finite and point-discrete family is compact-finite, it is sufficient to show $\{W_{n,m}(P) : P \in \mathcal{P}_n\}$ is point-finite. It is easy to see that $\{P - D_n : P \in \mathcal{P}_n\}$ is point-finite. So we only need to show $\{V_m(P) : P \in \mathcal{P}_n\}$ is point-finite. For $x \in X - I(X)$, if $\{P \in \mathcal{P}_n : x \in W_{n,m}(P)\}$ is infinite, then $\mathcal{P}_x(m) \cap \mathcal{P}_n$ is infinite. Pick $\{P_i : i \in \mathbb{N}\} \subset \mathcal{P}_x(m) \cap \mathcal{P}_n$. Since $L_{x,m}$ is eventually in each element of $\mathcal{P}_x(m)$, we can choose a subsequence $\{x_i\}_{i \in \mathbb{N}}$ of $L_{x,m}$ such that $x_i \in P_i$ for each $i \in \mathbb{N}$. This contradicts that $\mathcal{P}_n$ is point-discrete. Therefore $\{W_{n,m}(P) : P \in \mathcal{P}_n\}$ is compact-finite.

For each $x \in X$ and $m \in \mathbb{N}$, let

$$\mathcal{B}'_x(m) = \begin{cases} \{\{x\}\}, & x \in I(X), \\ \{W_{n,m}(P) : P \in \mathcal{P}_x(m) \cap \mathcal{P}_n, n \in \mathbb{N}\}, & x \in X - I(X) \end{cases}$$

and $\mathcal{B}_x(m) = \mathcal{B}'_x(m)^{<\omega}$. Then $\mathcal{B} = \cup\{\mathcal{B}_x(m) : x \in X, m \in \mathbb{N}\}$ is σ-compact-finite. To complete the proof, it is sufficient to show $\mathcal{B}$ is an $\aleph_0$-weak base for X.

To begin with, for each $x \in X$ and $m \in \mathbb{N}$, $\mathcal{B}_x(m)$ is a network at x. In fact, let U be an open neighborhood of x, there exists a $P \in \mathcal{P}_x(m) \cap \mathcal{P}_n$ for some $n \in \mathbb{N}$ with $P \subset U$. Then $x \in W_{n,m}(P) \subset P \subset U$. In addition, let L be a non-trivial sequence converging to $x \in X$. By Lemma 2.1, there exists an $m \in \mathbb{N}$ and a subsequence L' such that L' is eventually in each element of $\mathcal{P}_x(m)$. By Lemma 2.3, $(L' \cup \{x\}) \cap D_n$ is finite for each $n \in \mathbb{N}$. By Lemma 2.1, L' is eventually in each element of $\mathcal{B}_x(m)$. Therefore $\mathcal{B}$ is a σ-compact-finite $\aleph_0$-weak base for X. □

Remark 2.1. In [15], Lin and Shen proved that every space with a σ-point-discrete sn-network has a σ-compact-finite sn-network. However, this is not true for $\aleph_0$-weak bases. Indeed, Burke, Engelking and Lutzer [4] gave a space with a σ-point-discrete base which is not a k-space.

It is well-known that a space X has a σ-compact-finite weak base if and only if X is a k-space with a σ-compact-finite sn-network. The following two examples show that a space with a σ-compact-finite sn-network (even with a compact-finite sn-network) may not have a σ-point-discrete cs^*-network.

Example 2.1. There exists a space which has a σ-compact-finite sn-network, but dose not have any σ-point-discrete network.

Proof. Let X be an uncountable set and p be a fixed point in X. We endow X with the Fortissimo topology [18, 35]. That is, every point $x \in X - \{p\}$ is isolated and the neighborhood base at p is $\{U \subset X : p \in U \text{ and } X - U \text{ is countable}\}$. According to [18, Example 2.5.19], X satisfies the following two conditions.

(1) Every compact subset of X is finite.

(2) Every uncountable $A \subset X$ is not closed discrete.

By (1), there is no non-trivial convergent sequences in X. As a result, $\{\{x\} : x \in X\}$ is a compact-finite sn-network for X. Note that $\{\{x\} : x \in X\}$ is also a k-network for X by (1). Now we prove that X doesn't have any σ-point-discrete network. Suppose that X has a network $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, where $\mathcal{P}_n$ is point-discrete for each $n \in \mathbb{N}$. Since every $x \in X - \{p\}$ is isolated and $\mathcal{P}$ is a network for X, $\{x\} \in \mathcal{P}$ for every $x \in X - \{p\}$. So we can find an uncountable subset A of $X - \{p\}$ and an $n_0 \in \mathbb{N}$ such that $\{x\} \in \mathcal{P}_{n_0}$ for each $x \in A$. By (2), $\mathcal{P}_{n_0}$ can not be point-discrete. This is a contradiction. □

Example 2.2. There exists a space which has a compact-finite sn-network, but dose not have any σ-point-discrete network.

Proof. Let X be the infinite, completely regular and countably compact space in [11, Example 9.1] in which every compact subset is finite. Since every compact subset of X is finite, $\{\{x\} : x \in X\}$ is a compact-finite sn-network for X. It is easy to prove that a countably compact space with a σ-point-discrete network has a countable network. If X has a σ-point-discrete network, then X is metrizable, hence it is discrete. This is a contradiction. □

Corollary 2.1. *[24] Every strongly Fréchet-Urysohn space with a σ-point-discrete $\aleph_0$-weak base is metrizable.*

Proof. Let X be a strongly Fréchet-Urysohn space with a σ-point-discrete $\aleph_0$-weak base. By Theorem 2.4, X has a σ-compact-finite $\aleph_0$-weak base. Then X is $\aleph_0$-weakly first-countable. By [29, Lemma 2.14], X is first-countable. Since any compact-finite family of subsets of a first-countable space is locally finite, X has a σ-locally finite $\aleph_0$-weak base. So X is an $\aleph$-space [30, Theorem 2.4]. Therefore X is metrizable. □

The following questions remain open.

Question 2.1. Does every space with a σ-compact-finite $\aleph_0$-weak base have a σ-locally finite $\aleph_0$-weak base?

Note that this question is closely related to Liu's question [18, 21]: whether every space with a σ-compact-finite weak base is g-metrizable? Note that if the answer to Question 2.1 is affirmative, then the same to Liu's question.

Question 2.2. Does every weakly first-countable space with a σ-compact-finite $\aleph_0$-weak base have a σ-compact-finite weak base?

Question 2.3. Does every weakly first-countable (weakly quasi-first-countable) space with a σ-point-discrete cs^*-network have a σ-compact-finite weak base ($\aleph_0$-weak base)?

§3 Separable spaces with σ-point-discrete $\aleph_0$-weak bases

Lemma 3.1. *Suppose that a space X has a σ-point-discrete wcs^*-network. Then X has a σ-point-discrete k-network and a σ-compact-finite k-network.*

Proof. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-point-discrete wcs^*-network for X, where each $\mathcal{P}_n$ is a point-discrete family and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Put $D_n = \{x \in X : \mathcal{P}_n$ is not point-finite at $x\}$, $\mathcal{P}'_n = \{P - D_n : P \in \mathcal{P}_n\} \cup \{\{x\} : x \in D_n\}$ and $\mathcal{P}' = \cup\{\mathcal{P}'_n : n \in \mathbb{N}\}$. By Lemma 2.3, $\mathcal{P}'$ is σ-compact-finite. Note that the intersection of D_n and any compact subset of X is finite, we can see that $\mathcal{P}'$ refines $\mathcal{P}$ and is a wcs^*-network, hence any compact subset of X is metrizable and $\mathcal{P}'$ is a σ-compact-finite k-network (see [37, Proposition B(1)]). We prove that $\mathcal{P}$ is a k-network. Let $K \subset U$ with K compact and U open in X, it is easy to see that $\{P \in \mathcal{P} : P \subset U\}$ is a σ-point-discrete wcs^*-network of the space U. Without loss of generality, we assume $\{P \in \mathcal{P} : P \subset U\} = \mathcal{P}$. Since $\mathcal{P}'$ is a k-network, there is a finite subfamily $\mathcal{F} \subset \mathcal{P}'$

such that $K \subset \cup\mathcal{F} \subset U$. For each $F \in \mathcal{F}$, pick $P(F) \in \{P \in \mathcal{P} : P \subset U\}$ such that $F \subset P(F)$, then $K \subset \cup\{P(F) : F \in \mathcal{F}\} \subset U$. □

Since every k-network for a space X is a wcs^*-network for X, we have the following corollaries.

Corollary 3.1. *A space X has a σ-point-discrete k-network if and only if X has a σ-point-discrete wcs^*-network.*

Corollary 3.2. *Suppose that a space X has a σ-point-discrete k-network. Then X has a σ-compact-finite k-network.*

Corollary 3.3. *Suppose that a space X has a σ-point-discrete wcs^*-network. Then X has a σ-compact-finite wcs^*-network.*

We remark here that F. Lin also prove the same result of Corollary 3.3.

Example 3.1. [14, Example 2.2] The fan space S_{ω_1} has a σ-point-discrete cs^*-network, and S_{ω_1} does not have any σ-compact-finite cs^*-network.

Since S_{ω_1} does not have any σ-point-discrete cs-network [14, Theorem 2.8], the following question remains open.

Question 3.1. Suppose that a space X have a σ-point-discrete cs-network. Then does X have a σ-compact-finite cs-network?

Now we discuss separable spaces with σ-point-discrete $\aleph_0$-weak bases. We must remark that the main technique used in the proof of Theorem 3.1 and Lemma 3.2 comes from [22].

Theorem 3.1. *Under (CH), every separable space with a σ-point-discrete $\aleph_0$-weak base has a countable $\aleph_0$-weak base.*

Proof. Let X be a separable space with a σ-point-discrete $\aleph_0$-weak base. By (CH), the character of X is not greater than ω_1. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\} = \cup\{\mathcal{P}_x(m) : x \in X, m \in \mathbb{N}\}$ be a σ-point-discrete $\aleph_0$-weak base for X, where each $\mathcal{P}_n$ is a point-discrete family and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Without loss of generality, we may assume that for each $m \in \mathbb{N}$, $\{x\} \notin \mathcal{P}_x(m)$ for each $x \in X - I(X)$ and $\mathcal{P}_x(m) = \{\{x\}\}$ for each $x \in I(X)$. Now suppose $\mathcal{P}_x(m) \cap \mathcal{P}_n$ is uncountable for some $x \in X - I(x)$ and $n, m \in \mathbb{N}$. Let $\{V_\alpha : \alpha < \omega_1\}$ be the local base at x. Notice that for any neighborhood V of x, $V \cap (P - \{x\}) \neq \emptyset$ for each $P \in \mathcal{P}_x(m)$. Then, by induction, there exist a subset $S = \{x_\alpha : \alpha < \omega_1\}$ of X and a subfamily $\{P_\alpha : \alpha < \omega_1\}$ of $\mathcal{P}_x(m) \cap \mathcal{P}_n$ such that $x_\alpha \in V_\alpha \cap P_\alpha$, where $x_\alpha \neq x$ and $P_\alpha \neq P_\beta$ whenever $\alpha \neq \beta$. Thus $x \in \overline{S}$, which contradicts the point-discreteness of $\mathcal{P}_n$. Therefore X is $\aleph_0$-weakly first-countable, and thus sequential.

By Lemma 3.1, X has a σ-compact-finite k-network. Under (CH), a separable, sequential space with a σ-compact-finite k-network is an $\aleph_0$-space [25, Theorem 7]. Hence, X has a countable $\aleph_0$-weak base. □

Corollary 3.4. *Every closed map on a space with a σ-point-discrete $\aleph_0$-weak base is compact-covering under (CH).*

Proof. Let $f : X \to Y$ be a closed map and X have a σ-point-discrete $\aleph_0$-weak base. Assume that L is a compact subset of Y. Since X has a σ-point-discrete network, Y also has a σ-point-discrete network. By Lemma 2.3, Y has a σ-compact-finite network. So L is a compact metrizable subspace of Y. Then we can take a countable $D \subset L$ such that $L = \overline{D}$. For each $y \in D$, pick $x_y \in f^{-1}(y)$. Let $E = \{x_y : y \in D\}$. Then E is countable and $f(\overline{E}) = L$. Now $\overline{E}$ is a separable space with a σ-point-discrete $\aleph_0$-weak base. By Theorem 3.1, $\overline{E}$ has a countable $\aleph_0$-weak base. So $\overline{E}$ is a paracompact space. By [26], every closed map on a paracompact space is compact-covering. Therefore, there is a compact $K \subset \overline{E}$ such that $f(K) = L$. □

In the following, we shall prove that the assumption (CH) in Theorem 3.1 can be replaced by either of the following conditions: (1) X is $\aleph_1$-compact; (2) The sequential order of X is countable.

Recall a space X is $\aleph_1$-*compact* if each closed discrete subspace of X is countable. Let S be a subset for X. We define iterates of the operator seq cl inductively for a space X as follows: seq $cl^0(S)$=S; seq $cl(S)$=$\{x : x$ is a limit point of $S\}$; if α is an ordinal, let seq $cl^{\alpha+1}(S)$=seq cl(seq $cl^\alpha(S)$); if α is a limit ordinal, let seq $cl^\alpha(S)$=$\cup_{\beta<\alpha}$seq $cl^\beta(S)$. The *sequential order* of X is the least ordinal α such that for each subset S of X we have $cl(S)$=seq $cl^\alpha(S)$.

Lemma 3.2. *Suppose a space X has a σ-point-discrete $\aleph_0$-weak base. If $A \subset X$ is $\aleph_1$-compact, then seq $cl(A)$ is $\aleph_1$-compact.*

Proof. Assume to the contrary that there is a closed discrete subset $D = \{x_\alpha : \alpha < \omega_1\}$ in seq $cl(A) - A$. For each $\alpha < \omega_1$, let $\{x^\alpha_n\} \subset A$ be a sequence converging to x_α. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\} = \cup\{\mathcal{P}_x(m) : x \in X, m \in \mathbb{N}\}$ be a σ-point-discrete $\aleph_0$-weak base for X, where each $\mathcal{P}_n$ is a point-discrete family and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. We assign to each $\alpha < \omega_1$ an $m_\alpha \in \mathbb{N}$ and a subsequence $\{y^\alpha_n\}$ of $\{x^\alpha_n\}$ such that $\{y^\alpha_n\}$ is eventually in each element of $\mathcal{P}_{x_\alpha}(m_\alpha)$. Since D is closed discrete, we can take $P_\alpha \in \mathcal{P}_{x_\alpha}(m_\alpha)$ such that $P_\alpha \cap D = \{x_\alpha\}$ for each $\alpha < \omega_1$. Without loss of generality, we may assume that $\{y^\alpha_n : n \in \mathbb{N}\} \subset P_\alpha$ and $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}_{n_0}$ for each $\alpha < \omega_1$ and some $n_0 \in \mathbb{N}$.

If $\{y^\alpha_n : n \in \mathbb{N}, \alpha < \omega_1\}$ is uncountable, then we can take an uncountable $S = \{y_\beta : \beta < \omega_1\} \subset \{y^\alpha_n : n \in \mathbb{N}, \alpha < \omega_1\}$ such that $y_\beta \in P_\beta$. Thus S is a uncountable closed discrete subset of A. This is a contradiction.

Now suppose that $\{y^\alpha_n : n \in \mathbb{N}, \alpha < \omega_1\}$ is countable. For each $\alpha < \omega_1$, pick a $k(\alpha) \in \mathbb{N}$ such that $\{P_\alpha : \alpha < \omega_1\}$ is point-finite at $y^\alpha_{k(\alpha)}$. Then $T = \{y^\alpha_{k(\alpha)} : \alpha < \omega_1\}$ is countable. So T intersects at most countable elements of $\{P_\alpha : \alpha < \omega_1\}$. Thus $\{P_\alpha : \alpha < \omega_1\}$ is countable. This is a contradiction.

Therefore, seq $cl(A)$ is $\aleph_1$-compact. □

Theorem 3.2. *Let X be a separable space with a σ-point-discrete $\aleph_0$-weak base. If one of the followings holds, then X has a countable $\aleph_0$-weak base.*

(1) X is $\aleph_1$-compact;

(2) The sequential order of X is countable.

Proof. (1) By Lemma 3.1, X has a σ-compact-finite k-network. An $\aleph_1$-compact space with a σ-compact-finite k-network is an $\aleph_0$-space [16]. So we only need to show X is $\aleph_0$-weakly first-countable.

Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\} = \cup\{\mathcal{P}_x(m) : x \in X, m \in \mathbb{N}\}$ be a σ-point-discrete $\aleph_0$-weak base for X, where each $\mathcal{P}_n$ is a point-discrete family and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $x \in X - I(X)$ and $m \in \mathbb{N}$, without loss of generality, we may assume that $\{x\} \notin \mathcal{P}_x(m)$. Suppose $\mathcal{P}_x(m) \cap \mathcal{P}_n$ is uncountable for some $n, m \in \mathbb{N}$ and $x \in X - I(X)$. Then we can choose an uncountable $\{x_\alpha : \alpha < \omega_1\}$ and a $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}_x(m) \cap \mathcal{P}_n$ such that $\{x, x_\alpha\} \subset P_\alpha$ and the P_α's are distinct. Thus $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable, closed discrete subset of X, which is a contradiction with the $\aleph_1$-compactness of X. Therefore X is $\aleph_0$-weakly first-countable. The proof is complete.

(2) Since X is separable, we can pick a countable $D \subset X$ such that $X = \overline{D}$. Since the sequential order of X is countable, $X = \bigcup_{\alpha<\gamma} \text{seq } cl^{\alpha}(D)$ for some countable ordinal γ. By Lemma 3.2, seq $cl^{\alpha}(D)$ is $\aleph_1$-compact for each $\alpha < \gamma$. Hence, X is $\aleph_1$-compact. By (1), X has a countable $\aleph_0$-weak base. □

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