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A survey of the theory of ℵ-spaces

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In 1984, Gruenhage [13] wrote a beautiful survey of the theory of generalized metric spaces. It contains the most important results which has been achieved in the research of this field in more than twenty years. The class of ℵ-spaces, as a generalization of metric spaces, has many important properties, some of which have been included in the Gruenhage's survey. During the recent years, a great deal of progress has been made for the study of ℵ-spaces. This paper is devoted to a survey of this field, including China scholars' some works, and poses some interesting open problems. It contains four parts: Basic operation properties; Characterizations; Relationship; Mapping properties. A detailed list of references is given at the end.

1. Basic Operation Properties.

All spaces are assumed to be regular and T_1. $\mathbb{N}$ denotes the set of positive integers.

Definition 1.1 (Michael [25]) A collection $\mathcal{P}$ of subsets of X is a *pseudobase* for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. A space with a countable pseudobase is an *$\aleph_0$-space*.

Definition 1.2 (O'Meara [27]) A collection $\mathcal{P}$ of subsets of X is a *k-network* for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}' \subset U$ for some $\mathcal{P}' \subset \mathcal{P}$. A space with a σ-locally finite k-network is an *$\aleph$-space*.

Why did O'Meara introduce a new space by k-network rather than pseudobase? Except for the concept of k-networks is a natural generalization of one of bases, the following theorem clears our mind of doubts from another points of view.

Theorem 1.3 (Shou Lin [20, 22]) *A space with a point countable pseudobase (hence with a σ-locally finite pseudobase) is an $\aleph_0$-space.*

It is easy to check that (1) every metric space is an $\aleph$-space; (2) every subspace of a (paracompact) $\aleph$-space is also a (paracompact) $\aleph$-space; (3) a product of countably many (paracompact) $\aleph$-spaces is a (paracompact) $\aleph$-space; (4) a image of a $\aleph$-space under perfect mappings is an $\aleph$-space. These show that the class of $\aleph$-spaces is an important one of generalized metric spaces. Moreover, the following theorem makes clear the reason that one is interested in $\aleph$-spaces.

Theorem 1.4 (O'Meara [29]) *If X is an $\aleph_0$-space and Y is a paracompact $\aleph$-space, then $C(X, Y)$ with the compact-open topology is a paracompact $\aleph$-space.*

O'Meara [29] asked whether the Theorem 1.4 is true when "paracompact" is deleted. This question is a powerful stimulus for growth in the theory of $\aleph$-spaces. For solving it, Guthrie introduced the concept of cs-σ-spaces.

Definition 1.5 (Guthrie [14, 15]) A collection $\mathcal{P}$ of subsets of X is a *cs-network* (i.e., *convergent sequence network*) for X if, whenever $Z \subset U$ with Z convergent sequence and U open in X, then Z is eventually in P and $P \subset U$ for some $P \in \mathcal{P}$. A space with a σ-locally finite cs-network is a *cs-σ-space*.

The importance of cs-σ-spaces is showed by the next theorem.

Theorem 1.6 (Guthrie [15]) *If X is an $\aleph_0$-space and Y is an cs-σ-space, Then $C(X, Y)$ with the compact-open topology is a cs-σ-space.*

In [15], Guthrie proved also that every cs-σ-space is an $\aleph$-space, and that every paracompact $\aleph$-space is a cs-σ-space. An exact relationship between cs-σ-spaces and $\aleph$-spaces is related in the next section.

As is well known, the product of two k-spaces need not be a k-space. One of questions for study of general topology is to search for a proper class of spaces in order to obtain the condition under which the product of two k-spaces is a k-space. ℵ-spaces play an exceptional role in this field showed by the next theorem.

Theorem 1.7 (Tanaka [33]) *Let X and Y be k-and ℵ-spaces. Then $X \times Y$ is a k-space if and only if one of the following properties hold:*

(1) X and Y are metrizable spaces;

(2) X or Y is a locally compact spaces;

(3) X and Y are spaces of the class L', where a space Z is said to belong to the class L' if it is the union of countably many closed and locally compact subsets Z_n such that $A \subset Z$ is closed whenever $A \cap Z_n$ is closed for all n.

One of important properties of metrizable spaces or stratifiable spaces is that they satisfy the Dugundji Extension Theorem. Do normal ℵ-spaces have this property? Van Douwen answers negatively this question.

Definition 1.8 (Van Douwen [3]) For a space X, let $C^*(X)$ denote the set of all bounded, continuous, real valued functions on X. Put

$\|f\| = \max\{|f(x)|: x \in X\}$

for each $f \in C^*(X)$. A space X is said to have *property D^*_c*, where real number $c \geq 1$, if for each closed subspace F of X, there exists a linear transformation $\varphi: C^*(F) \to C^*(X)$ satisfying that if f

$\in C^*(F)$, then

$\varphi(f)|F = f$, and $\|\varphi(f)\| \le c\|f\|$.

Metrizable spaces and stratifiable spaces have the property D_1^* (Dugundji [4], Borges [2]).

Theorem 1.9 (Van Douwen [3]) *There exists an $\aleph_0$-space which has not any property D_c^*.*

Finally, it is not difficult to construct an example showing the adjunction space of two metrizable spaces need not be an $\aleph$-space.

2. Characterizations

One of the milestones in the development of the theory of $\aleph$-spaces is a series of important characterizations of $\aleph$-spaces given by the next theorem.

Theorem 2.1 (Foged [6], Gao [11]) *The following properties of a space X are equivalent:*

(a) X is an $\aleph$-space;

(2) X is a cs-σ-space;

(3) X has a σ-discrete k-network;

(4) X has a σ-locally finite cs-network.

By Theorem 1.6 and 2.1, we obtain an affirmative answer to the question posed by O'Meara in [29] as follows.

Corollary 2.2 *If* X *is an* $\aleph_0$*-space and* Y *is an* $\aleph$*-space, then* $C(X, Y)$ *with the compact-open topology is an* $\aleph$*-space.*

A space with a σ-hereditarily closure-preserving base (resp. network) is a space with a σ-locally finite base (network). And there exists a space with a σ-hereditarily closure-preserving k-network which has not a σ-locally finite k-network. However, we have the following questions.

Question 2.3. Is a space with a σ-hereditarily closure-preserving cs-network an $\aleph$-space?

Question 2.4 Is a space whose square has a σ-hereditarily closure-preserving k-network an $\aleph$-space?

The peculiarity of above-mentioned characterizations is depicting $\aleph$-spaces by collections of closed subsets of the space. A characterization given by collections of open subsets of the space is the following theorem.

Theorem 2.5 (Nagata [26]) *A space* (X, τ), *where* τ *denotes the topology of* X, *is an* $\aleph$*-space if and only if there is a function* $g: N \times X \to \tau$ *satisfying that*

(1) if $\{x_n: n \in N\} \to p \in X$ *and if* $x_n \in g(n, y_n)$ *for all* $n \in N$, *then* $\{y_n: n \in N\} \to p$, *where* $\to$ *denotes convergence.*

(2) if $y \in g(n, x)$, *then* $g(n, y) \subset g(n, x)$.

(3) for each $x \in X$ *and* $n \in N$,

$$|\{g(n,y): y \in g(b, x),\ x \notin g(n, y)\}| < \aleph_0.$$

Question 2.6 (Nagata [26]) Is it possible to drop (2) from Theorem 2.5?

3. Relationship

First, we recount the relations between $\aleph$-spaces and some generalized metric spaces.

Definition 3.1 (Arhangel'skii [1], Siwiec [31]) A collection $\mathcal{P}$ of subsets of X is a *weak base* for X provided that $\mathcal{P} = \cup\{\mathcal{P}_x: x \in X\}$ such that

(1) $x \in \cap\mathcal{P}_x$;

(2) if U, V $\in \mathcal{P}_x$, then there exists a $W \in \mathcal{P}_x$ with $W \subset U \cap V$;

(3) a subset F of X is closed if and only if for each point $x \in X - F$, there exists a $B \in \mathcal{P}_x$ with $B \cap F = \phi$. A space is *g-first countable* if X has a weak base $\cup\{\mathcal{P}_x: x \in X\}$ such that each $\mathcal{P}_x$ is countable. A space X is *g-metrizable* if X has a σ-locally finite weak base.

Obviously, every metrizable space is g-metrizable. An exact relation between $\aleph$-spaces and g-metrizable spaces is related in the next theorem which affirmatively answers a question posed by Siwiec [31].

Theorem 3.2 (Foged [5]) *A space X is g-metrizable if and only if it is a g-first countable $\aleph$-space.*

We have known that every g-metrizable space is a k- and $\aleph$-space and it contains no closed copy of sequential fan S_ω. Is its converse proposition true?

Question 3.3 Suppose X is a k- and $\aleph$-space. Is X a g-metrizable space if it contains no closed copy of S_ω?

By Definition 1.1 and 1.2, a space with a countable k-network is equivalent to a space with a countable pseudobase. Then, how any $\aleph$-space an $\aleph_0$-space?

Theorem 3.4 (O'Meara [27, 29]) *The following conditions of a space* X *are equivalent:*

(1) X *is an* $\aleph_0$*-space;*

(2) X *is an* $\aleph$*-space with the Lindelöf property;*

(3) X *is a hereditarily separable* $\aleph$*-space.*

Theorem 3.4 causes to me to discuss the relation between separable $\aleph$-spaces and $\aleph_0$-spaces.

Theorem 3.5 (Shou Lin [18]) *There exists a completely regular separable* $\aleph$*-space which is not an* $\aleph_0$*-space. Whether every normal separable* $\aleph$*-space is an* $\aleph_0$*-space is independent of the axioms of set theory.*

Question 3.6 Is there a normal CCC $\aleph$-space which is not an

$\aleph_0$-space?

Question 3.7 Is there a separable space with a σ-hereditarily closure-preserving k-network which is not an $\aleph$-space?

$\aleph$-spaces are not relative to stratifiable spaces. A common generalization of $\aleph$-spaces and stratifiable spaces is the concept of k-semistratifiable spaces introduced by Lutzer [23].

Definition 3.8 (Lutzer [23]) A space X is *k-semistratifiable* space if, for each open set $U \subset X$, one can assign a sequence $\{F(n, U)\}$ of closed subsets of X such that

(1) $U = \cup\{F(n, U): n \in N\}$;

(2) $F(n, U) \subset F(n, V)$ whenever $U \subset V$;

(3) if K is a compact subset of U, then $K \subset F(n, U)$ for some $n \in N$.

It is easy to check that every $\aleph$-space is a k-semistratifiable space. How a k-semistratifiable space is an $\aleph$-space?

Question 3.9 Is a k-semistratifiable space with a σ-locally countable k-network an $\aleph$-space?

An $\aleph$-space has a G_δ-diagonal. In fact, $\aleph$-spaces have the following strong properties.

Theorem 3.10 (Shou Lin [21]) *If X is an $\aleph$-space, then there* *is a sequence* $(\mathcal{U}_n)$ *of open covers of* X *such that for each compact subset* K *of* X, $K = \cap_n \overline{st(K, \mathcal{U}_n)}$.

Question 3.11 Has an $\aleph$-space a regular G_δ-diagonal?

In the next part, we relate some covering properties on $\aleph$-spaces. Every $\aleph$-space is a subparacompact space. But, an $\aleph$-space need not be a metalindelöf space, or not to be a normal space. One of reasons consists in lacking an appropriate weakly first countability axiom on $\aleph$-spaces. In fact, it was by adding first countable axiom to $\aleph$-spaces that O'Meara [28] obtained a metrizable theorem on $\aleph$-spaces. When some weakly first countability axioms are added to $\aleph$-spaces, we can obtain certain covering properties. Some outstanding achievements belong to Foged.

Theorem 3.12 (Foged [8]) *If a (normal)* $\aleph$*-space* X *is a k-space, then it is a (paracompact) metalindelöf space.*

Theorem 3.13 (Foged [7]) *If an* $\aleph$*-space* X *is a Fréchet space, then it is a Lašnev space (i.e., a image of a metrizable space under a closed continuous mapping).*

Corollary 3.14 (O'Meara [28]) *If an* $\aleph$*-space* X *satisfies the first countability axiom, then it is a metrizable space.*

4. Mapping Properties

In this section, all mappings are continuous and onto. Another milestone in the development of the theory of ℵ-spaces is to obtain a series of mapping theorems on ℵ-spaces which are parallel to mapping theorems on metric spaces. The motive force of mapping properties of studying ℵ-spaces comes of Soviet mathematician Alexandroff's plan which tries to express a space by a image of a metrizable space under a mapping.

First of all, we consider closed mappings on ℵ-spaces. Clearly, perfect mappings preserve ℵ-spaces. Further, we have the next theorem.

Theorem 4.1 (Shou Lin [21], Shu-hao Sun [32]) *ℵ-spaces are preserved under closed mappings with Lindelöf fibers.*

Corollary 4.2 (Zhi-min Gao, Y. Hattori [12], Lin [21]) *For a Fréchet space* X, X *is an ℵ-space if and only if it is a image of a metrizable space under a closed mapping with Lindelöf fibers.*

Theorem 4.3 (Zhi-min Gao [10]) *Under the Continuum Hypothesis, a Fréchet Lašnev space* X *is an ℵ-space if and only if the character* $\chi(X)$ *of* X *does not exceed* $\aleph_1$.

Corollary 4.2 and Theorem 4.3 correspond to the Hanai-Morita-Stone Theorem on metric spaces, by which it is not difficult to prove that metrizability is preserved under open and closed mappings. This suggests the following question.

Question 4.4 Are ℵ-spaces preserved under open and closed mappings ?

ℵ-spaces are not preserved under closed mappings [17]. How an internal characterization has a image of ℵ-spaces under a closed mapping ? It is easy to check a image of ℵ-spaces under a closed mapping has a σ-hereditarily closure-preserving k-network. The following question is interesting, which is analogous to the internal characterization on Lašnev spaces given by Foged [7].

Question 4.5 Is any space with a σ-hereditarily closure-preserving k-network a image of an ℵ-space under a closed mapping ?

Secondly, we consider open mappings on ℵ-spaces. ℵ-spaces are not preserved under a finite-to-one open mapping [19]. We have known that an image of a metric space under a (pseudo) open and compact mapping is a development space [24], and that an image of a k-semistratifiable space under a (pseudo) open and compact mapping is a semistratifiable space [16]. Hence the following question is very natural.

Question 4.6 Is an image of ℵ-spaces under a (pseudo) open and compact mapping a σ-space ?

Thirdly, we consider perfect inverse images on ℵ-spaces. Because a perfect inverse image of an ℵ-space need not be an ℵ-space [24], it is necessary to add an appropriate condition to ℵ-spaces in considering the question of inverse invariant of $\aleph$-spaces under perfect mappings.

theorem 4.7 (Shou Lin [21]) A perfect inverse image of an $\aleph$-space is an $\aleph$-space if and only if it satisfies any of the following:

(1) it has a G_δ-diagonal;

(2) it has a point-countable k-network.

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