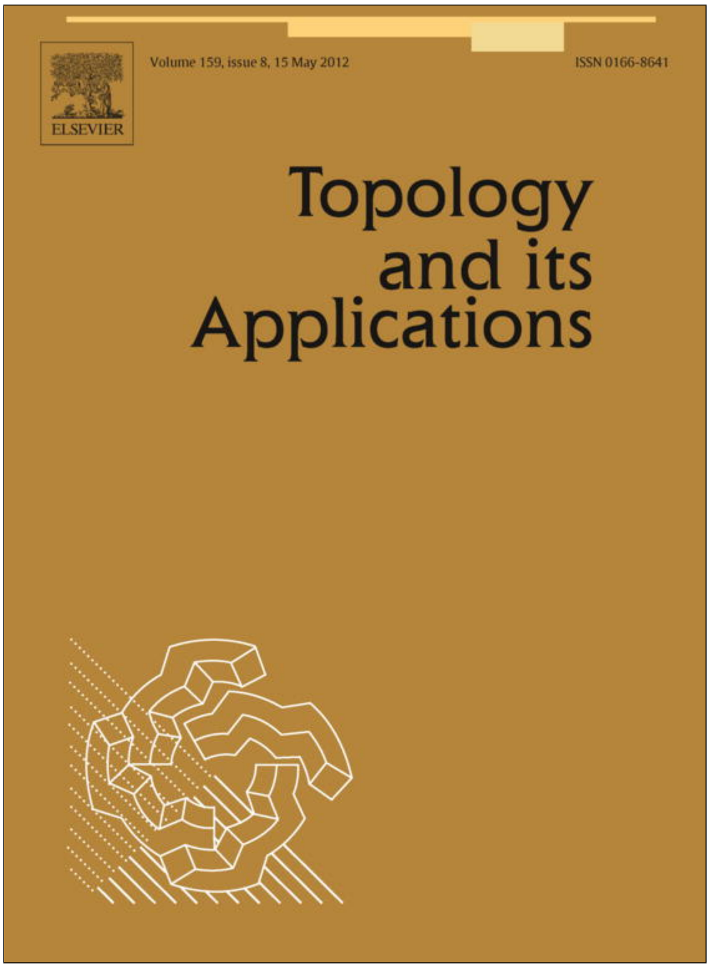
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A note on rectifiable spaces ☆

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ABSTRACT

In this paper, we firstly discuss the question: Is l_2^∞ homeomorphic to a rectifiable space or a paratopological group? And then, we mainly discuss locally compact rectifiable spaces, and show that a locally compact rectifiable space with the Souslin property is σ-compact, which gives an affirmative answer to A.V. Arhangel'skiĭ and M.M. Choban's question [A.V. Arhangel'skiĭ, M.M. Choban, On remainders of rectifiable spaces, Topology Appl. 157 (2010) 789–799]. Next, we show that a rectifiable space X is strongly Fréchet–Urysohn if and only if X is an α_4-sequential space. Moreover, we discuss the metrizabilities of rectifiable spaces, which gives a partial answer for a question posed in F.C. Lin and R.X. Shen (2011) [16]. Finally, we consider the remainders of rectifiable spaces, which improve some results in A.V. Arhangel'skiĭ (2005) [2], A.V. Arhangel'skiĭ and M.M. Choban (2010) [5], C. Liu (2009) [17].

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a *rectifiable space* [9] provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. In fact, for a topological group with the neutral element e, then it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line [11, Example 1.2.2] is such an example. Also, the 7-dimensional sphere S_7 is rectifiable but not a topological group [23, §3]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous.

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By a remainder of a space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X.

In Section 3, we show that l_2^∞ is homeomorphic to no rectifiable space or paratopological group, where l_2^∞ is the separable Hilbert space, which extends a result of T. Banakh in [8]. In Section 4, we mainly discuss locally compact rectifiable spaces, and show that a locally compact and separable rectifiable space is σ-compact, which give an affirmative answer for a question of A.V. Arhangel'skiĭ and M.M. Choban's. Moreover, we prove that under the set theory assumption a locally compact rectifiable space with the α_4-properties is metrizable. In Section 5, we show that a rectifiable space X is strongly Fréchet–Urysohn if and only if X is an α_4-sequential space. In Section 6, we mainly discuss the metrizability of rectifiable spaces which have a point-countable k-network. In Section 7, we mainly consider the question: When does a Tychonoff rectifiable space G have a Hausdorff compactification bG with a remainder belonging to the class of separable and metrizable spaces?

2. Preliminaries

In [20], E. Pentsak studied the topology of the direct limit $X^\infty = \varinjlim X^n$ of the sequence

$$X \subset X \times X \subset X \times X \times X \subset \cdots,$$

where $(X, \star)$ was a "nice" pointed space and X^n was identified with the subspace $X^n \times \{\star\}$ of X^{n+1}.

A space X is called an S_2-space (*Arens' space*) if $X = \{\infty\} \cup \{x_n\colon n \in \mathbb{N}\} \cup \{x_n(m)\colon m, n \in \mathbb{N}\}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m)\colon m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\bigcup\{V_n\colon n > k\})$ for some $k \in \mathbb{N}$, where V_n is a neighborhood of x_n.

T. Banakh defined the space K [8].

Let

$$K = \{(0,0)\} \cup \left\{\left(\frac{1}{n}, \frac{1}{nm}\right)\colon n, m \in \mathbb{N}\right\} \subset \mathbb{R}^2.$$

The space K is non-locally compact and metrizable. Also, the space K is a minimal space with these properties in the sense that each metrizable non-locally compact space contains a closed copy of K. For convenience, put $x_0 = (0,0)$ and $x_{n,m} = (\frac{1}{n}, \frac{1}{nm})$ for any $n, m \in \mathbb{N}$.

The space X is called S_ω if X is obtained by identifying all the limit points of ω many convergent sequences.

If A is a subset of a space X, then $[A]^{seq}$ denotes the *sequential closure* of A, i.e. the set of limits of convergent sequences in A. Clearly, we have $A \subset [A]^{seq}$. By induction on $\alpha \in \omega_1 + 1$, we can define $[A]_\alpha$ as follows: $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$ and $[A]_\alpha = \bigcup\{[A]_\beta\colon \beta < \alpha\}$ for a limit order α. One can easily verify that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space X is sequential iff $\overline{A} = [A]_{\omega_1}$ for every $A \subset X$. For a sequential space X we define $so(X)$, the *sequential order* of X, by $so(X) = \min\{\alpha \in \omega_1 + 1\colon \overline{A} = [A]_\alpha \text{ for each } A \subset X\}$.

Definition 2.1. A space X is said to be *Fréchet–Urysohn* if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and $\{x_n\colon n \in \mathbb{N}\} \subset A$. A space X is said to be *strongly Fréchet–Urysohn* if the following condition is satisfied

(SFU) For every $x \in X$ and each sequence $\eta = \{A_n\colon n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is a sequence $\zeta = \{a_n\colon n \in \mathbb{N}\}$ in X converging to x and intersecting infinitely many members of η.

Obviously, a strongly Fréchet–Urysohn space is Fréchet–Urysohn. However, the space S_ω is Fréchet–Urysohn and non-strongly Fréchet–Urysohn.

Let X be a space. For $P \subset X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P.

Definition 2.2. Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$; (b) the family $\mathscr{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathscr{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}_x$.

The family $\mathscr{P}$ is called a *weak base* for X [1] if, for every $G \subset X$, the set G must be open in X whenever for each $x \in G$ there exists $P \in \mathscr{P}_x$ such that $P \subset G$. The space X is *weakly first-countable* if the family $\mathscr{P}$ is a weak base for X such that each $\mathscr{P}_x$ is countable.

The following theorem for the first time there was announced in [9], and the readers can see the proof in [10,13,22].

Theorem 2.3. *([9]) A topological space G is rectifiable if and only if there exists $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G$, $y \in G$ the next identities hold*:

$$p\big(x, q(x,y)\big) = q\big(x, p(x,y)\big) = y, \quad q(x,x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 2.3. Fixed a point $x \in G$, then $f_x, g_x : G \to G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphism, respectively. We denote f_x, g_x with $p(x, G), q(x, G)$, respectively.

Let G be a rectifiable space, and let p be the multiplication on G. Further, we sometime write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with operation p, q, 0-ary operation e and identities as above. It is easy to see that this algebraic system need not to satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

All spaces are T_1 and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive natural numbers. The letter e denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. Readers may refer to [3,11,12] for notations and terminology not explicitly given here.

3. l_2^∞ is homeomorphic to no rectifiable space or paratopological groups

In this section, by a modification of the proof of Theorem 1 in [8], we show that l_2^∞ is homeomorphic to no rectifiable space or paratopological group.

We call a subset A of a rectifiable space (resp. paratopological group) G *multiplicative* if for any $a, b \in A$ we have $p(a, b) = a \cdot b \in A$ (resp. $ab \in A$).

We denote by $\text{conv}(K) = \{(0, 0)\} \cup \{(x, y)\colon 0 < y \leqslant x \leqslant 1\}$, where $\text{conv}(K)$ is the convex hull of K in $\mathbb{R}^2$.

In this section, we may assume that $S_\omega = \{y_0\} \cup \{y_{n,m} = (n, \frac{1}{m})\colon n, m \in \mathbb{N}\}$, where, for each $n \in \mathbb{N}$, the sequence $\{y_{n,m}\} \to y_0$ as $m \to \infty$. For each $k \in \mathbb{N}$, let $V_k = \{y_0\} \cup \{y_{n,m}\colon n \leqslant k,\ m \in \mathbb{N}\}$. It is easy to see that S_ω has the direct limit topology with respect to the sequence $V_1, V_2, \ldots$.

Theorem 3.1. *Let X be a normal k-space. If X contains closed copies of S_ω and K, then it is homeomorphic to no closed multiplicative subset of a rectifiable space G such that y_0 is the right neutral element of G.*

Proof. Suppose not, let X be a closed multiplicative subset of a rectifiable space G. Now, we define a map $f : K \times S_\omega \to X$ with $f(x, y) = p(x, y)$ for each $(x, y) \in K \times S_\omega$. Then the following (1) and (2) hold:

(1) the map $(\pi_1, f) : K \times S_\omega \to K \times X$ is a closed embedding, where $(\pi_1, f)(x, y) = (x, f(x, y))$ for each $(x, y) \in K \times S_\omega$;
(2) the map $g : K \to X$ defined by $g(x) = p(x, y_0) = x$ (that is, g is the identity map), for each $x \in K$, is a closed embedding.

Indeed, the statement (2) is obvious. Moreover, it is easy to see that the map $(\pi_1, f) : K \times S_\omega \to K \times X$ is injective continuous. We only show that the map (π_1, f) is relatively open. For each open subset $U \times V$ of $K \times S_\omega$, we have $(\pi_1, f)(U \times V) = \bigcup\{\{x\} \times (x \cdot V)\colon x \in U\}$, where U and V are open in K and S_ω respectively. Since V is open in S_ω, there exists an open subset W of X such that $W \cap S_\omega = V$. Therefore, we have

$$(\pi_1, f)(U \times V) = \bigcup\bigl\{\{x\} \times p(x, V)\colon x \in U\bigr\} = \bigl(U \times (U \cdot W)\bigr) \cap (\pi_1, f)(K \times S_\omega).$$

Since $p(U, W) = \bigcup\{p(x, W)\colon x \in U\}$ is open in X, the set $U \times p(U, W)$ is open in $K \times X$.

By the normality of X, let $h : X \to \text{conv}(K)$ be a continuous extension of the map $g^{-1} : g(K) \to K$.

For each $n, m \in \mathbb{N}$, let $\delta_{n,m} = \frac{1}{2nm(m+1)}$, and put

$$W_{n,m} = \text{conv}(K) \cap \left(\left(\frac{1}{n} - \delta_{n,m}, \frac{1}{n} + \delta_{n,m}\right) \times \left(\frac{1}{nm} - \delta_{n,m}, \frac{1}{nm} + \delta_{n,m}\right)\right).$$

Obviously, the collection $\{W_{n,m}\colon n, m \in \mathbb{N}\}$ consists of pairwise disjoint neighborhoods of the points $x_{n,m}$ in $\text{conv}(K)$. Since $y_{n,m} \to y_0$ as $m \to \infty$ and $h \circ f(x_{n,m}, y_0) = h(p(x_{n,m}, y_0)) = x_{n,m} = (\frac{1}{n}, \frac{1}{nm})$, for any $n, m \in \mathbb{N}$, there exists a $k(n, m) \in \mathbb{N}$ such that $h \circ f(x_{n,m}, y_{n,k(n,m)}) \in W_{n,m}$. Without loss of generality, we may assume that $k(n, m + 1) > k(n, m)$ for any $n, m \in \mathbb{N}$. Put

$$Z = \bigl\{p(x_{n,m}, y_{n,k(n,m)})\colon n, m \in \mathbb{N}\bigr\}.$$

For each $n, m \in \mathbb{N}$, it follows from $h \circ f(x_0, y_0) \notin W_{n,m}$ that $f(x_0, y_0) \notin Z$.

Claim. *Z is closed in X.*

Since X is a k-space, it suffices to prove that for each compact subset F of X the intersection $F \cap Z$ is closed in F. Let

$$F_1 = \{x_0\} \cup \bigl\{x_{n,m}\colon h(F) \cap W_{n,m} \neq \emptyset,\ n, m \in \mathbb{N}\bigr\} \quad \text{and} \quad F_2 = \pi_2\bigl((\pi_1, f)^{-1}(F_1 \times F)\bigr).$$

Since $h(F) \subset \text{conv}(K)$ is compact, the set F_1 is compact. It follows from (1) that

$$(\pi_1, f)^{-1}(F \times F_1) \subset K \times S_\omega$$

is compact, and hence F_2 is also compact. Because $S_\omega = \varinjlim V_n$, there exists an $n_0 \in \mathbb{N}$ such that $F_2 \subset V_{n_0}$. Since $F \cap Z \subset p(F_1, F_2)$, we have $F \cap Z \subset \{p(x_{n,m}, y_{n,k(n,m)})\colon n \leqslant n_0,\ y_{n,m} \in F_1\}$. By the compactness of F_1, it is easy to see that $\{p(x_{n,m}, y_{n,k(n,m)})\colon n \leqslant n_0,\ x_{n,m} \in F_1\}$ is finite. Therefore, the set $F \cap Z$ is closed in F.

Since $p(x_0, y_0) = x_0 \notin Z$ and p is continuous, it follows from Claim that there exist open neighborhoods $V(x_0) \subset K$ and $U(y_0) \subset S_\omega$ of x_0 and y_0 respectively such that $p(V(x_0), U(y_0)) \cap Z = \emptyset$. For every $m \in \mathbb{N}$, we can fix an $n \in \mathbb{N}$ such that $x_{n,m} \in V(x_0)$. Since $\{y_{n,m}\}_{m=1}^{\infty}$ converges to y_0 and $\{k(n,m)\}_{m=1}^{\infty}$ is increasing, there is an $m \in \mathbb{N}$ such that $y_{n,k(n,m)} \in U(y_0)$. Then $p(x_{n,m}, y_{n,k(n,m)}) \in p(V(x_0), U(y_0)) \cap Z$, which is a contradiction. □

Theorem 3.2. *Let X be a normal k-space. If X contains closed copies of S_ω and K, then it is homeomorphic to no closed multiplicative subset of a paratopological group.*

Proof. Suppose not, let X be a closed multiplicative subset of a rectifiable space G. Now, we define a map $f : K \times S_\omega \to X$ with $f(x, y) = xy$ for each $(x, y) \in K \times S_\omega$. Obviously, we can obtain the following (1) and (2):

(1) the map $(\pi_1, f) : K \times S_\omega \to K \times X$ is a closed embedding, where $(\pi_1, f)(x, y) = (x, xy)$ for each $(x, y) \in K \times S_\omega$;
(2) the map $g : K \to X$ defined by $g(x) = xy_0$, for each $x \in K$, is a closed embedding.

By the normality of X, let $h : X \to \mathrm{conv}(K)$ be a continuous extension of the map $g^{-1} : g(K) \to K$.

By the proof of Theorem 3.1, we can define the neighborhoods $W_{n,m}$ of the points $x_{n,m}$ in $\mathrm{conv}(K)$ and the closed set Z with $x_0 y_0 \notin Z$.

Since $f(x_0, y_0) = x_0 y_0 \notin Z$, G is joint continuous and Z is closed, it follows that there exist open neighborhoods $V(x_0) \subset K$ and $U(y_0) \subset S_\omega$ of x_0 and y_0 respectively such that $(V(x_0) \times U(y_0)) \cap Z = \emptyset$. For every $m \in \mathbb{N}$, we can fix an $n \in \mathbb{N}$ such that $x_{n,m} \in V(x_0)$. Since $\{y_{n,m}\}_{m=1}^{\infty}$ converges to y_0 and $\{k(n,m)\}_{m=1}^{\infty}$ is increasing, there is an $m \in \mathbb{N}$ such that $y_{n,k(n,m)} \in U(y_0)$. Then $x_{n,m} y_{n,k(n,m)} \in (V(x_0) \times U(y_0)) \cap Z$, which is a contradiction. □

It is well known that a space X contains a closed copy of S_ω, provided X can be written as a direct limit of a sequence

$$X_1 \subset X_2 \subset \cdots,$$

where each X_n is a closed metrizable subset of X, nowhere dense in X_{n+1}. In particular, the space l_2^∞ contains a topological closed copy of S_ω. Moreover, the space l_2^∞ is a normal k-space and contains a topological closed copy of K. Therefore, by the topological homogeneity of l_2^∞ and Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *l_2^∞ is homeomorphic to no rectifiable space or a paratopological group.*

Corollary 3.4. *l_2^∞ is homeomorphic to no topological loop.*

Corollary 3.5. *([8]) l_2^∞ is homeomorphic to no topological group.*

4. Locally compact rectifiable spaces

In [5], A.V. Arhangel'skiĭ and M.M. Choban posed the following question:

Question 4.1. ([5, Problem 5.10]) Is every rectifiable p-space with a countable Souslin number Lindelöf? What if we assume the space to be separable? Separable and locally compact?

Now, we give an affirmative answer for Questions 4.1 of the case of separable and locally compact rectifiable spaces.

Lemma 4.2. *Let G be a rectifiable space. If Y is a dense subset of G and U is an open neighborhood of the right neutral element e of G, then $G = Y \cdot U$.*

Proof. Fix an arbitrary $z \in G$. Since $q(z, z) = e \in U$, there exists an open neighborhood V of e such that $q(z \cdot V, z) \subset U$. Put $W = z \cdot V$. Then W is an open neighborhood of z in G. Since Y is a dense subset of G, we have $W \cap Y \neq \emptyset$. Take a point $y \in W \cap Y = z \cdot V \cap Y$. Then $y = z \cdot v$ for some $v \in V$.

$$z = p\big(z \cdot v, q(z \cdot v, z)\big) = p\big(y, q(z \cdot v, z)\big) \in p\big(y, q(z \cdot V, z)\big) \subset p(y, U) = y \cdot U \subset Y \cdot U.$$

By the choice of arbitrary point of z, we have $G = Y \cdot U$. □

It follows from Lemma 4.2, we have the following results, which give an answer for Question 4.1.

Theorem 4.3. *If G is a locally σ-compact*[1] *rectifiable space with the Souslin property, then G is σ-compact.*

Proof. Let G be a locally σ-compact rectifiable space with the Souslin property. For each $\alpha \in \Gamma$, let $\mathcal{A}_\alpha$ be the family consisting of disjoint open subsets of G such that each element of $\mathcal{A}_\alpha$ is covered by countably many compact subsets (since G is locally σ-compact). $\{\mathcal{A}_\alpha, \alpha \in \Gamma\}$ is a set with partial order by inclusion. It is easy to see that every chain of $\{\mathcal{A}_\alpha, \alpha \in \Gamma\}$ has an upper bound, by Zorn's Lemma, there is a maximal element $\mathcal{A} \in \{\mathcal{A}_\alpha, \alpha \in \Gamma\}$. Since G has Souslin property, we have $|\mathcal{A}| \leqslant \omega$, and hence we write $\mathcal{A} = \{A_i\}$, $A_i \subset \bigcup\{K_{i,j}\}$, where each $K_{i,j}$ is a compact subset of G. By maximality of $\mathcal{A}$, $\bigcup\{K_{i,j}\}$ is a dense subset of G. Let U be an open neighborhood of e, which is covered by countably many compact subsets $\{H_l\}$. By Lemma 4.2, $G = (\bigcup\{K_{i,j}\}) \cdot (\bigcup\{H_l\}) = \bigcup(K_{i,j} \cdot H_l)$, each $K_{i,j} \cdot H_l$ is compact, hence G is σ-compact. □

Corollary 4.4. *If G is a locally compact and separable rectifiable space, then G is σ-compact* (*and, hence, Lindelöf*).

Corollary 4.5. *If G is a locally Lindelöf and separable rectifiable space, then G is Lindelöf.*

Let A be a subspace of a rectifiable space G. Then A is called *a rectifiable subspace of G* if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.

Proposition 4.6. *Let G be a rectifiable space. If H is a rectifiable subspace of G, then $\overline{H}$ is also a rectifiable subspace of G.*

Proof. Take two points $x, y \in \overline{H}$. Then we shall show that $p(x, y) \in \overline{H}$ and $q(x, y) \in \overline{H}$.

Since $x, y \in \overline{H}$, there exist two nets $\{x_\alpha\}$, $\{y_\beta\}$ in H such that $x_\alpha \to x$, $y_\beta \to y$. Since p is continuous, $p(x, y)$ is a cluster point of $\{p(x_\alpha, y_\beta)\} \subset H$. Hence $p(x, y) \in \overline{H}$.

Similarly, we can show that $q(x, y) \in \overline{H}$. □

Lemma 4.7. *Let G be a rectifiable space. If V is an open rectifiable subspace of G, then V is closed in G.*

Proof. Suppose that V is non-closed in G. Then $\overline{V} \setminus V \neq \emptyset$. Take a point $x \in \overline{V} \setminus V$. Since $q(x, x) = e \in V$ and the continuity of q, there exists an open neighborhood W of e such that $q(x \cdot W, x) \subset V$. Put $U = x \cdot W$. Then U is an open neighborhood of x, and hence $U \cap V \neq \emptyset$ since $x \in \overline{V}$. Therefore, there exist $a \in W$ and $b \in V$ such that $x \cdot a = b$. Then we have

$$x = p\big(x \cdot a, q(x \cdot a, x)\big) = p\big(b, q(x \cdot a, x)\big) \subset p(V, V) = V,$$

where $p(V, V) = V$ since V is a rectifiable subspace of G. However, the point $x \notin V$, which is a contradiction. □

Theorem 4.8. *If H is a locally compact rectifiable subspace of a rectifiable space G, then H is closed in G.*

Proof. Let $K = \overline{H}$. Then K is a rectifiable subspace of G by Proposition 4.6. Since H is a dense locally compact subspace of K, it follows from [11, Theorem 3.3.9] that H is open in K. By Lemma 4.7, the set H is closed, and hence $K = H$. □

The following lemma maybe was proved somewhere.

Lemma 4.9. *Let F be a compact subset of a space X and have a countable base $\{U_n\}$ with $\overline{U_{n+1}} \subset U_n$ in X, and let $H = \bigcap_n V_n$ ($V_{n+1} \subset V_n$ and each V_n is open in F) is a compact G_δ-set of F. For $n \in \mathbb{N}$, let W_n be an open set in X such that $V_n = W_n \cap F$, $W_n \subset U_n$, $\overline{W_{n+1}} \subset W_n$, then $\{W_n\}$ is a countable base at H in X.*

Proof. $H = \bigcap_n W_n = \bigcap_n \overline{W_n}$. Suppose that $\{W_n\}$ is not a countable base at H, then there is an open subset U of X such that $H \subset U$ and $W_n \setminus U \neq \emptyset$. By induction, choose $x_n \in W_n \setminus U$ with $x_i \neq x_j$ if $i \neq j$. Since $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has a cluster point x. In fact, if $\{x_n\} \cap F$ is infinite, then $\{x_n\}$ has a cluster point in F since F is compact; if $\{x_n\} \cap F$ is finite, without loss generality, we assume $\{x_n\} \cap F = \emptyset$. Since $F \subset X \setminus \{x_n\}$ which is open in X, there is $n_0 \in \mathbb{N}$ such that $F \subset U_n \subset X \setminus \{x_n\}$ for $n > n_0$. This is a contradiction since $x_n \in U_n$. Therefore, we have $x \in \overline{W_n}$ for each n, then $x \in H \subset U$, and hence U contains infinitely many x_n's, which is a contradiction. □

Next, we shall show that, for each locally compact rectifiable space, there exists a compact rectifiable subspace with a countable character.

[1] A space X is *locally σ-compact* if, for each point x of X, there exists an open neighborhood U_x of x such that U_x can be covered by a countably many compact subsets of X.

Lemma 4.10. *Let G be a rectifiable space and F be a compact subset of G containing e and having a countable base $\{U_n\colon n \in \mathbb{N}\}$ in G. Assume that a sequence $\zeta = \{V_n\colon n \in \mathbb{N}\}$ of open neighborhoods of e in G such that $\overline{V_{n+1}} \cdot \overline{V_{n+1}} \subset V_n \cap U_n$ and $q(V_{n+1}, V_{n+1}) \subset V_n$. Then $H = \bigcap_{n\in\mathbb{N}} V_n$ is a compact rectifiable subspace of G, $H \subset F$ and ζ is a base for G at H.*

Proof. Obviously, we have $\overline{V_{n+1}} \subset V_n$ for each $n \in \mathbb{N}$. We first claim that H is a compact rectifiable subspace of G.

Indeed, it is easy to see that $H = \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} \overline{V_n}$, and hence H is closed in G. For each $x, y \in H$, we have $x, y \in V_n$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have $x \cdot y \in V_n$ since $\overline{V_{n+1}} \cdot \overline{V_{n+1}} \subset V_n$. Therefore, $x \cdot y \in H$. Since $q(V_{n+1}, V_{n+1}) \subset V_n$, we have $q(x, y) \in H$. Therefore, H is a rectifiable closed subspace. Obviously, $H \subset \bigcap_{n\in\mathbb{N}} U_n = F$. Thus H is compact. By Lemma 4.9, ζ is a base for G at H. □

Proposition 4.11. *Let G be a rectifiable space with point-countable type. If O is an open neighborhood of e, then there exists a compact rectifiable subspace H of countable character in G satisfying $H \subset O$.*

Proof. Since G is of point-countable type, there exists a compact subset of G having a countable base in G. By the homogeneity of G, we may assume that $e \in F$. Let $\{U_n\colon n \in \mathbb{N}\}$ be a countable base for G at F. We define by induction a sequence $\{V_n\colon n \in \mathbb{N}\}$ of open neighborhoods of e in G satisfying the following conditions:

(1) $V_1 \subset O$;
(2) $\overline{V_{n+1}} \cdot \overline{V_{n+1}} \subset V_n \cap U_n$ for each $n \in \mathbb{N}$;
(3) $q(V_{n+1}, V_{n+1}) \subset V_n$ for each $n \in \mathbb{N}$.

Put $H = \bigcap_{n\in\mathbb{N}} V_n$. Then $H \subset O$ by (1). It follows from Lemma 4.10 that H is a compact rectifiable subspace of G and that $\{V_n\colon n \in \mathbb{N}\}$ is a base of G at H. □

Since each locally compact space is of point-countable type, we have the following corollary.

Corollary 4.12. *Let G be a locally compact rectifiable space. If O is an open neighborhood of e, then there exists a compact rectifiable subspace H of countable character in G satisfying $H \subset O$.*

Definition 4.13. Let X be a topological space. For $i = 1, 4$ we say that X is an α_i-*space* if for each countable family $\{S_n\colon n \in \mathbb{N}\}$ of sequences converging to some point $x \in X$ there is a sequence S converging to x such that:

(α_1) $S_n \setminus S$ is finite for all $n \in \mathbb{N}$;
(α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

Obviously, we have $\alpha_1 \Rightarrow \alpha_4$.

Let ω^ω denote the family of all functions from $\mathbb{N}$ into $\mathbb{N}$. For $f, g \in \omega^\omega$ we write $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \mathbb{N}$. A family $\mathscr{F}$ is *bounded* if there is a $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$, and is *unbounded* otherwise. We denote by $\flat$ the smallest cardinality of an unbounded family in ω^ω. It is easy to see that $\omega < \flat \leqslant \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.

Lemma 4.14. *([19]) For $i \in \{1, 4\}$, D^τ is an α_i-space if and only if $\tau < \flat$, where D is the discrete two-points space $\{0, 1\}$.*

Theorem 4.15. *The following conditions are equivalent*:

(1) *Every compact rectifiable space with the α_1-property is metrizable*;
(2) *Every locally compact rectifiable space with the α_4-property is metrizable*;
(3) $\flat = \omega_1$.

Proof. The implication (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). Since D^{ω_1} is a non-metirzable compact group, so it cannot be an α_1-space by (1). It follows from Lemma 4.14 that $\flat \leqslant \omega_1$. Since $\flat > \omega$, it follows that $\flat = \omega_1$.

(3) $\Rightarrow$ (2). Suppose that $\flat = \omega_1$, and that G is a locally compact α_4-rectifiable space. Next, we shall prove that G is metrizable. By Proposition 4.11, there exists a compact rectifiable subspace F of G which has a countable character at F in G. We claim that F is metrizable. If not, as proved V.V. Uspenskij in [22,23], compact rectifiable spaces are dyadic, and hence the space F contains a subspace homeomorphic to D^{ω_1}. Since a subspace of an α_4-space is an α_4-space, the subspace D^{ω_1} is an α_4-space. Then, it follows from Lemma 4.14 that $\omega_1 < \flat$, which is a contradiction. Therefore, the space F is metrizable.

Let $\{U_n\colon n \in \mathbb{N}\}$ be a countable base of G at F, where $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$. Let $\{V_n\colon n \in \mathbb{N}\}$ be a countable neighborhoods base at the point e in F, where the closure $\mathrm{cl}_F\, V_{n+1} \subset V_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists an open subset W_n of G such that $V_n = W_n \cap F$, $W_n \subset \overline{U_n}$ and $\overline{W_{n+1}} \subset W_n$. Put $\gamma = \{W_n\colon n \in \mathbb{N}\}$. By Lemma 4.9, the family γ is a neighborhood base in G at point e. hence the space G is first-countable, and therefore, it is metrizable. □

Corollary 4.16. *([19]) The following conditions are equivalent*:

(1) *Every compact topological group with the α_1-property is metrizable*;
(2) *Every locally compact topological group with the α_4-property is metrizable*;
(3) $\flat = \omega_1$.

Question 4.17. Let G be a locally compact rectifiable α_4-space. Is the space G an α_1-space in ZFC?

5. Rectifiable α_4-spaces

In this section, we first give a new proof of the properties of Fréchet–Urysohn and strongly Fréchet–Urysohn are coincide in rectifiable spaces, which was proved in [16].

First, we recall a concept.

(AS) For any family $\{a_{m,n}\colon (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n a_{m,n} = a \in X$ for each $m \in \mathbb{N}$, it is possible to choose two strictly increasing sequences $\{i_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ and $\{j_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_l a_{i_l,j_l} = a$. Obviously, a space with AS-property is an α_4-space.

It is well known that a topological space X is a strongly Fréchet–Urysohn space if and only if it is Fréchet–Urysohn and has the double sequence property (α_4). Therefore, it is sufficient to show that a Fréchet–Urysohn rectifiable space has the double sequence property (α_4). Indeed, we have the following result.

Lemma 5.1. *A Fréchet–Urysohn Hausdorff rectifiable space G satisfies (AS) and hence (α_4) as well.*

Proof. Assume that G is a non-discrete space. Let $\{a_{m,n}\colon (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n a_{m,n} = e$ for each $m \in \mathbb{N}$. Since G is a Fréchet–Urysohn non-discrete space, there exists a sequence $\{s_m\}m \in \mathbb{N} \subset G$ with $\lim_m s_m = e$ such that $s_m \neq e$ for each $m \in \mathbb{N}$.

Put $z_{m,k} = q(s_m, a_{m,k+m})$ if $q(s_m, a_{m,k+m}) \neq e$, and $z_{m,k} = s_m$ if $q(s_m, a_{m,k+m}) = e$. Let $M = \{z_{m,k}\colon (m,k) \in \mathbb{N} \times \mathbb{N}\}$. Obviously, we have $e \notin M$ since $s_m \neq e$ for each $m \in \mathbb{N}$. However, we have $e \in \overline{M}$. Indeed, if $M \cap \{s_m\colon m \in \mathbb{N}\}$ is infinite, then it is easy to see that $e \in \overline{M}$. Therefore, suppose that $M \cap \{s_m\colon m \in \mathbb{N}\}$ is finite. Then there is an open neighborhood U of e such that $U \cap M \cap \{s_m\colon m \in \mathbb{N}\} = \emptyset$. Let V be any open neighborhood of e with $V \subset U$. Hence there is an open neighborhood W of e such that $q(W,W) \subset V$. It follows from $\lim_m s_m = e$ that there exists an $m \in \mathbb{N}$ such that $s_m \in W$. Since $\lim_n a_{m,n} = e$, there exists a $k \in \mathbb{N}$ such that $a_{m,k+m} \in W$. Therefore, we have $q(s_m, a_{m,k+m}) = z_{m,k} \in q(W,W) \subset V \subset U$.

Since $e \in \overline{M}$ and G is Fréchet–Urysohn, we can find a sequence $\{(m_l,k_l)\}_{l\in\mathbb{N}}$ in G such that $\lim_l z_{m_l,k_l} = e$.

Case 1. The sequence $\{k_l\}_{l\in\mathbb{N}}$ is bounded.

Without loss of generality, we may assume that $k_l = r$ for each $l \in \mathbb{N}$ for some $r \in \mathbb{N}$. Since $\lim_l z_{m_l,k_l} = \lim_l z_{m_l,r} = e$ and $z_{m_l,r} \neq e$ for each $l \in \mathbb{N}$, we have $\lim_l m_l = \infty$. Without loss of generality, suppose that $m_l < m_{l+1}$ for each $l \in \mathbb{N}$. Let $N_1 = \{l \in \mathbb{N}\colon z_{m_l,r} = s_{m_l}\}$.

Subcase 1.1. The set N_1 is infinite.

We denote N_1 by $\{p_i\colon i \in \mathbb{N}\}$, where $p_i < p_{i+1}$ for each $i \in \mathbb{N}$. Then it is easy to see that $q(s_{m_{p_l}}, a_{m_{p_l},r+m_{p_l}}) = e$ for each $l \in \mathbb{N}$. Since $\lim_l s_{m_{p_l}} = e$, we have

$$a_{m_{p_l},r+m_{p_l}} = p\big(s_{m_{p_l}}, q(s_{m_{p_l}}, a_{m_{p_l},r+m_{p_l}})\big) = p(s_{m_{p_l}}, e) = s_{m_{p_l}} \to e \quad \text{as } l \to \infty.$$

Therefore, we can set $i_l = m_{p_l}$ and $j_l = r + m_{p_l}$ for each $l \in \mathbb{N}$. Then we get the strictly increasing sequences $\{i_l\}_{l\in\mathbb{N}}$ and $\{j_l\}_{l\in\mathbb{N}}$ such that $\lim_l a_{i_l,j_l} = e$.

Subcase 1.2. The set N_1 is finite.

Let $N_2 = \{l \in \mathbb{N}\colon z_{m_l,r} \neq s_{m_l}\}$. Then N_2 is infinite. We may denote N_2 by $\{q_i\colon i \in \mathbb{N}\}$, where $q_i < q_{i+1}$ for each $i \in \mathbb{N}$. It follows that

$$z_{m_{q_l},k_{q_l}} = q(s_{m_{q_l}}, a_{m_{q_l},r+m_{q_l}}) \quad \text{for each } l \in \mathbb{N}.$$

Since $\lim_l z_{m_{q_l},k_{q_l}} = e$ and $\lim_l s_{q_l} = e$, we have

$$a_{m_{q_l},r+m_{q_l}} = p\big(s_{q_l}, q(s_{m_{q_l}}, a_{m_{q_l},r+m_{q_l}})\big) = p(s_{q_l}, z_{m_{q_l},k_{q_l}}) \to p(e,e) = e \quad \text{as } l \to \infty.$$

Therefore, we can set $i_l = m_{q_l}$ and $j_l = r + m_{q_l}$ for each $l \in \mathbb{N}$. Then we get the strictly increasing sequences $\{i_l\}_{l\in\mathbb{N}}$ and $\{j_l\}_{l\in\mathbb{N}}$ such that $\lim_l a_{i_l,j_l} = e$.

Case 2. The sequence $\{k_l\}_{l\in\mathbb{N}}$ is unbounded.

Without loss of generality, we may assume that $\{k_l\}_{l\in\mathbb{N}}$ is a strictly increasing sequence.

Claim. $\lim_l m_l = \infty$.

If not, we may assume that, for each $l \in \mathbb{N}$, $m_l = t$ for some $t \in \mathbb{N}$. Since $\{k_l\}_{l\in\mathbb{N}}$ is strictly increasing, we have $\lim_l a_{t,t+k_l} = e$. It follows from $\lim_l z_{t,k_l} = e$ that

$$a_{t,t+k_l} = a_{m_l,m_l+k_l} = p(s_{m_l}, z_{m_l+k_l}) = p(s_t, z_{t,k_l}) \to p(s_t, e) = s_t \quad \text{as } l \to \infty.$$

However, $a_{t,t+k_l} \to e$ as $l \to \infty$. Hence $s_t = e$, which is a contradiction.

It follows from Claim that there exists a strictly increasing sequence $\{n_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $m_{n_i} < m_{n_{i+1}}$ for each $i \in \mathbb{N}$. Therefore, we can set $i_l = n_l$ and $j_l = m_{n_l} + k_{n_l}$ for each $l \in \mathbb{N}$. Then we get the strictly increasing sequences $\{i_l\}_{l\in\mathbb{N}}$ and $\{j_l\}_{l\in\mathbb{N}}$ such that $\lim_l a_{i_l,j_l} = e$. □

It follows from Lemma 5.1, we have the following theorem, which was proved in [16].

Corollary 5.2. *A rectifiable space G is Fréchet–Urysohn if and only if it is strongly Fréchet–Urysohn.*

Lemma 5.3. *Let G be an α_4-rectifiable space. If G is a sequential space then G is strongly Fréchet–Urysohn.*

Proof. It follows from Corollary 5.2 that it suffices to show that G is Fréchet–Urysohn. Suppose that G is non-Fréchet–Urysohn. Then there exists a subset A of G such that $\hat{\hat{A}} \setminus \hat{A} \neq \emptyset$, where the set $\hat{A}$ is all the limit points of convergent sequences in A. Take a point $x \in \hat{\hat{A}} \setminus \hat{A}$. Without loss of generality, we may assume the $x = e$.

Since $e \in \hat{\hat{A}}$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \hat{A}$ such that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to e. For each $n \in \mathbb{N}$, there exists a sequence $\{x_{nj}\}_{j=1}^{\infty} \subset A$ such that the sequence $\{x_{nj}\}_{j=1}^{\infty}$ converges to x_n. Since G is a rectifiable space, the sequence $\{q(x_n, x_{nj})\}_{j=1}^{\infty}$ converges to $q(x_n, x_n) = e$ as $j \to \infty$. Moreover, since G is an α_4-rectifiable space, there are an increasing sequence $\{n_k\}_{k=1}^{\infty}$ and a sequence $\{j(n_k)\}_{k=1}^{\infty}$ such that $\{q(x_{n_k}, x_{n_k j(n_k)})\}_{k=1}^{\infty}$ converges to e. Then we have

$$x_{n_k j(n_k)} = p\big(x_{n_k}, q(x_{n_k}, x_{n_k j(n_k)})\big) \to p(e,e) = e \quad \text{as } k \to \infty.$$

However, we have $e \notin \hat{A}$, which is a contradiction. □

It follows from Lemmas 5.1, 5.3 and Corollary 5.2 that we have the following theorem.

Theorem 5.4. *Let G be a sequential rectifiable space. Then the following conditions are equivalent*:

(1) *The space G is an α_4-space*;
(2) *The space G is an AS-space*;
(3) *The space G is Fréchet–Urysohn*;
(4) *The space G is strongly Fréchet–Urysohn.*

Corollary 5.5. *([13]) If G is a weakly first-countable rectifiable space, then G is first-countable and hence it is metrizable.*

Proof. It is well known that a weakly first-countable space is a sequential α_4-space, by Lemma 5.3, G is a Fréchet–Urysohn space, then G is first-countable since a Fréchet–Urysohn weakly first-countable space is first-countable. Hence G is metrizable. □

6. Metrizabilities of rectifiable spaces

In [16], F.C. Lin and R.X. Shen posed the following question:

Question 6.1. ([16]) Is every sequential rectifiable space with a point-countable k-network[2] a paracompact space?

In this section, we shall give a partial answer for Question 6.1. Moreover, we also discuss the metrizability of rectifiable spaces.

Let X be a space and $x \in X$. The space X has property $P(x, U)$ [21] if $U \subset X$, $\{x_i\colon i \in \mathbb{N}\} \subset U$, $x_i \to x$ as $i \to \infty$ and $x_i \neq x_j$ if $i \neq j$, then there is $\Gamma = \{x(n,k)\colon n, k \in \mathbb{N}\} \subset U$ such that $x(n,k) \to x_n$ as $k \to \infty$, $t : \mathbb{N}^2 \to \Gamma$ is a bijection, where $t(n,k) = x(n,k)$ and $\Gamma \cup \{x_i\colon i \in \mathbb{N}\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2.

Lemma 6.2. *([21]) A sequential non-Fréchet–Urysohn space with a point-countable k-network contains a closed copy of S_2.*

Lemma 6.3. *([16]) Let G be a rectifiable space. Then G contains a (closed) copy of S_ω if and only if G has a (closed) copy of S_2.*

Lemma 6.4. *Let G be a non-Fréchet–Urysohn sequential rectifiable space with point-countable k-network. Then for any $x \in G$ and any open $U \subset G$, G has the property $P(x, U)$.*

Proof. By Lemma 6.2, there exists a closed subset of G homeomorphic to S_2. It follows from Lemma 6.3, G contains a closed subset homeomorphic to S_ω. Let $S_\omega = \{y(n,k)\colon n, k \in \mathbb{N}\} \cup \{e\}$, where $y(n,k) \to e$ as $k \to \infty$ and $y(n,k) \neq y(l,m)$ if $(n,k) \neq (l,m)$. We may assume that S_ω is a closed subset of G. Let U be open in G, $\{x_i\colon i \in \mathbb{N}\} \subset U$, $x_i \to x$ as $i \to \infty$ and $x_i \neq x_j$ if $i \neq j$. For any $i, k \in \mathbb{N}$, put $x(i,k) = p(x_i, y(i,k))$. Since $y(i,k) \to e$ as $k \to \infty$, it follows that $x(i,k) \to p(x_i, e) = x_i$ as $k \to \infty$. For every $i \in \mathbb{N}$, we can choose a $k_i \in \mathbb{N}$ such that $\{x(i,k)\colon i \in \mathbb{N},\ k \geqslant k_i\} \subset U$ and $x(i,k) \neq x(i',k')$ if $(i,k) \neq (i',k')$ and $k \geqslant k_i, k' \geqslant k_{i'}$. Then $\{x_i\colon i \in \mathbb{N}\} \cup \{x(i,k)\colon i \in \mathbb{N}, k \geqslant k_i\} \cup \{x\}$ is a closed subset in G and homeomorphic to S_2. If not, there exists a sequence $\{x(i_j,k_j)\}_{j=1}^{\infty}$ converging to some point $b \in G$ such that $i_j \neq i_{j'}$ if $j \neq j'$. Therefore, we have

$$y(i_j,k_j) = q\bigl(x_{i_j}, p\bigl(x_{i_j}, y(i_j,k_j)\bigr)\bigr) = q\bigl(x_{i_j}, x(i_j,k_j)\bigr) \to q(x,b) \quad \text{as } j \to \infty.$$

However, the set $\{y(i_j,k_j)\colon j \in \mathbb{N}\}$ is closed and discrete in G, which is a contradiction. □

Lemma 6.5. *([21]) Let X be a sequential space with a point-countable k-network such that for any $x \in X$ and $U \subset X$ the property $P(x, U)$ holds. Then for any $\alpha < \omega_1$, $x \in X$, $U \subset X$ open in X the following property holds:*

$Q(\alpha, x, U)$: *If $\{x_i\colon i \in \mathbb{N}\} \subset U$, $x_i \to x$ as $i \to \infty$ then there is $Q \subset U$ such that $\overline{Q}$ is countable, $\overline{Q} \setminus \{x\} = U$, $x \in [Q]_\alpha$, $x \notin [Q]_\beta$ for each $\beta < \alpha$.*

Lemma 6.6. *([16]) Let G be a sequential rectifiable space. If G has a point-countable k-network, then G is metrizable if and only if G contains no closed copy of S_2.*

Theorem 6.7. *Let G be a sequential rectifiable space with a point-countable k-network. If $so(G) < \omega_1$, then G is metrizable.*

Proof. Suppose that $so(G) = \alpha$.

Claim. *The space G is Fréchet–Urysohn.*

Suppose not, it follows from Lemmas 6.4 and 6.5 that G has property $Q(\alpha+1, e, G)$. Clearly, since G has the property $Q(\alpha+1, e, G)$, we have $so(G) \geqslant \alpha + 1 > \alpha$, which is a contradiction.

It follows from the claim that G is a Fréchet–Urysohn rectifiable space, and hence G contains no closed copy of S_2. Since G is a Fréchet–Urysohn rectifiable space with a point-countable k-network, the space G is metrizable by Lemma 6.6. □

Proposition 6.8. *Let $\mathcal{P}$ be a topological property that is productive and preserved by continuous maps. Then the following are equivalent for a rectifiable space G.*

(i) *Every subset with the property $\mathcal{P}$ of G has countable pseudocharacter.*
(ii) *Every subset with the property $\mathcal{P}$ of G has regular G_δ-diagonal.*[3]

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[2] Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *k-network* [18] if whenever K is a compact subset of X and $K \subset U \in \tau(X)$, there is a finite subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \bigcup \mathscr{P}' \subset U$.

[3] A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x,x)\colon x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$.

Proof. (ii) $\to$ (i) obvious.

(i) $\to$ (ii). Let A be a subset of G and have the property $\mathcal{P}$. Since $q: G \times G \to G$ is continuous and the property $\mathcal{P}$ is productive and preserved by continuous maps, then $q(A \times A) = B$ is a subset of G and B has the property $\mathcal{P}$. Then $e \in B$ since $q(x, x) = e$. Therefore, e is a G_δ-set of B, let $\{U_n: n \in \mathbb{N}\}$ be a family of countable open subsets with $e \in U_n$ and $\overline{U_{n+1}} \subset U_n$. Then $\Delta = \{(x, x): x \in A\} = \bigcap_{n\in\mathbb{N}} q^{-1}(U_n) = \bigcap_{n\in\mathbb{N}} \overline{q^{-1}(U_n)}$. In fact, let $(x, y) \in \bigcap_{n\in\mathbb{N}} q^{-1}(U_n)$. For each $n \in \mathbb{N}$, we have $(x, y) \in q^{-1}(U_n)$, and hence $q(x, y) \in U_n$, which follows that $\pi_2(\varphi(x, y)) \in U_n$, $\pi_2(x, y') \in U_n$, where $\varphi(x, y) = (x, y')$ and $y' \in U_n$ for each $n \in \mathbb{N}$. Therefore, $y' = e$. Since $\varphi(x, x) = e$, $\varphi(x, y) = e$ and φ is one-to-one, we have $x = y$. Therefore $\Delta = \bigcap_{n\in\mathbb{N}} q^{-1}(U_n)$. Since $\overline{q^{-1}(U_{n+1})} \subset q^{-1}(\overline{U_{n+1}}) \subset q^{-1}(U_n)$, then we can see that A has a regular G_δ-diagonal. $\square$

It is well known that a countably compact (compact) space with a G_δ-diagonal is metrizable, we have the following.

Corollary 6.9. *The following are equivalent for a rectifiable space G.*

(i) *Every compact* (*countably compact*) *subset is first-countable.*
(ii) *Every compact* (*countably compact*) *subset is metrizable.*

Corollary 6.10. *Let G be a rectifiable space of countable pseudocharacter. Then G has a regular G_δ-diagonal.*

7. Compactifications of rectifiable spaces

In this section, we assume that all spaces are Tychonoff.

Note that a rectifiable space is metrizable if its π-character is countable [13], by the same proof of [17, Lemma 2], we can prove the following.

Lemma 7.1. *Let G be a non-locally compact rectifiable space. If for each $y \in Y = bG \setminus G$, there exists an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is metrizable and $\pi\chi(U(y)) \leqslant \omega$, then G is metrizable and locally separable.*

A space X is called having the property $(*)$: if the cardinality of X is Ulam non-measure, then X is weakly HN-complete.[4] A paracompact space has the property $(*)$ since a paracompact space with Ulam non-measurable cardinality is HN-complete [6,11], and hence it is weakly HN-complete.

Proposition 7.2. *Let G be a non-locally compact rectifiable space with property $(*)$. If for each $y \in Y = bG \setminus G$, there exists an open neighborhood $U(y)$ of y such that* (i) *every compact subset of $U(y)$ is a G_δ-subset of $U(y)$;* (ii) *every countably compact or Lindelöf p-subspace of $U(y)$ is metrizable. Then G, bG are separable and metrizable.*

Proof. From condition (ii), we can see that Y is not locally countably compact, otherwise G is closed in bG and is compact.

By [5, Theorem 3.1], Y is pseudocompact or Lindelöf.

Case 1. The space Y is pseudocompact. Then Y is first-countable since each singleton of Y is a G_δ-set. Since Y is not locally countably compact, the rectifiable space G is locally separable and metrizable by Lemma 7.1. Y is Lindelöf [14] since G is of countable type. Therefore, Y is compact, and hence G is locally compact, which is a contradiction.

Case 2. The space Y is Lindelöf. Since Y is a space of countable pseudocharacter, it follows that the cardinality of Y is Ulam non-measurable [6]. The space G is not locally compact, then G is nowhere locally compact since G is homogeneous. It follows that G is a remainder of Y, so the cardinality of G is also Ulam non-measurable [6]. Then G is weakly HN-complete. By [4, Theorem 4], each G_δ-point of Y is a point of bisequentiality of Y, it follows that $\pi\chi(Y) \leqslant \omega$. Therefore, G is locally separable and metrizable by Lemma 7.1. We write $G = \bigoplus_{\alpha\in A} G_\alpha$, where G_α is a separable metrizable subset for each $\alpha \in A$. Let $\eta = \{G_\alpha: \alpha \in A\}$, and let F be the set of all points of bG at which η is not locally finite. Since η is discrete in X. Then $F \subset bG \setminus G$. It is easy to see that F is compact, we can find finitely many closed neighborhoods that satisfy (ii) to cover F, hence F is separable and metrizable, thus F has a countable network. Put $M = Y \setminus F$. For each point $y \in M$, there is an open neighborhood O_y satisfying (ii) in bG such that $\overline{O_y} \cap F = \emptyset$. Since η is discrete, the set $\overline{O_y}$ meets at most finitely many G_α. Let $L = \bigcup\{G_\alpha: G_\alpha \cap \overline{O_y} \neq \emptyset\}$. Then L is separable metrizable. It follows that $\bar{L} \setminus L$ is a remainder of L, and hence it is a Lindelöf p-space by [2, Theorem 2.1]. $Cl_Y(O_y) \subset \bar{L} \setminus L$, then $Cl_Y(O_y)$ is a Lindelöf p-space, hence it is separable and metrizable and $Y \setminus F$ is locally separable metrizable. Since F is compact, there are finite many $\{U(y_i): i \leqslant k\}$ that satisfy (i) and cover F. Moreover, since each compact subset of $U(y_i) (i \leqslant k)$ is a G_δ-set, the set F is a G_δ-set in $\bigcup\{U(y_i): i \leqslant k\}$. We write $F = \bigcap V_n$ with V_n open in Y and $Cl_Y(V_{n+1}) \subset V_n$. Let $K_1 = Y \setminus V_1$, $K_n = Cl(V_{n-1}) \setminus V_n$ $(n > 1)$. Since Y is Lindelöf

[4] A space X is *weakly HN-complete* if the remainder Z of X in the Čech–Stone compactification βX of X is a space of point-countable type.

and K_n is closed in Y, each K_n is Lindelöf and locally separable metrizable. Therefore, K_n has a countable base for each n. Since $Y = F \cup (\bigcup\{K_n \colon n \in \mathbb{N}\})$, it follows that Y has a countable network. Then $c(Y) \leqslant \omega$, hence $c(G) \leqslant \omega$. Since G is a metrizable space with countable Souslin number, the space G is separable and metrizable. It follows that bG is separable and metrizable since G and Y both have countable networks. □

Recall that a space X has a *quasi-G_δ-diagonal* provided there is a sequence $\{\mathcal{G}(n) \colon n \in \mathbb{N}\}$ of collections of open sets with property that, given distinct points $x, y \in X$, there is some n with $x \in st(x, \mathcal{G}(n)) \subset X \setminus \{y\}$. Obviously, "$X$ has a G_δ-diagonal" implies "X has a quasi-G_δ-diagonal".

Theorem 7.3. *Let G be a non-locally compact, paracompact rectifiable space, and $Y = bG \setminus G$ have locally quasi-G_δ-diagonal. Then G and bG are separable and metrizable.*

Proof. By [7, Proposition 2.3], for $y \in Y$, there exists an open neighborhood $U(y)$ such that each compact subset of $U(y)$ is a G_δ-set and every countably compact subset of $U(y)$ is metrizable. Moreover, every Lindelöf p-subspace of $U(y)$ is metrizable by [15, Corollary 3.6]. Then G and bG are separable and metrizable by Proposition 7.2. □

Corollary 7.4. *([5]) Let G be a non-locally compact, paracompact rectifiable space, and $Y = bG \setminus G$ have a G_δ-diagonal. Then G and bG are separable and metrizable.*

Proposition 7.5. *Let G be a non-locally compact rectifiable space. If for each $y \in Y = bG \setminus G$, there exists an open neighborhood $U(y)$ of y such that* (i) $\pi\chi(U(y)) \leqslant \omega$; (ii) *every countably compact or Lindelöf p-subspace of $U(y)$ is metrizable*; (iii) *every compact subset of $U(y)$ is a G_δ-set of $U(y)$. Then G, bG are separable and metrizable.*

Proof. By Lemma 7.1, G is metrizable and locally separable. Similar to the proof of Proposition 7.2, G and bG are separable and metrizable. □

A space with point-countable base satisfies (i), (ii) [12, Corollary 7.11(ii)] and (iii) in Proposition 7.5.

Corollary 7.6. *Let G be a non-locally compact rectifiable space, and $Y = bG \setminus G$ have locally point-countable base. Then G and bG are separable and metrizable.*

By [7, Proposition 2.1] and [12, Corollary 8.3(ii)], a space with a $\delta\theta$-base[5] satisfies (i), (ii) and (iii) in Proposition 7.5.

Corollary 7.7. *Let G be a non-locally compact rectifiable space, and $Y = bG \setminus G$ have locally $\delta\theta$-base. Then G and bG are separable and metrizable.*

By [12, Corollary 10.7(ii), Theorem 10.6], a γ-space[6] satisfies (i), (ii) and (iii) in Proposition 7.5.

Corollary 7.8. *Let G be a non-locally compact rectifiable space, and $Y = bG \setminus G$ be a locally γ-space. Then G and bG are separable and metrizable.*

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References

[1] A.V. Arhangel'skiĭ, Mappings and spaces, Russian Math. Surveys 21 (1966) 115–162.
[2] A.V. Arhangel'skiĭ, Remainders in compactifications and generalized metrizability properties, Topology Appl. 150 (2005) 79–90.
[3] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., 2008.
[4] A.V. Arhangel'skiĭ, G_δ-points in remainders of topological groups and some addition theorems in compacta, Topology Appl. 156 (2009) 2013–2018.
[5] A.V. Arhangel'skiĭ, M.M. Choban, On remainders of rectifiable spaces, Topology Appl. 157 (2010) 789–799.
[6] A.V. Arhangel'skiĭ, V. Ponomarev, Fundamentals of General Topology in Problems and Exercises, Reidel, Dordrecht, 1984 (translated from Russian).
[7] H. Bennett, R. Byerly, D. Lutzer, Compact G_δ-sets, Topology Appl. 153 (2006) 2169–2181.

[5] A *$\delta\theta$-base* for a space X is a base $\mathcal{B} = \bigcup\{\mathcal{B}(n) \colon n \geqslant 1\}$ with the additional property that U is open and $x \in U$, then there is some $n = n(x, U)$ with properties that (a) some $B \in \mathcal{B}(n)$ has $x \in B \subset U$, and (b) $\mathrm{ord}(x, \mathcal{B}(n)) \leqslant \omega$.

[6] A space (X, τ) is a *γ-space* if there exists a function $g : \omega \times X \to \tau$ such that (i) $\{g(n, x) \colon n \in \omega\}$ is a base at x; (ii) for each $n \in \omega$ and $x \in X$, there exists $m \in \omega$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$.

[8] T. Banakh, On topological groups containing a Fréchet–Urysohn fan, Matematychni Studii 9 (1998) 149–154.
[9] M.M. Čoban, On topological homogeneous algebras, in: Interim Reports of Prague Topol. Symp. II, Prague, 1987, pp. 25–26.
[10] M.M. Čoban, The structure of locally compact algebras, Serdica 18 (1992) 129–137.
[11] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[12] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.
[13] A.S. Gul'ko, Rectifiable spaces, Topology Appl. 68 (1996) 107–112.
[14] M. Henriksen, J. Isbell, Some properties of compactifications, Duke Math. J. 25 (1958) 83–106.
[15] R. Hodel, Metrizability of topological spaces, Pacific J. Math. 55 (1974) 441–459.
[16] F.C. Lin, R.X. Shen, On rectifiable spaces and paratopological groups, Topology Appl. 158 (4) (2011) 597–610.
[17] C. Liu, Remainders in compactification of topological groups, Topology Appl. 156 (2009) 849–854.
[18] P. O'Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc. 29 (1971) 183–189.
[19] T. Nogura, D. Shakhmatov, Amalgamation of convergent sequences in locally compact groups, C. R. Acad. Sci. Paris, Ser. 1 320 (1995) 1349–1354.
[20] E. Pentsak, On manifolds modeled on direct limits of C-universal *ANR*'s, Matematychni Studii 5 (1995) 107–116.
[21] A. Shibakov, Metrizability of sequential topological groups with point-countable k-networks, Proc. Amer. Math. Soc. 126 (1998) 943–947.
[22] V.V. Uspenskij, The Mal'tsev operation on countably compact spaces, Comment. Math. Univ. Carol. 30 (1989) 395–402.
[23] V.V. Uspenskij, Topological groups and Dugundji compacta, Mat. Sb. 180 (8) (1989) 1092–1118 (in Russian); English transl. in: Math. USSR Sb. 67 (2) (1990) 555–580.