Studia Scientiarum Mathematicarum Hungarica 48 (3), 320–330 (2011)
DOI: 10.1556/SScMath.48.2011.3.1171

INSERTIONS OF k-SEMI-STRATIFIABLE SPACES BY SEMI-CONTINUOUS FUNCTIONS

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Communicated by I. Juhász

(Received July 4, 2009; accepted December 2, 2009)

Abstract

In this paper, we characterize k-semi-stratifiable spaces by semi-continuous functions and give some applications. Also we give the similar characterizations of MCM spaces and K-MCM spaces.

1. Introduction

One of the questions in general topology is how to characterize the generalized metric spaces [3, 6]. Recently, the problem of monotone insertions of generalized metric spaces is studied [5]. Lane, Nyikos and Pan [8] proved that a topological space X is stratifiable if and only if there is an order-preserving map $\psi : UL(X) \to C(X)$ such that for any $(g, h) \in UL(X)$, $g \leqq \psi\big((g, f)\big) \leqq h$ and $g(x) < \psi\big((g, h)\big)(x) < h(x)$ whenever $g(x) < h(x)$. Yang and Yan [14] proved that a topological space X is semi-stratifiable if and only if there is an order-preserving map $\varphi : LSC(X) \to USC(X)$ such that for any $h \in LSC(X)$, $0 \leqq \varphi(h) \leqq h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$. It is well-known that stratifiable spaces are k-semi-stratifiable, and k-semi-stratifiable spaces are semi-stratifiable.

2000 *Mathematics Subject Classification.* Primary 54E20, 54C08; Secondary 26A15.

Key words and phrases. k-semi-stratifiable spaces, order-preserving maps, upper semi-continuous, lower semi-continuous, g-functions.

The project is supported by NNSF (No. 10971185, No. 10971186) and FNSF (2009J01013) of China.

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QUESTION 1.1. *How to characterize k-semi-stratifiable spaces by the monotone insertion functions?*

In Section 2, we describe the characterizations of k-semi-stratifiable spaces similar to that of stratifiable spaces in [8] and semi-stratifiable spaces in [14]. We then give some applications of this result.

MCM spaces and K-MCM spaces are natural generalizations of semi-stratifiable spaces and k-semi-stratifiable spaces, respectively. As the complements of our main results, we also characterize them by semi-continuous functions in Section 3.

All spaces in this paper are assumed to be T_1. For a topological space X, τ denotes the topology on X, and $\tau^c = \{X - O : O \in \tau\}$. We refer the reader to [1, 6] for undefined terms.

A real-valued function f defined on a space X is *lower (upper) semi-continuous* if for each $x \in X$ and each real number r with $f(x) > r$ $\big(f(x) < r\big)$, there exists a neighborhood U of x in X such that $f(x') > r$ $\big(f(x') < r\big)$ for every $x' \in U$. Equivalently, f is lower (upper) semi-continuous if and only if $\{x : f(x) > r\}$ $(\{x : f(x) < r\})$ is open for each real number r. We write $LSC(X)$ $\big(USC(X)\big)$ for the set of all real-valued lower (upper) semi-continuous functions on X into $I = [0, 1]$.

Let X be a space, if $A \subset X$, we write χ_A for the *characteristic function* on A, that is, a function $\chi_A : X \to [0, 1]$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then $\chi_A \in USC(X)$ if A is a closed subset of X, and $\chi_A \in LSC(X)$ if A is an open subset in X.

2. On k-semi-stratifiable spaces

DEFINITION 2.1 (see [10]). A space X is *k-semi-stratifiable* if and only if there exists a map $\rho : \mathbb{N} \times \tau \to \tau^c$ satisfying the following conditions:

(1) $U = \bigcup_{n \in \mathbb{N}} \rho(n, U)$ for all $U \in \tau$;

(2) If $U \subset V$ with $U, V \in \tau$, then $\rho(n, U) \subset \rho(n, V)$ for all $n \in \mathbb{N}$;

(3) For each compact subset $K \subset U \in \tau$, $K \subset \rho(m, U)$ for some $m \in \mathbb{N}$.

In Definition 2.1, we can assume the map ρ satisfies the following condition: $\rho(n, U) \subset \rho(n + 1, U)$ for all $U \in \tau$ and all $n \in \mathbb{N}$. For a space X, if there is a map $\rho : \mathbb{N} \times \tau \to \tau^c$ satisfying the conditions (1) and (2) in Definition 2.1, X is called a *semi-stratifiable space* [6].

First, we give a new characterization of k-semi-stratifiable spaces by partially ordered sets.

Lemma 2.2. *A space X is k-semi-stratifiable if and only if for any partially ordered set $(H, \leqq)$ and any map $F : \mathbb{N} \times H \to \tau^c$ satisfying the following conditions* (1) *and* (2)*:*

(1) $F(n+1, h) \subset F(n, h)$ *for all* $h \in H$ *and all* $n \in \mathbb{N}$,

(2) *For any* $h_1, h_2 \in H$, *if* $h_1 \leqq h_2$, *then* $F(n, h_2) \subset F(n, h_1)$,

there is a map $G : \mathbb{N} \times H \to \tau$ *such that* (1) *and* (2) *above hold for* G, $F(n, h) \subset G(n, h)$ *for all* $n \in \mathbb{N}$, $h \in H$, *and* (3) *hold:*

(3) *If* K *is compact and* $K \cap F(n, h) = \emptyset$ *for some* $n \in \mathbb{N}$ *and* $h \in H$, *then* $K \cap G(m, h) = \emptyset$ *for some* $m \in \mathbb{N}$.

Proof. Necessity. Suppose that X is a k-semi-stratifiable space and $F : \mathbb{N} \times H \to \tau^c$ is a map which satisfies the conditions (1) and (2) of the lemma. Let $\rho : \mathbb{N} \times \tau \to \tau^c$ be the map in Definition 2.1, which satisfies $\rho(n, U) \subset \rho(n+1, U)$ for all $U \in \tau$, $n \in \mathbb{N}$. We shall show that a map $G : \mathbb{N} \times H \to \tau$ defined by

$$G(n, h) = X - \rho\big(n, X - F(n, h)\big), \quad n \in \mathbb{N}, \quad h \in H$$

is desired. By the properties of ρ and F, one can easily verify that (1) and (2) hold for G. By (1) of Definition 2.1, $\rho(n, U) \subset U$ for all $U \in \tau$. Thus $\rho\big(n, X - F(n, h)\big) \subset X - F(n, h)$, and then $F(n, h) \subset G(n, h)$ for all $n \in \mathbb{N}$ and $h \in H$.

Next, we shall show that (3) hold. Let K be a compact subset of X and $K \cap F(m_0, h) = \emptyset$ for some $m_0 \in \mathbb{N}$ and $h \in H$. By Definition 2.1,

$$X - F(m_0, h) = \bigcup_{n \in \mathbb{N}} \rho\big(n, X - F(m_0, h)\big)$$

and

$$K \subset \rho\big(k, X - F(m_0, h)\big) \quad \text{for some} \quad k \in \mathbb{N}.$$

Let $m = \max\{m_0, k\}$, then

$$K \subset \rho\big(k, X - F(m_0, h)\big) \subset \rho\big(m, X - F(m, h)\big) = X - G(m, h).$$

Hence $K \cap G(m, h) = \emptyset$.

Sufficiency. Let $H = \tau$ and define a partial order $\leqq$ on H by $h_1 \leqq h_2 \Leftrightarrow h_1 \subset h_2$ for each pair $h_1, h_2 \in H$. For each $U \in \tau$, we define that $F(n, U) = X - U$, then the map $F : \mathbb{N} \times \tau \to \tau^c$ satisfies (1) and (2) of the lemma. So there exists a map $G : \mathbb{N} \times \tau \to \tau$ such that (1) $\sim$ (3) hold for G, and $F(n, U) \subset G(n, U)$ for all $n \in \mathbb{N}$ and all $U \in \tau$. Let $\rho(n, U) = X - G(n, U)$, then the map $\rho : \mathbb{N} \times \tau \to \tau^c$ satisfies the conditions (1) and (2) in Definition 2.1. For a compact subset $K \subset U \in \tau$, $K \cap F(n, U) = K - U = \emptyset$, then $K \subset \rho(m, U)$ for some $m \in \mathbb{N}$ by (3). Hence X is a k-semi-stratifiable space. $\square$

In the necessity of Lemma 2.2, it can be obtained that

$$\bigcap_{n\in\mathbb{N}} F(n,h) = \bigcap_{n\in\mathbb{N}} G(n,h) \text{ for all } h \in H.$$

In fact, since $F(n, h) \subset G(n, h)$ for all $n \in \mathbb{N}$ and $h \in H$, $\bigcap_{n\in\mathbb{N}} F(n, h) \subset \bigcap_{n\in\mathbb{N}} G(n, h)$. On the other hand, if a point $x \notin \bigcap_{n\in\mathbb{N}} F(n, h)$, then $\{x\} \cap \big(\bigcap_{n\in\mathbb{N}} F(n, h)\big) = \emptyset$, thus there exists an $m \in \mathbb{N}$ such that $\{x\} \cap G(m, h) = \emptyset$ by (3), i.e., $x \notin \bigcap_{n\in\mathbb{N}} G(n, h)$.

Let $(X, <)$ and $(Y, <')$ be partially ordered sets. A map $\psi : X \to Y$ is said to be *order-preserving* [8] if $\psi(x) <' \psi(y)$ for every pair $x, y \in X$ with $x < y$.

THEOREM 2.3. *A space X is k-semi-stratifiable if and only if there is an order-preserving map $\varphi : LSC(X) \to USC(X)$ satisfying the following conditions:*

(1) *For any $h \in LSC(X)$, $0 \leqq \varphi(h) \leqq h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$;*

(2) *If K is a compact subset of X and there exists $h \in LSC(X)$ such that $h(x) > 0$ for all $x \in K$, then $\varphi(h)(x) > r$ for all $x \in K$, for some $r > 0$.*

PROOF. Suppose X is k-semi-stratifiable. Define a map $F : \mathbb{N} \times LSC(X) \to \tau^c$ by

$$F(n,h) = \left\{x \in X : h(x) \leqq 1/2^{n-1}\right\}, \quad n \in \mathbb{N},\ h \in LSC(X),$$

Then F satisfies (1) and (2) in Lemma 2.2. By Lemma 2.2, there exists a map $G : \mathbb{N} \times LSC(X) \to \tau$ such that (1) $\sim$ (3) in Lemma 2.2 hold for G, and $F(n, h) \subset G(n, h)$ for each $n \in \mathbb{N}$, $h \in LSC(X)$. Thus

$$\bigcap_{n\in\mathbb{N}} G(n,h) = \bigcap_{n\in\mathbb{N}} F(n,h) = \left\{x \in X : h(x) = 0\right\}.$$

Let $f(n, h) = \chi_{G(n,h)}$ for each $n \in \mathbb{N}$, $h \in LSC(X)$. Then $f(n, h) \in LSC(X)$ since $G(n, h)$ is open. Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(n,h) \in LSC(X)$$

by Theorem 2.4 in [14]. Let

$$\varphi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} f(n,h)(x),$$

then $\varphi(h) \in USC(X)$. Suppose that $h_1 \leqq h_2$ with $h_1, h_2 \in LSC(X)$ and $n \in \mathbb{N}$, then $G(n,h_2) \subset G(n,h_1)$, so $\chi_{G(n,h_2)} \leqq \chi_{G(n,h_1)}$, i.e., $f(n,h_2) \leqq f(n,h_1)$. Therefore, $\varphi(h_1) \leqq \varphi(h_2)$ by the definition of φ. Hence φ is order-preserving.

It remains to show that the map φ satisfies the conditions (1) and (2) in Theorem.

Let $h \in LSC(X)$. If a point $x \in X$ with $h(x) = 0$, then for each $n \in \mathbb{N}$, $x \in G(n,h)$ and so $f(n,h)(x) = 1$, thus $\varphi(h)(x) = 0$. If $h(x) > 0$, then $x \notin \bigcap_{n\in\mathbb{N}} G(n,h)$. Let

$$m = \min\left\{ n \in \mathbb{N} : \, x \notin G(n,h) \right\}.$$

Then for all $n < m$, $x \in G(n,h)$ and so $f(n,h)(x) = 1$. But $x \notin G(m,h)$, then $f(m,h)(x) = 0$ and so $\varphi(h)(x) > 0$. On the other hand, $x \notin F(m,h)$ since $F(m,h) \subset G(m,h)$, thus $h(x) > 1/2^{m-1}$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2^n} f(n,h)(x) &= \sum_{n=1}^{m-1} \frac{1}{2^n} f(n,h)(x) + \sum_{n=m}^{\infty} \frac{1}{2^n} f(n,h)(x) \\ &= 1 - \frac{1}{2^{m-1}} + \sum_{n=m}^{\infty} \frac{1}{2^n} f(n,h)(x).\end{aligned}$$

Consequently, $0 < \varphi(h)(x) \leqq 1/2^{m-1} < h(x)$ by the definition of φ.

Let K be a compact subset of X, and suppose that there exists an $h \in LSC(X)$ such that $h(x) > 0$ for all $x \in K$. Then $K \cap \left(\bigcap_{n\in\mathbb{N}} F(n,h)\right) = \emptyset$ by the definition of $F(n,h)$. By (3) of Lemma 2.2, there exists $m \in \mathbb{N}$ such that $K \cap G(m,h) = \emptyset$. Thus $f(m,h)(x) = 0$ for all $x \in K$, so $f(n,h)(x) = 0$ for all $n \geqq m$ and all $x \in K$. Take

$$2r = 1 - \sum_{n=1}^{m-1} \frac{1}{2^n}.$$

Then $r > 0$ and for each $x \in K$,

$$\varphi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} f(n,h)(x) = 1 - \sum_{n=1}^{m-1} \frac{1}{2^n} f(n,h)(x)$$
$$\geqq 1 - \sum_{n=1}^{m-1} \frac{1}{2^n} > r.$$

Conversely, suppose that there is an order-preserving map $\varphi : LSC(X) \to USC(X)$ that satisfies the conditions (1) and (2) in Theorem. For any $U \in \tau$, let $h_U = \chi_U$, then $h_U \in LSC(X)$ and so $\varphi(h_U) \in USC(X)$. For each $n \in \mathbb{N}$, let

$$\rho(n,U) = \left\{ x \in X : \varphi(h_U)(x) \geqq 1/2^n \right\},$$

then the equality above defines a map $\rho : \mathbb{N} \times \tau \to \tau^c$. We shall show that the map ρ satisfies the conditions (1) $\sim$ (3) in Definition 2.1. It can be easily checked that the map satisfies (2) in Definition 2.1. We now show that the map also satisfies (1) and (3) in Definition 2.1.

For each $n \in \mathbb{N}$, if $x \in \rho(n,U)$, then $1/2^n \leqq \varphi(h_U)(x) \leqq h_U(x)$. So

$$\chi_U(x) = h_U(x) \geqq 1/2^n > 0,$$

thus $x \in U$. This implies that $\bigcup_{n\in\mathbb{N}} \rho(n,U) \subset U$. On the other hand, if $x \in U$, then $h_U(x) = \chi_U(x) = 1 > 0$, and so $\varphi(h_U)(x) > 0$. There exists $m \in \mathbb{N}$ such that $\varphi(h_U)(x) \geqq 1/2^m$, hence $x \in \rho(m,U)$. Therefore $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$.

Suppose a compact subset $K \subset U \in \tau$ in X, let $h_U = \chi_U$, then $h_U \in LSC(X)$ and $h_U(x) = 1$ for all $x \in K$. By (2), there exists $m_0 \in \mathbb{N}$ such that $\varphi(h_U)(x) > 1/m_0$ for all $x \in K$, then $K \subset \rho(m_0, U)$. Hence ρ satisfies all conditions in Definition 2.1, so X is k-semi-stratifiable. □

Another characterization of k-semi-stratifiable spaces by semi-continuous functions is given in [13].

Corollary 2.4. *A space X is k-semi-stratifiable if and only if for each pair (A,U) of subsets of X with A closed, U open and $A \subset U$, there is $f_{U,A} \in LSC(X)$ such that*

(1) $A = f_{U,A}^{-1}(0)$, $X - U = f_{U,A}^{-1}(1)$;

(2) $f_{U,A} \geqq f_{V,B}$ *whenever* $A \subset B$ *and* $U \subset V$;

(3) *If a compact subset $K \subset U \in \tau$ in X, then $f_{U,\emptyset}(x) < r$ whenever $x \in K$ for some $r < 1$.*

PROOF. Suppose X is a k-semi-stratifiable space, there is an order-preserving map $\varphi : LSC(X) \to USC(X)$ that satisfies the conditions of Theorem 2.3. For each pair (A, U) of subsets of X with A closed, U open and $A \subset U$, let

$$g_A = 1 - \chi_A \quad \text{and} \quad h_U = \varphi(\chi_U).$$

Then $g_A \in LSC(X)$ and $h_U \in USC(X)$. For each $x \in X$, put

$$f_{U,A}(x) = \frac{g_A(x)}{1 + h_U(x)},$$

then $f_{U,A} \in LSC(X)$ by Proposition 2.2 in [14]. One can easily check that the conditions (1) and (2) hold.

Let K be a compact subset of X such that $K \subset U \in \tau$. Then $\chi_U \in LSC(X)$ and $\chi_U(x) > 0,\ x \in K$. Thus $\varphi(\chi_U)(x) > t$ whenever $x \in K$ for some $t > 0$ by (2) in Theorem 2.3. Let $r = \frac{1}{1+t}$, then $r < 1$ and $f_{U,\emptyset}(x) < r$ whenever $x \in K$.

Conversely, let $h_U = 1 - f_{U,\emptyset}$, $U \in \tau$. Then $h_U \in USC(X)$. It can easily be verified that $h_U \leqq h_V$ whenever $U \subset V \in \tau$, and $h_U(x) = 0$ if and only if $x \notin U$. For each $n \in \mathbb{N}$, let

$$\rho(n, U) = \left\{ x \in X : h_U(x) \geqq 1/2^n \right\}.$$

Similar to the proof of the sufficiency in Theorem 2.3, we can prove that ρ satisfies the conditions of Definition 2.1. Hence X is k-semi-stratifiable. □

COROLLARY 2.5. *A space X is k-semi-stratifiable if and only if for each $U \in \tau$ in X there is $f_U \in USC(X)$ such that*

(1) $X - U = f_U^{-1}(0)$;

(2) $f_U \leqq f_V$ *whenever* $U \subset V$;

(3) *If a compact subset $K \subset U \in \tau$, then $f_U(x) > r$ whenever $x \in K$ for some $r > 0$.*

PROOF. Suppose X is k-semi-stratifiable. For each $U \in \tau$, let $f_U = 1 - f_{U,\emptyset}$, where $f_{U,\emptyset}$ is the function given in Corollary 2.4. Obviously, $f_U \in USC(X)$ satisfies the conditions $(1) \sim (3)$. The sufficiency has been proved in Corollary 2.4. □

Before we go to the next corollary recall that a *g-function* [9] for a topological space X is a function $g : \mathbb{N} \times X \to \tau$ such that for each $x \in X$ and each $n \in \mathbb{N}$, $x \in g(n, x)$, $g(n+1, x) \subset g(n, x)$.

COROLLARY 2.6 (see [2]). *A Hausdorff space X is k-semi-stratifiable if and only if it has a g-function g such that if $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $y_n \to x$.*

PROOF. The proof of the sufficiency is given in [2]. We need only to prove the necessity. Suppose X is k-semi-stratifiable. For each $x \in X$, let

$$f_x = \varphi\big(1 - \chi_{\{x\}}\big), \text{ and } g(n,x) = \big\{ t \in X : f_x(t) < 1/n \big\},$$

where φ is the map given in Theorem 2.3. We show that g is the desired g-function.

Let $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $x_n \to x$. For each $U \in \tau$, $x \in U$, there is an $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geqq n_0$. Since $x \notin X - U$, then $\varphi\big(1 - \chi_{(X-U)}\big)(x) > 0$ and so $\varphi\big(1 - \chi_{(X-U)}\big)(x) > 1/m_1$ for some $m_1 \in \mathbb{N}$. Let $K = \{x\} \cup \{x_n : n \geqq n_0\}$. Then K is a compact subset of X and $K \subset U$. Thus $\big(1 - \chi_{(X-U)}\big)(t) > 0$ for all $t \in K$. By (2) in Theorem 2.3, there is an $m_0 \in \mathbb{N}$ such that $\varphi\big(1 - \chi_{(X-U)}\big)(t) > 1/m_0$ for all $t \in K$. For each $y \in X - U$, since $\chi_{\{y\}} \leqq \chi_{(X-U)}$, we have $1 - \chi_{\{y\}} \geqq 1 - \chi_{(X-U)}$. Then $f_y = \varphi\big(1 - \chi_{\{y\}}\big) \geqq \varphi\big(1 - \chi_{(X-U)}\big)$ since φ is order-preserving. Thus

$$f_y(t) \geqq \varphi\big(1 - \chi_{(X-U)}\big)(t) > 1/m_0$$

for all $t \in K$. Let $m = \max\{n_0, m_0\}$, $f_{y_n}(x_n) < 1/n$ by $x_n \in g(n, y_n)$ for each $n \geqq m$, then $y_n \in U$. Therefore $y_n \to x$. □

3. On MCM spaces and K-MCM spaces

Let $\{A_n\}$ and $\{B_n\}$ be two sequences of subsets of a space X. For convenience, we write $\{A_n\} \prec \{B_n\}$ if $A_n \subset B_n$ for every $n \in \mathbb{N}$, $LSC^+(X) = \big\{ f \in LSC(X) : f > 0 \big\}$; $USC^+(X) = \big\{ f \in USC(X) : f > 0 \big\}$.

DEFINITION 3.1. A space X is said to be *MCM* [4] if there is an operator U assigning to each decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection, a sequence $\big\{U\big(n, \{F_n\}\big)\big\}$ of open subsets of X such that

(i) $\{F_n\} \prec \big\{U\big(n, \{F_n\}\big)\big\}$,

(ii) $\bigcap_{n\in\mathbb{N}} U\big(n, \{F_n\}\big) = \emptyset$,

(iii) if $\{F_n\} \prec \{F'_n\}$, then $\big\{U\big(n, \{F_n\}\big)\big\} \prec \big\{U\big(n, \{F'_n\}\big)\big\}$.

X is said to be *K-MCM* [11] if, in addition,

(ii)$'$ for each compact subset $K \subset X$, $K \cap U\big(n, \{F_n\}\big) = \emptyset$ for some $n \in \mathbb{N}$.

The operation U is called an *MCM operator (K-MCM operator)* if it satisfies the respective conditions.

REMARK 3.2 (see [12]). MCM spaces are equivalent to β-spaces which are defined in [7]. Every k-semi-stratifiable space is K-MCM, and every K-MCM space is MCM.

THEOREM 3.3. *A space X is MCM if and only if there is an order-preserving map $\varphi : LSC^+(X) \to USC^+(X)$ satisfying that $\varphi(h) < h$ for each $h \in LSC^+(X)$.*

PROOF. Necessity. Let U be an MCM operator on X. For each $h \in LSC^+(X)$, put $F_n(h) = \big\{ x : h(x) \leqq 1/(n-1) \big\}$ for each $n > 1$ and $F_1(h) = X$. Then $\big\{ F_n(h) \big\}$ is a decreasing sequence of closed sets with empty intersection. Define $\varphi(h) : X \to \mathbb{R}^+$ as $\varphi(h)(x) = 1/n$ whenever $x \in U\big(n, \big\{ F_n(h) \big\} \big) - U\big(n+1, \big\{ F_n(h) \big\} \big)$. Then $\varphi(h) \in USC^+(X)$. Indeed, for each real number $r > 0$, there is an $n \in \mathbb{N}$ such that $\big\{ x : \varphi(h)(x) < r \big\} = \big\{ x : \varphi(h)(x) \leqq 1/n \big\} = U\big(n, \big\{ F_n(h) \big\} \big)$ is an open set.

Now suppose that $h, h' \in LSC^+(X)$ with $h \leqq h'$. Since $h'(x) \leqq 1/(n-1)$ implies that $h(x) \leqq 1/(n-1)$, we have $F_n(h') \subset F_n(h)$ for each $n \in \mathbb{N}$. Then $U\big(n, \big\{ F_n(h') \big\} \big) \subset U\big(n, \big\{ F_n(h) \big\} \big)$ for each $n \in \mathbb{N}$. For each $x \in X$, pick $n \in \mathbb{N}$ such that $x \in U\big(n, \big\{ F_n(h') \big\} \big) - U\big(n+1, \big\{ F_n(h') \big\} \big)$. Then $x \in U\big(n, \big\{ F_n(h) \big\} \big)$, which implies that $\varphi(h)(x) \leqq 1/n = \varphi(h')(x)$. So $\varphi(h) \leqq \varphi(h')$. It implies that φ is order-preserving.

For each $x \in X$, there is an $n \in \mathbb{N}$ such that $x \in U\big(n, \big\{ F_n(h) \big\} \big) - U\big(n+1, \big\{ F_n(h) \big\} \big)$ since $\bigcap_{n\in\mathbb{N}} U\big(n, \big\{ F_n(h) \big\} \big) = \emptyset$. Then $\varphi(h)(x) = 1/n$ and $x \notin F_{n+1}(h)$, thus $h(x) > 1/n$. Therefore $\varphi(h) < h$.

Sufficiency. Assume $\{F_n\}$ is a decreasing sequence of closed subsets of X with empty intersection. Without loss of generality, we may assume that $F_1 = X$. Define $h_{\{F_n\}} : X \to [0,1]$ as $h_{\{F_n\}}(x) = 1/n$ whenever $x \in F_n - F_{n+1}$. Then $h_{\{F_n\}} \in LSC^+(X)$. Put

$$U\big(n, \{F_n\}\big) = \Big\{ x : \varphi\big(h_{\{F_n\}}\big)(x) < 1/n \Big\}.$$

It is sufficient to show U is an MCM operator on X.

Obviously, each $U\big(n, \{F_n\}\big)$ is open. Since $x \in F_n$ if and only if $h_{\{F_n\}}(x) \leqq 1/n$, then $F_n \subset U\big(n, \{F_n\}\big)$ for each $n \in \mathbb{N}$. For each $x \in X$, pick $n \in \mathbb{N}$ such that $\varphi\big(h_{\{F_n\}}\big)(x) > 1/n$. Then $x \notin U\big(n, \{F_n\}\big)$, which shows that $\bigcap_{n\in\mathbb{N}} U\big(n, \{F_n\}\big) = \emptyset$.

Now assume that $\{F_n\}$ and $\{F'_n\}$ are two decreasing sequences of closed subsets of X with empty intersections and $\{F_n\} \prec \{F'_n\}$. Then $h_{\{F_n\}} \leqq h_{\{F'_n\}}$. Consequently, $\varphi\big(h_{\{F_n\}}\big) \leqq \varphi\big(h_{\{F'_n\}}\big)$, which implies that $U\big(n, \{F_n\}\big)$

$\subset U\big(n, \{F'_n\}\big)$ for each $n \in \mathbb{N}$. Therefore, U is an MCM operator on X. □

THEOREM 3.4. *A space X is K-MCM if and only if there is an order-preserving map $\varphi : LSC^+(X) \to USC^+(X)$ satisfying the following conditions:*

(1) *$\varphi(h) < h$ for each $h \in LSC^+(X)$;*

(2) *For any compact subset K of X, there is a real number $r > 0$ such that $\varphi(h)(x) > r$ for all $x \in K$.*

PROOF. Necessity. Let U be a K-MCM operator on X. For each $h \in LSC^+(X)$, put $F_n(h) = \big\{x : h(x) \leqq 1/(n-1)\big\}$ for each $n > 1$ and $F_1(h) = X$. Then $\big\{F_n(h)\big\}$ is a decreasing sequence of closed sets with empty intersection. Define $\varphi(h) : X \to \mathbb{R}^+$ as $\varphi(h)(x) = 1/n$ whenever $x \in U\big(n, \big\{F_n(h)\big\}\big) - U\big(n+1, \big\{F_n(h)\big\}\big)$. By the proof of Theorem 3.3, we know that $\varphi : LSC^+(X) \to USC^+(X)$ is an order-preserving map and satisfies the condition (1).

Now let K be a compact subset of X. Then $K \cap U\big(n, \big\{F_n(h)\big\}\big) = \emptyset$ for some $n \in \mathbb{N}$. Put $r = 1/n$, by the definition of $\varphi(h)$, we have $\varphi(h)(x) > r$ for all $x \in K$.

Sufficiency. Assume $\{F_n\}$ is a decreasing sequence of closed subsets of X with empty intersection. Without loss of generality, we may assume that $F_1 = X$. Define $h_{\{F_n\}} : X \to [0,1]$ as $h_{\{F_n\}}(x) = 1/n$ whenever $x \in F_n - F_{n+1}$. Then $h_{\{F_n\}} \in LSC^+(X)$. Put

$$U\big(n, \{F_n\}\big) = \Big\{x : \varphi\big(h_{\{F_n\}}\big)(x) < 1/n\Big\}.$$

By the proof of Theorem 3.3, U is an MCM operator on X.

Suppose K is a compact subset of X, there is a $r > 0$ such that $\varphi\big(h_{\{F_n\}}\big)(x) > r$ for all $x \in K$. Pick $n \in \mathbb{N}$ such that $1/n < r$. Then $K \cap U\big(n, \{F_n\}\big) = \emptyset$. □

Acknowledgements. Prof. Shengxiang Xia took part in the discussion of this paper and raised the question of how to characterize MCM spaces by semi-continuous functions. The authors would like to thank him for his valuable suggestions.

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