

UNIFORM BASES AT NON-ISOLATED POINTS AND MAPS

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ABSTRACT. In this paper, the authors mainly discuss the images of spaces with an uniform base at non-isolated points, and obtain the following main results: (1) Perfect maps preserve spaces with an uniform base at non-isolated points; (2) Open and closed maps preserve regular spaces with an uniform base at non-isolated points; (3) Spaces with an uniform base at non-isolated points don't satisfy the decomposition theorem.

1. INTRODUCTION

Recently, spaces with an uniform base or spaces with a sharp base bring some topologist attention and interesting results about certain bases are obtained [2, 3, 14]. In [9], the authors define the notion of uniform bases at non-isolated points and obtain some related matters. For example, it is proved that a space X has an uniform base at non-isolated points if and only if X is the open boundary-compact image of a metric space. It is well known that the class of spaces under the open and compact images of metric spaces are preserved by perfect maps or closed and open maps(see [14]). Hence a question arises: "What kind of maps preserve spaces with a uniform base at non-isolated points?" In this paper we shall consider the invariance of spaces with an uniform base at non-isolated points under perfect maps or closed and open maps.

By \mathbb{R}, \mathbb{N} , denote the set of all real numbers and positive integers, respectively. For a topological space X , let $\tau(X)$ denote the topology for X , and let

$$I(X) = \{x : x \text{ is an isolated point of } X\},$$

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$$\begin{aligned} X^d &= X - I(X), \\ \mathcal{I}(X) &= \{\{x\} : x \in I(X)\}, \\ \mathcal{I}_\Delta(X) &= \{(\{x\}, \{x\}) : x \in I(X)\}. \end{aligned}$$

In this paper all spaces are Hausdorff, all maps are continuous and onto. Recall some basic definitions.

Definition 1.1. Let \mathcal{P} be a base of a space X . \mathcal{P} is an *uniform base* [1] (resp. *uniform base at non-isolated points* [9]) for X if for each (resp. non-isolated) point $x \in X$ and \mathcal{P}' is a countably infinite subset of $\{P \in \mathcal{P} : x \in P\}$, \mathcal{P}' is a neighborhood base at x in X .

In the definition, “at non-isolated points” means “at each non-isolated point of X ”.

Definition 1.2. [8] Let $f : X \rightarrow Y$ be a map.

- (1) f is a *boundary-compact map*, if each $\partial f^{-1}(y)$ is compact in X ;
- (2) f is a *compact map* if each $f^{-1}(y)$ is compact in X ;
- (3) f is a *perfect map* if f is a closed and compact map.

Definition 1.3. Let X be a space and $\{\mathcal{P}_n\}_n$ a sequence of collections of open subsets of X .

- (1) $\{\mathcal{P}_n\}_n$ is called a *quasi-development* [4] for X if for every $x \in U$ with U open in X , there exists $n \in \mathbb{N}$ such that $x \in \text{st}(x, \mathcal{P}_n) \subset U$.
- (2) $\{\mathcal{P}_n\}_n$ is called a *development* [13] (resp. *development at non-isolated points* [9]) for X if $\{\text{st}(x, \mathcal{P}_n)\}_n$ is a neighborhood base at x in X for each (resp. non-isolated) point $x \in X$.
- (3) X is called *quasi-developable* (resp. *developable*, *developable at non-isolated points*) if X has a quasi-development (resp. development, development at non-isolated points).

Obviously, in the definition about developments at non-isolated points we can assume that each \mathcal{P}_n is a cover for X . Also, it is easy to see that a space which is developable at non-isolated points is quasi-developable, but a space with a development at non-isolated points may not have a development, see Example in [9].

Definition 1.4. Let \mathcal{P} be a family of subsets of a space X . \mathcal{P} is called *point-finite at non-isolated points* [9] if for each non-isolated point $x \in X$, x belongs to at most finite elements of \mathcal{P} . Let $\{\mathcal{P}_n\}_n$ be a development (resp. a development at non-isolated points) for X . $\{\mathcal{P}_n\}_n$ is said to be a *point-finite development* (resp.

a point-finite development at non-isolated points) for X if each \mathcal{P}_n is point-finite at each (resp. non-isolated) point of X .

Readers may refer to [8, 10] for unstated definitions and terminology.

2. DEVELOPMENTS AT NON-ISOLATED POINTS

In this section some characterizations of spaces with a development at non-isolated points are established.

Let X be a topological space. $g : \mathbb{N} \times X \rightarrow \tau(X)$ is called a g -function, if $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$ for any $x \in X$ and $n \in \mathbb{N}$. For $A \subset X$, put

$$g(n, A) = \bigcup_{x \in A} g(n, x).$$

Theorem 2.1. *Let X be a topological space. Then the following conditions are equivalent:*

- (1) X has a development at non-isolated points;
- (2) There exists a g -function for X such that, for every $x \in X^d$ and sequences $\{x_n\}_n, \{y_n\}_n$ of X , if $\{x, x_n\} \subset g(n, y_n)$ for every $n \in \mathbb{N}$, then $x_n \rightarrow x$.
- (3) X is a quasi-developable space, and X^d is a perfect subspace of X .

PROOF. (1) \Rightarrow (2). Let $\{\mathcal{U}_n\}_n$ be a development at non-isolated points for X . We can assume that $\mathcal{I}(X) \subset \mathcal{U}_n$ for every $n \in \mathbb{N}$.

For every $x \in X$ and $n \in \mathbb{N}$, fix $U_n \in \mathcal{U}_n$ with $x \in U_n$, where $U_n = \{x\}$ when $x \in I(X)$. Let $g(n, x) = \bigcap_{i < n} U_i$. Then $g : \mathbb{N} \times X \rightarrow \tau(X)$ is a g -function for X . For every $x \in X^d$, if sequences $\{x_n\}_n, \{y_n\}_n$ satisfy $\{x, x_n\} \subset g(n, y_n)$ for every $n \in \mathbb{N}$, then $x_n \rightarrow x$ because $\{\mathcal{U}_n\}_n$ is a development at non-isolated points.

(2) \Rightarrow (3). Let g be a g -function with (2). Put $\mathcal{U}_n = \{g(n, x) : x \in X^d\}$ for every $n \in \mathbb{N}$. Then $\{\mathcal{U}_n\}_n \cup \{\mathcal{I}(X)\}$ is a quasi-development for X . Otherwise, there exist $x \in X^d$ and an open neighborhood U of x in X such that $\text{st}(x, \mathcal{U}_n) \not\subset U$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose $x_n \in \text{st}(x, \mathcal{U}_n) - U$, then there exists $y_n \in X$ such that $\{x_n, x\} \subset g(n, y_n)$. Thus $x_n \rightarrow x$, a contradiction as $X - U$ is closed. Hence X has a quasi-development.

For any closed subset B of X^d , it is obvious that $B \subset \bigcap_{n \in \mathbb{N}} (g(n, B) \cap X^d)$. If a point $x \in \bigcap_{n \in \mathbb{N}} (g(n, B) \cap X^d) - B$, then $x \in g(n, B) \cap X^d$ for every $n \in \mathbb{N}$. There exists a sequence $\{y_n\}_n$ in B such that $\{x, y_n\} \subset g(n, y_n)$, so $y_n \rightarrow x$ by (2). Since X^d is closed in X , B is closed in X , then $x \in B$, a contradiction. Thus $B = \bigcap_{n \in \mathbb{N}} (g(n, B) \cap X^d)$, and X^d is a perfect subspace for X .

(3) \Rightarrow (1). Let $\{\mathcal{U}_n\}_n$ be a quasi-development for X , and X^d be a perfect subspace of X . For any $n \in \mathbb{N}$, there exists a sequence $\{F_{n,j}\}_j$ of closed subsets

of X^d such that $(\cup \mathcal{U}_n) \cap X^d = \cup_{j \in \mathbb{N}} F_{n,j}$. For each $n, j \in \mathbb{N}$, put

$$\mathcal{H}_{n,j} = \mathcal{U}_n \cup \{X - F_{n,j}\}.$$

Then $\{\mathcal{H}_{n,j}\}_{n,j}$ is a development at non-isolated points for X . Indeed, for any $x \in X^d$ and $x \in U \in \tau$, since $\{\mathcal{U}_n\}_n$ is a quasi-development for X , there exists $n \in \mathbb{N}$ such that $x \in \text{st}(x, \mathcal{U}_n) \subset U$. Hence there exists $j \in \mathbb{N}$ such that $x \in F_{n,j}$. Thus $x \in \text{st}(x, \mathcal{H}_{n,j}) \subset U$ because $x \notin X - F_{n,j}$. \square

Let \mathcal{P} be a pair-family of subsets of X . For any $P \in \mathcal{P}$, we denote $P = (P', P'')$. For any $\mathcal{R} \subset \mathcal{P}$, denote

$$\begin{aligned} \mathcal{R}' &= \{P' : P \in \mathcal{R}\}, \\ \mathcal{R}'' &= \{P'' : P \in \mathcal{R}\}, \\ \text{st}(x, \mathcal{R}) &= \cup \{P'' : P \in \mathcal{R}, x \in P'\}, \quad x \in X, \\ \text{st}(A, \mathcal{R}) &= \cup \{P'' : P \in \mathcal{R}, A \cap P' \neq \emptyset\}, \quad A \subset X. \end{aligned}$$

For each $i \leq n$ and $\mathcal{R}_i \subset \mathcal{P}$, denote

$$\mathcal{R}_1 \wedge \mathcal{R}_2 \cdots \wedge \mathcal{R}_n = \{(\bigcap_{i \leq n} P'_i, \bigcap_{i \leq n} P''_i) : P_i \in \mathcal{R}_i, i \leq n\}.$$

Definition 2.2. [5] Let X be a topological space and \mathcal{P} a pair-family for X . \mathcal{P} is called a *pair-network* if \mathcal{P} satisfies the following conditions:

- (i) $P' \subset P''$ for any $(P', P'') \in \mathcal{P}$;
- (ii) For any $x \in U \in \tau(X)$, there exists $(P', P'') \in \mathcal{P}$ such that $x \in P' \subset P'' \subset U$.

Theorem 2.3. *Let X be a space. Then the following conditions are equivalent:*

- (1) X is a developable space at non-isolated points;
- (2) There exists a pair-network $\cup_{n \in \mathbb{N}} \mathcal{P}_n$ for X satisfying the following conditions:
 - (i) For every $n \in \mathbb{N}$, $\mathcal{P}'_n|_{X^d}$ is a closed and locally finite family in X^d , and \mathcal{P}''_n is open in X ;
 - (ii) For every compact subset K and $K \subset U \in \tau(X)$, there exists $m \in \mathbb{N}$ such that $K \subset \text{st}(K, \mathcal{P}_m) \subset U$.
- (3) There exists a pair-network $\cup_{n \in \mathbb{N}} \mathcal{P}_n$ for X satisfying the following conditions:
 - (i) For every $n \in \mathbb{N}$, $\mathcal{P}'_n|_{X^d}$ is a closed and locally finite family in X^d ;
 - (ii) For every $x \in U \in \tau(X)$, there exists $m \in \mathbb{N}$ such that $x \in \text{st}^\circ(x, \mathcal{P}_m) \subset U$.

PROOF. We only need to prove that (3) \Rightarrow (1) \Rightarrow (2).

(3) \Rightarrow (1). Let X had a pair-network $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ with (3). Then $\bigcup_{n \in \mathbb{N}} \mathcal{R}'_n|_{X^d}$ is a closed and σ -locally finite network in X^d , X^d is a perfect subspace of X .

For any $n, k \in \mathbb{N}$, let

$$\phi_{n,k} = \{\mathcal{F} \subset \mathcal{R}'_n|_{X^d} : |\mathcal{F}| = k\};$$

$$U(\mathcal{F}) = (\cup\{R'' : R \in \mathcal{R}_n, R' \cap X^d \in \mathcal{F}\})^\circ - \cup(\mathcal{R}'_n|_{X^d} - \mathcal{F}), \text{ where } \mathcal{F} \in \phi_{n,k};$$

$$\mathcal{U}_{n,k} = \{U(\mathcal{F}) : \mathcal{F} \in \phi_{n,k}\}.$$

We should prove that $\{\mathcal{U}_{n,k}\}_{n,k} \cup \{\mathcal{I}(X)\}$ is a quasi-development for X . For any $x \in X^d$ and $x \in U \in \tau(X)$, there exists $m \in \mathbb{N}$ such that $x \in \text{st}^\circ(x, \mathcal{R}_m) \subset U$.

Let

$$\mathcal{F} = \{R' \cap X^d : R \in \mathcal{R}_m, x \in R'\}, |\mathcal{F}| = k.$$

It is easy to see $\mathcal{F} \in \phi_{m,k}$. Hence $x \in U(\mathcal{F}) \subset \text{st}^\circ(x, \mathcal{R}_m) \subset U$. If $\mathcal{G} \in \phi_{m,k} - \{\mathcal{F}\}$, then $x \in \cup(\mathcal{R}'_m|_{X^d} - \mathcal{G})$. Thus $x \notin U(\mathcal{G})$. So $x \in U(\mathcal{F}) = \text{st}(x, \mathcal{U}_{m,k}) \subset U$. Hence $\{\mathcal{U}_{n,k}\}_{n,k} \cup \{\mathcal{I}(X)\}$ is a quasi-development for X .

In a word, X has a development at non-isolated points by Theorem 2.1.

(1) \Rightarrow (2). Let $\{\mathcal{U}_n\}_n$ be a development at non-isolated points for X . We can also assume that $\{\mathcal{U}_n\}_n$ satisfies the following conditions (a)-(c) for every $n \in \mathbb{N}$:

(a) \mathcal{U}_{n+1} refines \mathcal{U}_n ;

(b) $\mathcal{I}(X) \subset \mathcal{U}_n$;

(c) $U_1 \cap X^d \neq U_2 \cap X^d$ for any distinct $U_1, U_2 \in \mathcal{U}_n - \mathcal{I}(X)$.

Put $\mathcal{U}_n - \mathcal{I}(X) = \{U_\alpha : \alpha \in \Lambda_n\}$. Since X^d is a developable subspace of X , it is a subparacompact subspace, then there exists a collection $\mathcal{F}_n = \bigcup_{k \in \mathbb{N}} \mathcal{F}_{n,k}$ of subsets of X^d such that each $\mathcal{F}_{n,k} = \{F_{k,\alpha} : \alpha \in \Lambda_n\}$ is closed and discrete in X^d and $F_{k,\alpha} \subset U_\alpha \cap X^d$ for every $k \in \mathbb{N}, \alpha \in \Lambda_n$. Let

$$\mathcal{P}_{n,k} = \{(F_{k,\alpha}, U_\alpha) : \alpha \in \Lambda_n\} \cup \mathcal{I}_\Delta(X).$$

Then $\bigcup_{n,k \in \mathbb{N}} \mathcal{P}_{n,k}$ is a pair-network for X . Let

$$\mathcal{H}(k_1, k_2, \dots, k_n) = \bigwedge_{i \leq n} \mathcal{P}_{i,k_i}, \quad k_i \in \mathbb{N}, i \leq n.$$

Then $\mathcal{H}(k_1, k_2, \dots, k_n)$ satisfies the condition (i) in (2). Suppose that $K \subset U$ with K compact and U open in X . If $x \in K \cap X^d$, there exists a sequence $\{k_i\}_i$ in \mathbb{N} such that $x \in \cup \mathcal{F}_{i,k_i}$ for any $i \in \mathbb{N}$. For every $n \in \mathbb{N}$, put

$$A_n = \cup\{H' : H \in \mathcal{H}(k_1, k_2, \dots, k_n), H' \cap K \neq \emptyset, H'' \not\subset U\}.$$

Since X^d is closed in X , $\{A_n\}_n$ is a decreasing sequence of closed subsets of X . Then there exists $m \in \mathbb{N}$ such that $A_m = \emptyset$. Otherwise, there exist a non-isolated

point $y \in K \cap (\bigcap_{n \in \mathbb{N}} A_n)$ and $j \in \mathbb{N}$ such that $\text{st}(y, \mathcal{U}_j) \subset U$. Thus

$$\text{st}(y, \mathcal{H}(k_1, k_2, \dots, k_j)) \subset \text{st}(y, \mathcal{U}_j) \subset U.$$

This is a contradiction with the definition of A_j . Hence $A_m = \emptyset$ for some $m \in \mathbb{N}$, and

$$x \in \text{st}(K, \mathcal{H}(k_1, k_2, \dots, k_m)) \subset U.$$

By the compactness of K , $\cup\{\mathcal{H}(k_1, \dots, k_n) : n, k_i \in \mathbb{N}, i \leq n\}$ satisfies the condition (ii) of (2). \square

Corollary 2.4. *X is a developable space at non-isolated points if and only if X has a pair-network $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ satisfying the following conditions:*

- (i) *For any $n \in \mathbb{N}$, $\mathcal{I}_\Delta(X) \subset \mathcal{P}_n$, and $P' \subset X^d$ for any $P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)$;*
- (ii) *For every $n \in \mathbb{N}$, $\mathcal{P}'_n|_{X^d}$ is a closed and hereditarily closure-preserving family in X^d ;*
- (iii) *There exists $m \in \mathbb{N}$ such that $x \in \text{st}^\circ(x, \mathcal{P}_m) \subset U$ for any $x \in U \in \tau(X)$.*

PROOF. Necessity. It is easy to see by the proof of (1) \Rightarrow (2) in Theorem 2.3.

Sufficiency. Let $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a pair-network for X satisfying the condition (i)-(iii). For any $n \in \mathbb{N}$, put

$$D_n = \{x \in X : |(\mathcal{P}'_n)_x| \geq \aleph_0\},$$

$$\mathcal{R}_n = \{(\overline{P' - D_n}, P'') : P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)\}$$

$$\cup\{(\{x\}, \text{st}(x, \mathcal{P}_n)) : x \in D_n\} \cup \mathcal{I}_\Delta(X).$$

Then $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ is a pair-network for X . We shall show that $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ satisfies the condition (3) in Theorem 2.3. Since X is a first-countable space by (iii), it is easy to see that $\mathcal{R}'_n|_{X^d}$ is a closed and locally finite family in X^d by [10, Lemma 3.2.16]. Suppose $x \in U \in \tau(X)$. If $x \in I(X) \cup (\bigcup_{n \in \mathbb{N}} D_n)$, it is obvious that there exists $m \in \mathbb{N}$ such that $x \in \text{st}^\circ(x, \mathcal{R}_m) \subset U$. If $x \in X - (I(X) \cup (\bigcup_{n \in \mathbb{N}} D_n))$, then $x \in \text{st}(x, \mathcal{R}_n) = \text{st}(x, \mathcal{P}_n)$. Thus X is a developable space at non-isolated points by Theorem 2.3. \square

Example 2.5. Let $X = \mathbb{N} \cup \{p\}$, here $p \in \beta\mathbb{N} - \mathbb{N}$, endowed with the subspace topology of Stone-Ćech compactification $\beta\mathbb{N}$. Then $X^d = \{p\}$ is a metrizable subspace of X . Since X is not first-countable, then X does not have a development at non-isolated points.

3. THE IMAGES OF SPACES WITH AN UNIFORM BASE AT NON-ISOLATED POINTS

In this section invariant properties of spaces with a development at non-isolated points and spaces with an uniform base at non-isolated points are discussed under perfect maps or closed and open maps.

A space X is called *metacompact* if every open cover of X has a point-finite open refinement.

Lemma 3.1. *For a space X , X^d is a metacompact subspace of X if and only if every open cover of X has an open refinement which is point-finite at non-isolated points.*

PROOF. Sufficiency is obvious. We only prove the necessity.

Necessity. Let X^d be a metacompact subspace of X . For every open cover \mathcal{U} for X , it is easy to see that $\mathcal{U}|_{X^d}$ is an open cover for subspace X^d . Since X^d is a metacompact subspace, there exists an open and point-finite refinement \mathcal{V} (in X^d) for $\mathcal{U}|_{X^d}$. For every $V \in \mathcal{V}$, there exist $U \in \mathcal{U}$ and $W(V) \in \tau(X)$ such that $V = W(V) \cap X^d$ and $W(V) \subset U$. Put

$$\mathcal{W} = \{W(V) : V \in \mathcal{V}\}.$$

Then \mathcal{W} is an open refinement for \mathcal{U} and also point-finite at non-isolated points. \square

Lemma 3.2. *Let X be a topological space. Then the following conditions are equivalent:*

- (1) X is an open boundary-compact image of a metric space;
- (2) X has an uniform base at non-isolated points;
- (3) X has a point-finite development at non-isolated points;
- (4) X has a development at non-isolated points, and X^d is a metacompact subspace of X .

PROOF. (1) \Leftrightarrow (2) \Leftrightarrow (3) was proved in [9]. We only need to prove (1) \Rightarrow (4) \Rightarrow (3).

(1) \Rightarrow (4). Let $f : M \rightarrow X$ be an open boundary-compact mapping, where M is a metric space. Let \mathcal{U} be an open cover for X . Then $f^{-1}(\mathcal{U})$ is an open cover for M . Since M is paracompact, there exists a locally finite open refinement \mathcal{V} of $f^{-1}(\mathcal{U})$. It is easy to see that $f(\mathcal{V})$ is point-finite at non-isolated points, and refines \mathcal{U} . Hence X^d is metacompact by Lemma 3.1.

(4) \Rightarrow (3). Let $\{\mathcal{U}_n\}_n$ be a development at non-isolated points of X . For every $n \in \mathbb{N}$, since X^d is metacompact, \mathcal{U}_n has an open refinement \mathcal{V}_n which is

point-finite at non-isolated points. Hence $\{\mathcal{V}_n\}_n$ is a point-finite development at non-isolated points. \square

Let $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a pair-network for a space X . We say that $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ satisfies (\star) if it has the (i) of Corollary 2.4. That is, let (\star) be the condition:

(\star) For any $n \in \mathbb{N}$, $\mathcal{I}_\Delta(X) \subset \mathcal{P}_n$ and $P' \subset X^d$ for any $P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)$.

Theorem 3.3. *Spaces with a development at non-isolated points are preserved by perfect maps.*

PROOF. Let $f : X \rightarrow Y$ be a perfect map, where X is developable at non-isolated points. Let $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a pair-network which satisfies the condition (2) in Theorem 2.3 for X . It is easy to see that we can suppose that $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ satisfies the condition (\star) by the proof of (1) \Rightarrow (2) in Theorem 2.3.

For any $n \in \mathbb{N}$, put

$$\mathcal{B}_n = \{(f(P'), f(P'')) : P \in \mathcal{P}_n\};$$

$$\mathcal{R}_n = \{(f(P') \cap Y^d, f(P'')) : P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)\} \cup \mathcal{I}_\Delta(Y).$$

Since f is closed, $Y^d \subset f(X^d)$. It is easy to check that $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ is a pair-network for Y . Next, we shall show that it satisfies the condition (3) of Theorem 2.3 for Y .

(i) It is well-known that a locally finite family is preserved by a perfect map. Since $f|_{X^d} : X^d \rightarrow f(X^d)$ is a perfect map and $\mathcal{P}'_n|_{X^d}$ is closed and locally finite in X^d , $\{f(P' \cap X^d) : P \in \mathcal{P}_n\}$ is closed and locally finite in $f(X^d)$, then

$$\{f(P' \cap X^d) \cap Y^d : P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)\} = \mathcal{R}'_n|_{Y^d}$$

is closed and locally finite in Y^d by the condition (\star) .

(ii) Let $y \in U \in \tau(Y)$. We can suppose that $y \in Y^d$. Since $f^{-1}(y)$ is compact for X , there exists $m \in \mathbb{N}$ such that

$$f^{-1}(y) \subset \text{st}(f^{-1}(y), \mathcal{P}_m) \subset f^{-1}(U).$$

Since f is closed and $\text{st}(f^{-1}(y), \mathcal{P}_m)$ is open in X , then

$$y \in \text{st}^\circ(y, \mathcal{B}_m) \subset \text{st}(y, \mathcal{B}_m) \subset U.$$

If $y \in f(P') \cap Y^d$ with $P \in \mathcal{I}_\Delta(X)$, $f(P'') = \{y\} \subset \text{st}(y, \mathcal{R}_m)$. Thus $\text{st}(y, \mathcal{B}_m) = \text{st}(y, \mathcal{R}_m)$, hence $y \in \text{st}^\circ(y, \mathcal{R}_m) \subset U$. \square

Corollary 3.4. *Spaces with an uniform base at non-isolated points are preserved by perfect maps.*

PROOF. Since metacompactness is preserved by closed maps, it is easy to see by Lemma 3.2 and Theorem 3.3. \square

Let Ξ be a topological property. Ξ is said to satisfy *the decomposition theorem* if, for any space X with the property Ξ and any closed map $f : X \rightarrow Y$, there exists a σ -closed discrete subset $Z \subset Y$ such that $f^{-1}(y)$ is compact in X for any $y \in Y - Z$.

In [6, Theorem 1.1], J. Chaber proved that each regular σ -space satisfies the decomposition theorem.

Theorem 3.5. *Let $f : X \rightarrow Y$ be a closed map, where X is a regular space having a development at non-isolated points. If Y is a first-countable space, then Y is developable at non-isolated points.*

PROOF. Since subspace X^d is a Moore space, there exists a subspace $Z = \bigcup_{n \in \mathbb{N}} Z_n \subset Y^d$ such that, for any $y \in Y^d - Z$, $f^{-1}(y) \cap X^d$ is a compact subset of X^d by [6, Theorem 1.1], where each Z_n is closed and discrete in Y^d . Hence $f^{-1}(y) \cap X^d$ is a compact subset of X for any $y \in Y^d - Z$. For any $y \in Z$, let $\{U(y, n) : n \in \mathbb{N}\}$ be a neighborhood base of y in Y . Let $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a pair-network for X satisfying the condition (2) of Theorem 2.3, and the condition (\star) by the proof of (1) \Rightarrow (2) in Theorem 2.3.

For any $n, j \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{W}_n &= \{(f(P'), f(P'')) : P \in \mathcal{P}_n\}, \\ \mathcal{R}_n &= \{(f(P') \cap Y^d, f(P'')) : P \in \mathcal{P}_n - \mathcal{I}_\Delta(X)\} \cup \mathcal{I}_\Delta(Y), \\ \mathcal{H}_{n,j} &= \{(\{y\}, U(y, j)) : y \in Z_n\} \cup \mathcal{I}_\Delta(Y). \end{aligned}$$

Then

$$\left(\bigcup_{n \in \mathbb{N}} \mathcal{R}_n\right) \cup \left(\bigcup_{n,j \in \mathbb{N}} \mathcal{H}_{n,j}\right) \cup \mathcal{I}_\Delta(Y)$$

is a pair-network for Y and satisfies the conditions (i) and (ii) of Corollary 2.4 because a hereditarily closure-preserving family is preserved by a closed map. We only need to prove that it also satisfies (iii) in Corollary 2.4. For any $y \in U \in \tau(Y)$, we discuss the following three cases respectively.

(a) If $y \in Z$, then there exist $n \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $y \in Z_n$ and $U(y, j) \subset U$. Hence $y \in \text{st}^\circ(y, \mathcal{H}_{n,j}) \subset U(y, j) \subset U$.

(b) If $y \in Y^d - Z$, then $f^{-1}(y) \cap X^d$ is a compact subset for X . There exists $m \in \mathbb{N}$ such that

$$f^{-1}(y) \cap X^d \subset \text{st}(f^{-1}(y) \cap X^d, \mathcal{P}_m) \subset f^{-1}(U),$$

then

$$\begin{aligned} f^{-1}(y) &\subset \text{st}(f^{-1}(y), \mathcal{P}_m) \\ &= \text{st}(f^{-1}(y) \cap X^d, \mathcal{P}_m) \cup \text{st}(f^{-1}(y) \cap I(X), \mathcal{P}_m) \subset f^{-1}(U), \end{aligned}$$

thus $y \in \text{st}^\circ(y, \mathcal{W}_m) \subset U$. Since $\text{st}(y, \mathcal{R}_m) = \text{st}(y, \mathcal{W}_m)$, $y \in \text{st}^\circ(y, \mathcal{R}_m) \subset U$.

(c) If $y \in I(Y)$, then $y \in \text{st}(y, \mathcal{I}_\Delta(Y)) = \{y\} \subset U$.

Hence Y is a developable space at non-isolated points by Corollary 2.4. \square

Corollary 3.6. *Regular spaces with an uniform base at non-isolated points are preserved by open and closed maps.*

PROOF. Let $f : X \rightarrow Y$ be an open and closed map, where X is a regular space having an uniform base at non-isolated points. Since f is open and closed, Y is regular and first-countable space, thus Y has an uniform base at non-isolated points by Theorem 3.5. \square

A collection \mathcal{C} of subsets of an infinite set D is said to be *almost disjoint* if $A \cap B$ is finite whenever $A \neq B \in \mathcal{C}$. Let \mathcal{A} be an almost disjoint collection of countably infinite subsets of D and maximal with respect to the properties. Isbell-Mrówka space $\psi(D)$ is the set $\mathcal{A} \cup D$ endowed with a topology as follows [12]: The points of D are isolated. Basic neighborhoods of a point $A \in \mathcal{A}$ are the sets of the form $\{A\} \cup (A - F)$ where F is a finite subset of D .

Example 3.7. There exists a closed map $f : X \rightarrow Y$, where X is a regular space with an uniform base at non-isolated points and Y is a first-countable space. However, f is not a boundary-compact map.

PROOF. Let \mathcal{A} be an almost disjoint collection of countably infinite subsets of \mathbb{N} and maximal with respect to the properties. Let $\psi(\mathbb{N}) = \mathcal{A} \cup \mathbb{N}$ be the Isbell-Mrówka space. Then $\psi(\mathbb{N})$ is a regular space with an uniform base at non-isolated points.

Define $f : \psi(\mathbb{N}) \rightarrow \psi(\mathbb{N})/\mathcal{A}$ by a quotient map, then f is a closed map and the quotient space $\psi(\mathbb{N})/\mathcal{A}$ is a first-countable space. Since $\partial f^{-1}(\{\mathcal{A}\}) = \mathcal{A}$ is discrete in $\psi(\mathbb{N})$, f is not boundary-compact. \square

Since a regular space with an uniform base is a σ -space, regular spaces with an uniform base satisfy the decomposition theorem. But regular spaces with an uniform base at non-isolated points don't satisfy the decomposition theorem.

Example 3.8. There are a regular space X with an uniform base at non-isolated points and a closed map $f : X \rightarrow Y$ such that f does not satisfy the decomposition theorem.

Let Y be the Isbell-Mrówka space $\psi(D)$, where D is an uncountable set. Let $S_1 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be the subspace of the real line \mathbb{R} . Put

$$X = Y \times S_1 - (D \times \{0\}),$$

endowed with the subspace topology of product topology. Then X is a regular space. Let $f : X \rightarrow Y$ be the projective map. Then f is a closed map.

Let $\psi(D) = \mathcal{A} \cup D$, where $\mathcal{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ and each $A_\alpha = \{x(\alpha, n) : n \in \mathbb{N}\} \subset D$. Put

$$\begin{aligned} V_n(\alpha) &= \{x(\alpha, m) : m \geq n\} \cup \{A_\alpha\}, \\ U_n(0) &= \{0\} \cup \{1/m : m \geq n\}, \\ \mathcal{B} &= \{\{(x, y)\} : (x, y) \in D \times (S_1 - \{0\})\} \\ &\quad \cup \{V_n(\alpha) \times U_n(0) : n \in \mathbb{N}, \alpha \in \Lambda\} \cup \{V_m(\alpha) \times \{1/n\} : m, n \in \mathbb{N}\}. \end{aligned}$$

It is easy to see that \mathcal{B} is an uniform base at non-isolated points for X . However, $f^{-1}(y) = \{y\} \times (S_1 - \{0\})$ is not compact in X for any $y \in D$. Since any closed (in Y) subset contained in D is finite, D is not a σ -discrete subspace for Y . Thus $f : X \rightarrow Y$ does not satisfy the decomposition theorem.

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