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ON DISCRETE SPACES AND AP-SPACES

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ABSTRACT. In this paper, it is proved that a space Y is discrete if and only if every sequentially quotient mapping onto Y is bi-quotient (weak-open). Also, we discuss AP-spaces which are important generalizations of Fréchet-Urysohn spaces. We give a new characterization of AP-spaces and prove that every space is an almost-open image of some AP-space.

1. INTRODUCTION

It is well-known that mappings are powerful tools in characterizing topological spaces. There are many classic results on mappings and spaces. For example, J. R. Boone and F. Siwiec [4] have shown that a space Y is sequential (resp. Fréchet-Urysohn, strongly Fréchet-Urysohn) if and only if every sequentially quotient mapping onto Y is quotient (resp. pseudo-open, countably bi-quotient). Recently, M. Sakai discussed spaces Y with the property: every sequence-covering mapping onto Y is bi-quotient (weak-open) in [16] [17]. We are wondering about what will happen if we replace sequence-covering mappings by sequentially quotient mappings. In section 2, we shall prove that such spaces are all discrete.

AP-spaces are important generalizations of Fréchet-Urysohn spaces, for their interesting applications in categorical topology and function spaces [19]. AP-spaces can be characterized to be spaces Y with the property: every quotient mapping onto Y is pseudo-open. The systemical study of AP-spaces appeared in [5] and [19]. In the past years, AP-spaces have been defined in different forms and have many different names, such as, accessibility spaces [20], Whyburn spaces [14]. For a brief history, see [15]. But we also find a new characterization of them

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in section 3. Also we prove that every space can be an almost-open image of some AP-space. This sheds a light of solution to the open problem that whether open mappings preserve the AP-spaces [19].

In this paper all spaces are Hausdorff and all mappings are continuous. By $\mathbb{R}, \mathbb{N}$, we denote the set of real numbers and positive integers, respectively. For a space X, we denote the topology of X by $\tau(X)$. We recall some basic definitions.

Definition 1.1. [7] Let X be a space. $P \subset X$ is called a *sequential neighborhood* of x in X, if each sequence converging to $x \in X$ is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. X is called a *sequential space* if each sequentially open subset of X is open.

Definition 1.2. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space X. $\mathcal{P}$ is called a *weak-base* [3] for X if it satisfies (a) for each $x \in X$ and $U, V \in \mathcal{P}_x$, there is a $W \in \mathcal{P}_x$ such that $W \subset U \cap V$; (b) for each $x \in X$, $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \cap \mathcal{P}_x$, and if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$; (c) whenever $G \subset X$ satisfies that for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, G is open in X.

For a space, it is obvious that any base is a weak-base. And if $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ is a weak-base for a space X, then any element of $\mathcal{P}_x$ is a sequential neighborhood of x for each $x \in X$.

Definition 1.3. Let $f : X \to Y$ be a mapping.

(1) f is called to be *quotient* [6] if in case $f^{-1}(U)$ is an open subset of X then U is an open subset of Y;

(2) f is called to be *sequentially quotient* [4] if in case L is a convergent sequence in Y then there is a convergent sequence S in X such that $f(S)$ is a subsequence of L;

(3) f is called to be *sequence-covering* [18] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$;

(4) f is called to be *pseudo-open* [2] if for each $y \in Y$ and an open subset $U \subset X$ with $f^{-1}(y) \subset U$, then $y \in \mathrm{Int}(f(U))$;

(5) f is called to be *bi-quotient* [11, 12] if for each $y \in Y$ and a family $\mathcal{U}$ of open subsets of X with $f^{-1}(y) \subset \cup\{U : U \in \mathcal{U}\}$, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $y \in \mathrm{Int}(\cup\{f(U) : U \in \mathcal{F}\})$;

(6) f is called to be *almost-open* [1] if there exists a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of x_y, $f(U)$ is a neighborhood of y.

(7) f is called to be *weak-open* [8, 21] if there exist a weak-base $\mathcal{P} = \bigcup_{y\in Y} \mathcal{P}_y$ for Y and a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of x_y, $f(U)$ contains some element of $\mathcal{P}_y$.

Remark 1.4. The following implications hold.
(1) open ⇒ almost-open ⇒ bi-quotient ⇒ pseudo-open ⇒ quotient;
(2) almost open ⇒ weak-open ⇒ quotient.

Readers may refer to [6] for unstated definitions and terminology.

2. On sequentially quotient mappings and discrete spaces

Lemma 2.1. [17] *For a space Y, the following are equivalent.*
(1) Every sequence-covering mapping onto Y is bi-quotient;
(2) Every non-isolated point of Y has an open neighborhood which is a non-trivial convergent sequence.

A collection $\mathcal{C}$ of subsets of an infinite set is said to be *almost disjoint* if $A \cap B$ is finite whenever $A \neq B \in \mathcal{C}$. Take an infinite maximal almost disjoint collection $\mathcal{A}$ consisting of infinite subsets of $\mathbb{N}$. Then $|\mathcal{A}| > \omega$ [10]. The Isbell-Mrówka space $\psi(\mathbb{N})$ [13] is the set $\mathcal{A} \cup \mathbb{N}$ endowed with a topology as follows: The points of $\mathbb{N}$ are isolated. Basic neighborhoods of a point $A \in \mathcal{A}$ are the sets of the form $\{A\} \cup (A - F)$, where F is a finite subset of $\mathbb{N}$.

Theorem 2.2. *For a space Y, the following are equivalent.*
(1) Y is discrete;
(2) Every sequentially quotient mapping onto Y is bi-quotient;
(3) Every sequentially quotient mapping onto Y is open.

Proof. (1)⇒(3)⇒(2) is obvious. We now prove (2)⇒(1).

Suppose that Y has a non-isolated point y. By Lemma 2.1, there exists a non-trivial sequence $\{y_n\}$ converging to y such that $K = \{y\} \cup \{y_n : n \in \mathbb{N}\}$ is an open subset of Y. Then we have $Y = K \bigoplus (Y - K)$. Let $\psi(\mathbb{N}) = \mathcal{A} \cup \mathbb{N}$ be the Isbell-Mrówka space and $X = \psi(\mathbb{N}) \bigoplus (Y - K)$. Define $f : X \to Y$ as

$$f(x) = \begin{cases} y, & \text{if } x = A \in \mathcal{A}, \\ y_n, & \text{if } x = n \in \mathbb{N}, \\ x, & \text{if } x \in Y - K. \end{cases}$$

Obviously f is continuous. Also, we have the following claims.

Claim 1. f is sequentially quotient.

Suppose that L is a convergent sequence in Y. Without loss of generality, we can assume that L converges to y and $L \subset K$. Denote $L = \{y_{n_k}\}_{k \in \mathbb{N}}$. Since $\mathcal{A}$ is a maximal almost disjoint collection, there is an $A \in \mathcal{A}$ such that $A \cap \{n_k : k \in \mathbb{N}\}$ is infinite. It is easy to see that $A \cap \{n_k : k \in \mathbb{N}\}$ is a sequence converging to $A \in \mathcal{A}$ in X and $f(A \cap \{n_k : k \in \mathbb{N}\})$ is a subsequence of L. Therefore f is sequentially quotient.

Claim 2. f is not bi-quotient.

Suppose f is bi-quotient. Since $\{A\} \cup A$ is open in X for each $A \in \mathcal{A}$ and

$$f^{-1}(y) \subset \cup\{\{A\} \cup A : A \in \mathcal{A}\},$$

there is a finite $\mathcal{F} \subset \mathcal{A}$ such that

$$y \in \mathrm{Int} f(\cup\{\{A\} \cup A : A \in \mathcal{F}\}) = \mathrm{Int}(\{y\} \cup f(\cup \mathcal{F})).$$

So $\mathbb{N} - \cup\mathcal{F}$ is finite. Since $\mathcal{A}$ is uncountable, we can pick a $B \in \mathcal{A} - \mathcal{F}$. Then

$$\mathbb{N} - \cup\mathcal{F} \supset B - \cup\mathcal{F} = B - \cup\{B \cap A : A \in \mathcal{F}\}$$

is infinite. This contradiction shows that f is not bi-quotient. □

Lemma 2.3. [16] *For a space Y, the following are equivalent.*

(1) Every sequence-covering mapping onto Y is weak-open;

(2) Y is sequential and for each $y \in Y$, there exists a sequence L_y converging to y such that for any sequence L converging to y, $L - L_y$ is finite.

Theorem 2.4. *For a space Y, the following are equivalent.*

(1) Y is discrete;

(2) Every sequentially quotient mapping onto Y is weak-open.

Proof. (1)⇒(2) is obvious. We now prove (2)⇒(1).

By Lemma 2.3, Y is sequential. So we only need to show that Y has no non-trivial convergent sequences. Suppose that Y has a non-trivial sequence $\{y_n\}$ converging to y. By Lemma 2.3, we may assume that $K = \{y\} \cup \{y_n : n \in \mathbb{N}\}$ is a sequential neighborhood of y. Take an infinite maximal almost disjoint collection $\mathcal{A}$ consisting of infinite subsets of $\{y_n : n \in \mathbb{N}\}$. For each $x \in Y - \{y\}$, put $\mathcal{B}_x = \{B \in \tau(Y) : B \cap K \subset \{x\}\}$. Put $X = \mathcal{A} \cup (Y - \{y\})$ and endow X with the topology as follows: for each $x \in Y - \{y\}$, take $\mathcal{B}_x$ as a neighborhood base of x; for each $A \in \mathcal{A}$, take

$$\{\{A\} \cup \bigcup_{x \in A'} B_x : B_x \in \mathcal{B}_x, A' \subset A \text{ and } A - A' \text{ is finite}\}$$

as a neighborhood base of A. Define $f : X \to Y$ as

$$f(x) = \begin{cases} y, & \text{if } x \in \mathcal{A}, \\ x, & \text{if } x \in Y - \{y\}. \end{cases}$$

Claim 1. f is continuous.

Obviously f is continuous at each $x \in Y - \{y\}$. Suppose $A \in \mathcal{A}$ and U is an open neighborhood of y in Y. There is an n_0 such that $\{y_n : n \geq n_0\} \subset U$. Also for each $n \geq n_0$, there is a $B_{y_n} \in \mathcal{B}_{y_n}$ such that $B_{y_n} \subset U$. Put

$$V = \{A\} \cup (\cup\{B_{y_n} : n \geq n_0,\ y_n \in A\}).$$

Then V is an open neighborhood of A and $f(V) \subset U$.

Claim 2. f is sequentially quotient.

Suppose that L is a convergent sequence in Y. Without loss of generality, we can assume that L converges to y and $L \subset K$. Since $\mathcal{A}$ is a maximal almost disjoint collection, there is an $A \in \mathcal{A}$ such that $A \cap L$ is infinite. So $A \cap L$ is a sequence converging to $A \in \mathcal{A}$ in X and $f(A \cap L)$ is a subsequence of L. Therefore f is sequentially quotient.

Claim 3. f is not weak-open.

For each $A \in \mathcal{A}$, it is easy to see that $K - A$ is infinite. Pick any $B_x \in \mathcal{B}_x$ for each $x \in A$. Then $U = \{A\} \cup \bigcup_{x \in A} B_x$ is an open neighborhood of A but $f(U)$ cannot to be a sequential neighborhood of y. Therefore f is not weak-open. □

3. On AP-spaces

A space X is called to be an *AP-space* [19] (called *accessibility space* in [20]) if for any non-closed subset $A \subset X$ and $x \in \overline{A} - A$ there is an almost closed subset $F \subset A$ which converges to x, where by the almost closed set F converging to x we understand $\overline{F} - F = \{x\}$. Any subspace of an AP-space is AP and the ordinal space $\omega_1 + 1$ is not AP [19]. In [20], the author proved that a space X is an AP-space if and only if every quotient mapping onto X is pseudo-open. Now we obtain a new characterization of AP-spaces.

Recall that a space X is *determined* [9] by a cover $\mathcal{P}$ if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. For each $x \in X$ and a family $\mathcal{P}$ of subset of X, we denote $\mathrm{st}(x, \mathcal{P}) = \cup\{P \in \mathcal{P} : x \in P\}$.

Theorem 3.1. *For a regular space Y, the following are equivalent.*

(1) Y is an AP-space;

(2) Every quotient mapping onto Y is pseudo-open;

(3) If Y is determined by a cover $\mathcal{P}$, then $y \in Int(st(y, \mathcal{P}))$ for each $y \in Y$.

Proof. The equivalence of (1) and (2) is due to Whyburn [20].

(2)⇒(3). Suppose Y is determined by a cover $\mathcal{P} = \{P_\alpha : \alpha < \kappa\}$. Put $X = \bigoplus_{\alpha<\kappa} P_\alpha$ and let $f : X \to Y$ be the natural mapping. Then f is quotient, and thus pseudo-open. For each $y \in Y$, $\{P_\alpha : y \in P_\alpha\}$ is a family of open subsets of X and $\{P_\alpha : y \in P_\alpha\}$ covers $f^{-1}(y)$. So $y \in \mathrm{Int} f(\cup\{P_\alpha : y \in P_\alpha\}) = \mathrm{Int}(\mathrm{st}(y, \mathcal{P}))$.

(3)⇒(2). Suppose $f : X \to Y$ is a quotient mapping. For each $y \in Y$ and an open subset $U \subset X$, if $f^{-1}(y) \subset U$, then $\mathcal{U} = \{U, X - f^{-1}(y)\}$ is an open cover of X. So X is determined by $\mathcal{U}$. By [9, Lemma 1.7], Y is determined by $f(\mathcal{U}) = \{f(U), Y - \{y\}\}$. Therefore $y \in \mathrm{Int}(\mathrm{st}(y, f(\mathcal{U}))) = \mathrm{Int}(f(U))$, which shows that f is pseudo-open. □

Put $X^d = \{x : x \text{ is a non-isolated point of } X\}$ for a space X.

Proposition 3.2. For a regular space X, if X^d is discrete, then X is an AP-space.

Proof. Suppose $x \in \overline{A} - A$. Since X^d is discrete, there is an open neighborhood U of x such that $\overline{U} \cap X^d = \{x\}$. Put $F = U \cap A$. It is easy to verify that $\overline{F} - F = \{x\}$. Therefore, X is an AP-space. □

It is well-known that AP-spaces are preserved by closed mappings [19]. And whether AP-spaces are preserved by open mappings is still an open problem [19]. The following corollary shows that AP-spaces are not always preserved by almost-open mappings.

Corollary 3.3. *Every space is an almost-open image of an AP-space.*

Proof. For any space X and each $x \in X$, put $X_x = X$ and endow X_x with the topology as follows: all points but x are isolated and take $\{U : x \in U \in \tau(X)\}$ as a neighborhood base of x. Then $Y = \bigoplus_{x\in X} X_x$ is a regular space and Y^d is discrete. By Proposition 3.2, Y is an AP-space. Let $f : Y \to X$ be the natural mapping. Obviously f is almost-open. The proof is finished. □

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