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On Finite Subsequence-covering Maps

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Abstract: In this paper, some topological properties which are preserved by finite subsequence-covering maps are discussed, and an example is provided to answer the following questions: (1) Is every 1-sequentially quotient map sequence-covering? (2) Is every open map of a sequential space 1-sequence-covering? (3) Does not every LSC set-valued map from zero-dimensional paracompct spaces to a Lasnev space have continuous selection?

Key words: finite subsequence-covering maps; sequence-covering maps; sn-first countable spaces; g-metrizable spaces; $\aleph$-spaces

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1 Definitions

It is well known that first countability is preserved by open continuous maps. The following question was posed by Y. Tanaka[12]: Is g-first countability preserved by open continuous maps? It is an interesting question what maps preserve g-first countability. In recent years, sequence-covering maps arouse attention and new developments about them are witnessed[6]. In this paper, some relationships among sequence-covering maps are discussed. Throughout this paper, all spaces are T_2, all maps are continuous and onto.

Let us recall some definitions of sequence-covering maps.

Definition 1 Let $f: X \to Y$ be a map. f is *sequence-covering*[11] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$; f is *sequentially quotient*[2] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$; f is *1-sequence-covering*[6] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$; f is *1-sequentially quotient*[5] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_k\}$ converging to x in X with each $x_k \in f^{-1}(y_{n_k})$; f is *pseudo sequence-covering*[4] if whenever S is a convergent sequence(containing limit point) in Y there is a compact subset L in X such that $f(L) = S$; f is *finite subsequence-covering*[10] if for each $y \in Y$ there is a finite subset

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K of $f^{-1}(y)$ such that for any sequence S converging to y, there exists a sequence L converging to some point in K and $f(L)$ is a subsequence of S.

It is obvious that

1-sequentially quotient maps $\longrightarrow$ finite subsequence-covering maps

1-sequence-covering maps $\nearrow$ 1-sequentially quotient maps; finite subsequence-covering maps $\searrow$ sequentially quotient maps

1-sequence-covering maps $\longrightarrow$ sequence-covering maps $\longrightarrow$ sequentially quotient maps

sequence-covering maps $\searrow$ pseudo sequence-covering maps.

Definition 2 Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$,

(a) $\mathcal{P}_x$ is a network of x in X;

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a *weak base*[1] for X if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in X; $\mathcal{P}$ is called an *sn-network*[8] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a weak base(resp. an sn-network) defined the above-mentioned. Each $\mathcal{P}_x$ is called a *local weak base*(resp. a *local sn-network*) at x. A space X is called a *g-first countable space*(resp. an *sn-first countable space*) if X has a weak base(resp. an sn-network) such that the local weak base(resp. the local sn-network) of each point in X is a countable family.

Definition 3 Let $\mathcal{P}$ be a collection of subsets of a space X. $\mathcal{P}$ is called a *k-network*[4] if for every compact subset K and a neighborhood V of K in X there exists a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset V$.

A regular space with a σ-locally finite k-network(resp. sn-network, weak base) is called an $\aleph$-*space*(resp. *sn-metrizable space, g-metrizable space*). HCP means a family which is hereditarily closure-preserving. For an infinite cardinal α, S_α is the quotient space obtained from the topological sum of α many non-trivial convergent sequences by identifying all the limit points to a single point.

It is known that for regular spaces[3, 13],

metrizable spaces $\rightleftarrows$ $\aleph$-spaces + first countable spaces

$\downarrow$

g-metrizable spaces $\rightleftarrows$ $\aleph$-spaces + g-first countable spaces

$\downarrow$

sn-metrizable spaces $\rightleftarrows$ $\aleph$-spaces + sn-first countable spaces

$\downarrow$

$\aleph$-spaces $\rightleftarrows$ σ-HCP k-networks + no any (closed) copy of S_{ω_1}

2 Properties of Finite Subsequence-covering Maps

In this section, some properties of sequence-covering maps are discussed. It is our main interests what topological spaces are preserved by finite subsequence-covering maps.

Theorem 4 sn-first countability is preserved by finite subsequence-covering maps.

Proof Let $f: X \to Y$ be a finite subsequence-covering map and X be an sn-first countable space. Let $\mathcal{P}_x$ be a countable local sn-network at each point $x \in X$. Without loss of generality, denote $\mathcal{P}_x$ by $\{P_{x,n}\}_{n \in Z_+}$ with every $P_{x,n+1} \subset P_{x,n}$. For each $y \in Y$ there is a finite subset $K = \{x_1, x_2, \cdots, x_k\}$ of $f^{-1}(y)$ which satisfies the condition of finite subsequence-covering maps in Definition 1. Let $\mathcal{Q}_y = \{f(B_n)\}_{n \in Z_+}$, here each $B_n = \bigcup_{i \le k} P_{x_i,n}$.

Next we will prove that $\mathcal{Q}_y$ is a countable local sn-network at $y \in Y$ from (1) and (2) as follows.

(1) $\mathcal{Q}_y$ is a network of y in Y. Obviously, $y \in \bigcap \mathcal{Q}_y$. For any open set U with $y \in U$ and $1 \le i \le k$, $f^{-1}(U)$ is an open neighborhood of x_i in X, thus $x_i \in P_{x_i,n_i} \subset f^{-1}(U)$ for some $n_i \in Z_+$. Take $m = \max\{n_i : 1 \le i \le k\}$, then $B_m \subset f^{-1}(U)$, therefore $f(B_m) \subset U$.

(2) Every $f(B_m)$ is a sequential neighborhood of y in Y. If not, there exists a sequence $\{y_n\}$ in Y converging to y and each $y_n \notin f(B_m)$. There is a sequence $\{z_j\}$ in X converging to some point in K such that $\{f(z_j)\}$ is a subsequence of $\{y_n\}$ because f is finite subsequence-covering. We can assume that all $z_j \in B_m$ because B_m is a sequential neighborhood of K in X. Thus all $f(z_j) \in f(B_m)$, a contradiction with each $y_n \notin f(B_m)$.

Hence, Y is an sn-first countable space.

Corollary 5 A space is an sn-first countable space if and only if it is a finite subsequence-covering image of a metric space.

Proof It is known that a space is an sn-first countable space if and only if it is a 1-sequence-covering image of a metric space[6]. The corollary holds by Theorem 4.

Corollary 6 g-first countability is preserved by finite subsequence-covering and quotient maps.

Proof It is known that a space is g-first countable if and only if it is a sequential space and an sn-first countable space[6]. Let $f: X \to Y$ be a finite subsequence-covering and quotient map with X g-first countable. Y is an sn-first countable space by Theorem 4. On the other hand, Y is a sequential space since sequential spaces are preserved by quotient maps. So Y is a g-first countable space.

Theorem 4 and its corollaries are some finer results since there are a metric space M and a sequence-covering and quotient map $f: X \to Y$ such that Y is not sn-first countable by [6, Example 3.4.7(7)].

All spaces in the final of this section are assumed to be regular, and we shall further discuss some topological properties which are preserved by finite subsequence-covering maps. A question posed by Gu[5] whether $\aleph$-spaces are preserved by 1-sequentially quotient and closed maps is affirmatively answered by Theorem 7.

Theorem 7 $\aleph$-spaces are preserved by finite subsequence-covering and closed maps.

Proof Let $f: X \to Y$ be a finite subsequence-covering and closed map, here X is an $\aleph$-space. Since f is closed, Y has a σ-HCP k-network[13]. If Y is not an $\aleph$-space, Y has a closed subspace Y_t which is homeomorphic to S_{ω_1}. Let $Y_t = \{t\} \cup (\bigcup_{\alpha < \omega_1} T_\alpha)$, here the family $\{T_\alpha\}_{\alpha < \omega_1}$ is disjoint and each T_α is a non-trivial sequence converging to t in Y. Since f is finite subsequence-covering, there is a finite subset $K \subset f^{-1}(t)$ such that whenever T is a sequence converging to t in Y there is a sequence L converging to a point $l \in K$ in X such that $f(L)$ is a subsequence of T. For each $\alpha < \omega_1$ there is a sequence L_α converging to $l_\alpha \in K$ such that $f(L_\alpha)$ is a subsequence of the sequence T_α. Since K is finite, without losing generality, suppose that all $l_\alpha = l \in K$, and let $X_l = \{l\} \cup (\bigcup_{\alpha<\omega_1} L_\alpha)$. Then X_l is homeomorphic to S_{ω_1}. In fact, it is obvious that the family $\{L_\alpha\}_{\alpha<\omega_1}$ is disjoint. Since each point of T_α is isolated in Y_t, each point of L_α is also isolated in X_l. For each $\alpha < \omega_1$ and each finite subset F_α of L_α, then $\{f(F_\alpha)\}_{\alpha<\omega_1}$ is discrete in Y. By the continuity of f, $\{F_\alpha\}_{\alpha<_1}$ is discrete in X. Thus the subspace X_l of X is homeomorphic to S_{ω_1}, a contradiction with $\aleph$-spaces. Therefore, Y is an $\aleph$-space.

Corollary 8 *sn*-metrizability, *g*-metrizability or metrizability is preserved by finite sub -sequence-covering and closed maps.

Proof *sn*-metrizability is preserved by finite subsequence-covering and closed maps from Theorems 4 and 7. *g*-metrizability is preserved by finite subsequence-covering and closed maps from Theorem 7 and Corollary 6. Since every first countable space is equivalent to a *g*-first countable and Fréchet space[6], and each Fréchet space is preserved by a closed map, metrizability is preserved by finite subsequence-covering and closed maps.

Corollary 9 Spaces with a point-countable base are preserved by finite subsequence-covering and closed maps.

Proof It has been proved that a space with a point-countable base is equivalent to a first countable space with a point-countable k-network[6, Corollary 2.1.7]. Let $f : X \to Y$ be a finite subsequence-covering and closed map, here X has a point-countable base. Then Y is a *g*-first countable, Fréchet space, thus Y is a first countable space. On the other hand, Y has a point-countable k-network by [7, Theorem 5]. Hence, Y has a point-countable base.

Remark 10 Some related results are generalized by finite subsequence-covering maps. For example,

(1) *sn*-first countability is preserved by 1-sequentially quotient maps[5];

(2) *g*-first countability is preserved by 1-sequence-covering and quotient maps[6];

(3) *g*-metrizability is preserved by finite subsequence-covering and closed maps[10];

(4) Spaces with a point-countable base are preserved by open and closed maps[7].

Question 11 Are spaces with a point-countable weak base preserved by finite-to-one and closed maps?

3 Examples

There are two questions about sequence-covering maps as follows.

Question 12 (1) Is every 1-sequentially quotient map sequence-covering[5, Question 2]?

(2) Is every open map of a sequential space 1-sequence-covering[14, Question 3.9]?

In this section the questions are negatively answered by the following example. It is proved that let $f : X \to Y$ be an open map with X first countable, then f is 1-sequence-covering[14].

Example 13 Let $Y = \{\frac{1}{n} : n \in Z_+\} \bigcup \{0\}$ endow usual subspace topology of real line R. There are a closed image X of a metric space and a map $f : X \to Y$ such that

(1) f is 1-sequentially quotient;

(2) f is open;

(3) f is not pseudo sequence-covering.

In fact, a family $\mathcal{D}$ of subsets of $Y\setminus\{0\}$ is said to be *almost disjoint* if $A\cap B$ is finite whenever $A, B\in\mathcal{D}, A\neq B$. Using Zorn's Lemma, there exists a family $\mathcal{A}$ of infinite subsets of $Y\setminus\{0\}$ such that $\mathcal{A}$ is a almost disjoint family and maximal with respect to these properties. Then $\mathcal{A}$ must be uncountable and denote it by $\{A_\alpha:\alpha\in A\}$. For each $\alpha\in A$, put $B_\alpha=\{0_\alpha\}\cup A_\alpha$ with subspace topology of Y. Define $M=\oplus_{\alpha\in A}B_\alpha$, and let X be the quotient space $M/\{0_\alpha:\alpha\in A\}$. Then X is the closed image of metric space M, and so it is a Fréchet space and homeomorphic to $S_{|A|}$. Define a map $f:X\to Y$ satisfying each $f|_{B_\alpha}$ is natural map.

(1) f is a 1-sequentially quotient map. First, f is continuous. For each $y\in Y\setminus\{0\}$, $f^{-1}(y)=\oplus\{y:y\in A_\alpha\}$ is an open and closed subspace of X. If U is a neighborhood of 0 in Y, then $B_\alpha\setminus f^{-1}(U)$ is finite for each $\alpha\in A$, thus $f^{-1}(U)$ is open in X. Now, let $\{y_n\}$ be a non-trivial convergent sequence in Y. Then $y_n\to 0$. Put $S=\{y_n:n\in Z_+\}\setminus\{0\}$. Since $\mathcal{A}$ is a almost disjoint and maximal family of $Y\setminus\{0\}$, there is $\alpha\in A$ such that $S\cap A_\alpha$ is infinite, i.e., there is a sequence $\{x_k\}$ converging to 0 in X with each $x_k\in f^{-1}(y_{n_k})$. Hence, f is 1-sequentially quotient.

(2) f is an open map. If not, there is an open set V in X such that $f(V)$ is not open in Y. Then $0\in V$. Put $L=Y\setminus f(V)$, then L is infinite. For each $\alpha\in A$, $f|_{B_\alpha}:B_\alpha\to f(B_\alpha)$ is one-to-one, so $L\cap A_\alpha=f(A_\alpha\setminus V)$ is finite. Hence, $L\notin\mathcal{A}$ and $\mathcal{A}\cup\{L\}$ is almost disjoint, a contradiction.

(3) f is not a pseudo sequence-covering map. If not, there exists a compact subset K of X such that $f(K)=Y$. Since X is homeomorphic to $S_{|A|}$, there is a finite subset F of A such that $K\subset\bigcup_{\alpha\in F}B_\alpha$, thus $Y\setminus\{0\}=\bigcup_{\alpha\in F}A_\alpha$. Take $\beta\in A\setminus F$, then $A_\beta=A_\beta\cap(Y\setminus\{0\})=\bigcup_{\alpha\in F}(A_\beta\cap A_\alpha)$ is finite, a contradiction.

Example 14 There is a metric space X and a finite subsequence-covering and finite-to-one map $f:X\to Y$ such that

(1) f is not 1-sequentially quotient;

(2) f is not sequence-covering;

(3) Y has no any point-countable weak base.

Let X be the topological sum of a family $\{I\}\cup\{X_\alpha:\alpha\in I\}$, where I is the closed unit interval, and each X_α is a non-trivial convergent sequence. Let Y be the space obtained from X by identifying the limit point of X_α with $\alpha\in I$ for each $\alpha\in I$. Let $f:X\to Y$ be the obvious map. Then Y is the quotient and finite-to-one image of a locally compact metric space X under f, and Y has no any point-countable weak base[7, Remark 14]. It is easy to check that f is a finite subsequence-covering map. It is also easy to check that f is neither 1-sequentially quotient nor sequence-covering.

We know that every sequence-covering and compact map of a metric space is 1-sequence-covering[8], and every 1-sequentially quotient map of an sn-first countable space is 1-sequence-covering[9]. Hence, finite subsequence-covering maps are different from 1-sequentially quotient maps and sequence-covering maps.

It has been shown[15] that every LSC set-valued map from zero-dimensional paracompct spaces to a BCO space has continuous selections. A natural question is that the BCO spaces can be replaced by stratifiable spaces or not? We give the following example which negatively answers this question.

Example 15 Let X and Y be spaces, $f: X \to Y$ in Example 13. Define $\varphi: Y \to F(X)$ by $\varphi(y) = f^{-1}(y)$ for each $y \in Y$, then φ is lsc which has no continuous selection. If not, let g be a continuous selection of φ, then $g(Y)$ is a compact set of X, but f is not pseudo sequence-covering, a contradiction.

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关于有限子序列覆盖映射

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摘要: 本文研究了几类拓扑性质在有限子序列覆盖映射下的保持性, 并给出一个反例回答了如下问题: (1) 1- 序列商映射是序列覆盖映射吗? (2) 序列空间上的开映射是 1 序列覆盖映射吗? (3) 从零维仿紧空间到 Lasnev 空间的 LSC 的集值映射一定存在连续选择吗?

关键词: 有限子序列覆盖映射; 序列覆盖映射; *sn* 第一可数空间; *g* 可度量化空间; ℵ 空间