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MAPS ON SUBMETRIZABLE SPACES

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Abstract. A space X is called a submetrizable space if it can be mapped onto a metric space by a one-to-one map. In this paper, the internal characterizations on certain compact or K-images of submetrizable spaces are discussed. We obtain some characterizations of compact-covering compact images, compact-covering and sequence-covering compact images, sequence-covering K-images, perfect images and pseudo-sequence-covering compact images of submetrizable spaces, and establish some relations between these. Moreover, we discuss the sequence-covering compact maps or closed sequence-covering maps on submetrizable spaces of countable type. Some questions about maps on submetrizable spaces are posed.

1. Introduction

A study of images of topological spaces under certain maps is an important question in general topology. In particular, A.V. Arhangel'skii published the famous paper "Mappings and spaces"[1] in 1966, and then a number of topologists devote to the study of certain images on metrizable spaces. In recent years, some noticeable results have been obtained by using sequence-covering maps to systematically study metrizable spaces and generalized metrizable spaces, see [4, 5, 6, 7, 8, 9, 11, 12, 14, 19, 20]. In [19], P.F. Yan and C. Lu obtained the internal characterizations on sequence-covering compact images and sequentially quotient compact images of submetirzable spaces, where a space X is *submetrizable* if it has a coarser metric topology. In this paper, the internal characterizations on certain compact or K-images of submetrizable spaces are discussed. We mainly discuss compact-covering compact maps, compact-covering and sequence-covering compact maps, sequence-covering K-maps, perfect maps and pseudo-sequence-covering

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compact maps on submetrizable spaces, and obtain their internal characterizations, respectively. Moreover, we discuss the sequence-covering compact maps or closed sequence-covering maps on submetrizable spaces of countable type.

Definition 1.1. Let X be a space, and $P \subset X$.

(1) P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P;
(2) P is a *k-closed* subset of X if for any compact subset K of X, $P \cap K$ is closed in X.

Definition 1.2. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a *cs-cover* [10] for X, if for any convergent sequence S in X, there exists a $P \in \mathcal{P}$ such that S is eventually in P;
(2) $\mathcal{P}$ is called a *cs*-cover* [10] for X, if for any convergent sequence S in X, there exists a $P \in \mathcal{P}$ such that some subsequence of S is eventually in P;
(3) $\mathcal{P}$ is called an *sn-cover* [10] for X, if every element of $\mathcal{P}$ is a sequential neighborhood of some point of X and for every $x \in X$, there exists a $P \in \mathcal{P}$ such that P is the sequential neighborhood of x;
(4) $\mathcal{P}$ is called a *k-cover* [15] for X, if for every compact subset K of X, there exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \cup\mathcal{P}'$;
(5) Let K be a compact subset of X. $\mathcal{F}$ is called a *cfp-cover of* K [10], if $\mathcal{F}$ is a cover of K in X such that it can be precisely refined by some finite cover of K consisting of compact subsets of K;
(6) $\mathcal{P}$ is called a *cfp-cover* [20] for X, if for every compact subset K of X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ is a cfp-cover of K.

Definition 1.3 ([10]). Let $f : X \to Y$ be a map.

(1) f is a *compact map* if each $f^{-1}(y)$ is compact in X;
(2) f is a *boundary-compact map* if each $\partial f^{-1}(y)$ is compact in X;
(3) f is a *K-map* if $f^{-1}(K)$ is compact in X for each compact subset K in Y;
(4) f is a *compact-covering map* if, for each compact subset K in Y, there exists a compact subset L in X such that $f(L) = K$;
(5) f is a *sequence-covering map* if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$;
(6) f is a *1-sequence-covering map* if for each $y \in Y$ there is a point $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$;
(7) f is a *sequentially quotient map* if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$;
(8) f is a *pseudo-sequence-covering map* if for each convergent sequence L in Y there is a compact subset K in X such that $f(K) = \overline{L}$;

It is obvious that

$$\begin{tikzcd}[row sep=small]
 & & \text{pseudo-sequence-covering maps} \\
\text{1-sequence-covering maps} \arrow[r] & \text{sequence-covering maps} \arrow[ur] \arrow[dr] & \\
 & & \text{sequentially quotient maps.}
\end{tikzcd}$$

Throughout this paper all spaces are assumed to be T_1 and regular, all maps are continuous and onto. The letter $\mathbb{N}$ will denote the set of positive integers. Readers may refer to [2, 3, 10] for unstated definitions and terminology.

2. Compact maps on submetrizable spaces

Lemma 2.1 ([19])**.** *Let X be a submetrizable space. Then there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{P}_i) = K$ for each compact subset $K \subset X$.*

Remark 1. By the regularity, we can assume that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K,\overline{\mathcal{P}_i}) = K$ for each compact subset $K \subset X$ in Lemma 2.1. In this paper, we always assume this when using Lemma 2.1.

Theorem 2.2. *Y is a compact-covering compact image of a submetrizable space if and only if Y has a sequence $\{\mathcal{F}_i\}_i$ of point-finite cfp-covers such that $\bigcap_{i\in\mathbb{N}} st(y,\mathcal{F}_i) = \{y\}$ for each $y \in Y$.*

Proof. Necessity. Let $f : X \to Y$ be a compact-covering compact map, where X is a submetrizable space. Since X is a submetrizable space, it follows from Lemma 2.1 that there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K,\mathcal{P}_i) = K$ for every compact subset K of X. For every $i \in \mathbb{N}$, let $\mathcal{F}_i = f(\mathcal{P}_i)$. Since f is a compact map, $\mathcal{F}_i$ is point-finite cover for Y. Next, we prove that $\mathcal{F}_i$ is a cfp-cover for each $i \in \mathbb{N}$. For each compact subset K of Y, since f is a compact-covering map, there exists a compact subset L of X such that $f(L) = K$. For each $i \in \mathbb{N}$, since $\mathcal{P}_i$ is an open cover for X, there exists a finite subfamily $\mathcal{P}'_i \subset \mathcal{P}_i$ such that $L \subset \cup\mathcal{P}'_i$. Clearly, $\mathcal{P}'_i$ is a cfp-cover of L. Therefore, it is obvious that $f(\mathcal{P}'_i)$ is a cfp-cover of K. Moreover, it is easy to see that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(y,\mathcal{F}_i) = \{y\}$ for each $y \in Y$.

Sufficiency. For each $i \in \mathbb{N}$, let $\mathcal{F}_i = \{F_\alpha : \alpha \in \Lambda_i\}$, and endow Λ_i with the discrete topology. Put

$$M = \{\{\alpha_i\} \in \prod_{i\in\mathbb{N}} \Lambda_i : \text{There is a point } y \in Y \text{ such that } \bigcap_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}\},$$

and let

$$X = \{(y,\{\alpha_i\}) \in Y \times M : y \in \bigcap_{i\in\mathbb{N}} F_{\alpha_i}\}.$$

Obviously, M is a metrizable space. Let f and p be the restrictions to X of the projections of $Y \times M$ onto Y and M. For any $\{\alpha_i\} \in M$, there exists only one point $y \in Y$ such that $p^{-1}(\{\alpha_i\}) = (y, \{\alpha_i\})$. Therefore, p is a one-to-one map, and hence X is a submetrizable space. Clearly, f is compact by the point-finiteness of $\mathcal{F}_i$ for each $i \in \mathbb{N}$.

Next we show that f is a compact-covering map.

Let K be a compact subset of Y. For each $i \in \mathbb{N}$, since $\mathcal{F}_i$ is a point-finite cfp-cover, there exists a finite subfamily $\mathcal{P}_i \subset \mathcal{F}_i$ such that $\mathcal{P}_i$ is a cfp-cover of K. For each $i \in \mathbb{N}$, let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Gamma_i\}$, and let $\{K_\alpha : \alpha \in \Gamma_i\}$ be a family of compact subsets and precisely refine $\mathcal{P}_i$. Put

$$L = \{(y, \{\alpha_i\}) \in K \times \prod_{i \in \mathbb{N}} \Gamma_i : y \in \bigcap_{i \in \mathbb{N}} K_{\alpha_i}\}.$$

Claim 1: L is compact.

Let $(y, \{\alpha_i\}) \in (K \times \prod_{i \in \mathbb{N}} \Gamma_i) \setminus L$. Then $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$ or $y \notin \bigcap_{i \in \mathbb{N}} K_{\alpha_i}$.

Case 1: $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$.

Then there is an $i_0 \in \mathbb{N}$ such that $\bigcap_{i \leq i_0} K_{\alpha_i} = \emptyset$. Put

$$W = \{(k \times \{\beta_i\}) \in K \times \prod_{i \in \mathbb{N}} \Gamma_i : k \in K \text{ and } \beta_i = \alpha_i \text{ whenever } i \leq i_0\}.$$

Then W is an open subset with $(y, \{\alpha_i\}) \in W$ and $W \cap L = \emptyset$. Hence L is closed subset of $K \times \prod_{i \in \mathbb{N}} \Gamma_i$, and therefore, L is compact.

Case 2: $y \notin \bigcap_{i \in \mathbb{N}} K_{\alpha_i}$.

Then there is an $i_0 \in \mathbb{N}$ such that $y \notin K_{\alpha_{i_0}}$. Put

$$W = \{k \times \{\beta_i\} \in K \times \prod_{i \in \mathbb{N}} \Gamma_i : k \in K \setminus K_{\alpha_{i_0}} \text{ and } \beta_{i_0} = \alpha_{i_0}\}.$$

Then W is an open subset with $(y, \{\alpha_i\}) \in W$ and $W \cap L = \emptyset$. Hence L is a closed subset of $K \times \prod_{i \in \mathbb{N}} \Gamma_i$, and therefore, L is compact.

Claim 2: $L \subset X$ and $f(L) = K$.

For every $(y, \{\alpha_i\}) \in L$, since $\bigcap_{i \in \mathbb{N}} \mathrm{st}(y, \mathcal{F}_i) = \{y\}$, $\bigcap_{i \in \mathbb{N}} F_{\alpha_i} = \{y\}$. Hence $L \subset X$ and $f(L) \subset K$. For each $y \in K$ and $i \in \mathbb{N}$, we can choose $\alpha_i \in \Gamma_i$ such that $y \in K_{\alpha_i}$. Let $\alpha = \{\alpha_i\}$. Then $(y, \alpha) \in L$ and $f(y, \alpha) = y$. Hence $K \subset f(L)$.

Therefore, f is a compact-covering map by Claims 1 and 2. $\square$

Theorem 2.3. *For a space Y, the following conditions are equivalent:*

(1) *Y is a compact-covering, 1-sequence-covering and compact image of a submetrizable space;*
(2) *Y is a compact-covering, sequence-covering and compact image of a submetrizable space;*
(3) *Y has a sequence $\{\mathcal{F}_i\}_i$ of point-finite sn-covers such that $\bigcap_{i \in \mathbb{N}} st(y, \mathcal{F}_i) = \{y\}$ for each $y \in Y$, and every compact subset of Y is metrizable;*

(4) *Y has a sequence $\{\mathcal{F}_i\}_i$ of point-finite cs-covers such that $\bigcap_{i\in\mathbb{N}} st(y,\mathcal{F}_i) = \{y\}$ for each $y \in Y$, and every compact subset of Y is metrizable.*

Proof. Obviously, (1) ⇒ (2) and (3) ⇒ (4).

(2) ⇒ (3). It follows from [19, Theorem 2] that we only need to prove that every compact subset of Y is metrizable. Let $f : X \to Y$ be a compact-covering, sequence-covering and compact map, where X is submetrizable. For each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$. It is easy to see that $f|_L : L \to K$ is a compact-covering, compact and continuous map. Since X is submetrizable and L is compact, L is metrizable by [3, Theorems 2.5 and 2.13]. Therefore, K is the compact-covering compact image of a compact metrizable space, and hence K is metrizable since K is a compact space with a countable-network.

(4) ⇒ (1). It is sufficient to show that, for each $i \in \mathbb{N}$, $\mathcal{F}_i$ is a cfp-cover by Theorem 2.2 and [19, Theorem 2]. Let K be a compact subset of Y. Then K is metrizable, and hence K is first-countable. Fix a point $y \in K$, let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing local base at the point y. For every $i \in \mathbb{N}$, put

$$\mathcal{P}_i = \{P \cap K : \text{There exists an } n \in \mathbb{N} \text{ such that } V_n \subset P \cap K, P \in \mathcal{F}_i\}.$$

Clearly, $\mathcal{P}_i$ is finite and every member of $\mathcal{P}_i$ is a neighborhood of y in K.

Claim: $\mathcal{P}_i \neq \emptyset$.

Suppose not, let $\mathcal{P}_i = \emptyset$. Denote $(\mathcal{F}_i)_y = \{F_j : 1 \leq j \leq j_0\}$. For each $1 \leq j \leq j_0, n \in \mathbb{N}$, we can choose a point $x_{n,j} \in V_n \setminus F_j$. Then denote $y_k = x_{n,j}$, where $k = (n-1)j_0 + j, 1 \leq j \leq j_0, n \in \mathbb{N}$. Hence $y_k \to y$ as $k \to \infty$. However, for each $i \in \mathbb{N}$, $\mathcal{F}_i$ is a cs-cover, which is a contradiction.

By the Claim, it is easy to see that $\mathcal{F}_i$ is a cfp-cover for each $i \in \mathbb{N}$. □

Remark 2. (1) We can not omit the condition "Every compact subset of Y is metrizable" in Theorem 2.3. In fact, let Y be the Stone-Čech compactificatioon $\beta\mathbb{N}$ of $\mathbb{N}$. Then Y is obviously the image of the discrete space of cardinality of $\beta\mathbb{N}$ under the identity map, which is a sequence-covering compact map. However, Y is a non-metrizable compact space, and hence f is not a compact-covering map by Theorem 2.3.

(2) From the proof of (4) ⇒ (1) in Theorem 2.3, it is easy to see that we can replace the condition "Every compact subset of Y is metrizable" in Theorem 2.3 by "Every compact subset of Y is a first-countable subspace".

The following lemma is an easy exercise.

Lemma 2.4. *Let $\mathcal{P}$ be a point-finite cs^*-cover for a space X. Then, for each $x \in X$, $st(x,\mathcal{P})$ is a sequential neighborhood at the point x.*

Theorem 2.5. *For a space Y, the following conditions are equivalent:*

(1) *Y is a sequentially-quotient compact image of a submetrizable space;*

(2) *Y is a pseudo-sequence-covering compact image of a submetrizable space;*

(3) *Y has a sequence $\{\mathcal{F}_i\}_i$ of point-finite cs^*-covers such that $\bigcap_{i\in\mathbb{N}} st(y,\mathcal{F}_i)=\{y\}$ for each $y\in Y$.*

Proof. Obviously, (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (3) by [19, Theorem 3]. So we only need to show (3) $\Rightarrow$ (2).

For each $i\in\mathbb{N}$, let $\mathcal{F}_i=\{F_\alpha:\alpha\in\Lambda_i\}$, and endow Λ_i with the discrete topology. Put

$$M=\{\{\alpha_i\}\in\prod_{i\in\mathbb{N}}\Lambda_i:\text{There is a point } y\in Y \text{ such that } \bigcap_{i\in\mathbb{N}}F_{\alpha_i}=\{y\}\},$$

and let

$$X=\{(y,\{\alpha_i\})\in Y\times M: y\in\bigcap_{i\in\mathbb{N}}F_{\alpha_i}\}.$$

Obviously, M is a metrizable space. Let f and p be the restrictions to X of the projections of $Y\times M$ onto Y and M. It is easy to see that X is submetrizable and f is compact by the proof of Theorem 2.2.

Next we prove that f is a pseudo-sequence-covering map.

For each $y\in Y$ and sequence $\{y_n\}$ converging to y, put $K=\{y\}\cup\{y_n:n\in\mathbb{N}\}$. For each $i\in\mathbb{N}$, it follows from Lemma 2.4 that $\mathrm{st}(y,\mathcal{F}_i)$ is a sequential neighborhood at the point y. Therefore, there is a finite subfamily $\mathcal{F}_i'\subset(\mathcal{F}_i)_K$ such that $K\subset\cup\mathcal{F}_i'$ and $(\mathcal{F}_i)_y\subset\mathcal{F}_i'$. Then there exists a finite subset $\Gamma_i\subset\Lambda_i$ such that $\mathcal{F}_i'=\{F_\alpha:\alpha\in\Gamma_i\}$. For each $\alpha\in\Gamma_i$, we take

$$K_\alpha=\begin{cases} K\cap F_\alpha, & \text{if } y\in F_\alpha,\\ (K\setminus\mathrm{st}(y,\mathcal{F}_i))\cap F_\alpha, & \text{if } y\notin F_\alpha.\end{cases}$$

Then $\{K_\alpha:\alpha\in\Gamma_i\}$ is a family of compact subsets and $K=\bigcup_{\alpha\in\Gamma_i}K_\alpha$. Hence $\mathcal{F}_i'$ is a cfp-cover of K. Put

$$L=\{(y,\{\alpha_i\})\in K\times\prod_{i\in\mathbb{N}}\Gamma_i: y\in\bigcap_{i\in\mathbb{N}}K_{\alpha_i}\}.$$

From the proof of Theorem 2.2, L is compact and $f(L)=K$. $\square$

3. K-maps on submetrizable spaces

Theorem 3.1. *For a space Y, the following conditions are equivalent:*

(1) *Y is a 1-sequence-covering K-image of a submetrizable space;*

(2) *Y is a sequence-covering K-image of a submetrizable space;*

(3) *Y has a sequence $\{\mathcal{F}_i\}_i$ of compact-finite sn-covers of k-closed subsets such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{F}_i)=K$ for each compact subset $K\subset Y$;*

(4) *Y has a sequence $\{\mathcal{F}_i\}_i$ of compact-finite cs-covers of k-closed subsets such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{F}_i)=K$ for each compact subset $K\subset Y$.*

Proof. (1)⇒(2) and (3)⇒(4) are obvious. The proof of (4)⇒(3) is similar to the proof of (4)⇒(3) in [19, Theorem 2].

(2)⇒(4) Let $f : X \to Y$ be a sequence-covering K-map, where X is a submetrizable space. Since X is a submetrizable space, it follows from Lemma 2.1 that there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K, \overline{\mathcal{P}_i}) = K$. For every $i \in \mathbb{N}$, let $\mathcal{F}_i = f(\overline{\mathcal{P}_i})$. Since f is a K-map and $\overline{\mathcal{P}_i}$ is locally finite for each $i \in \mathbb{N}$, it is easy to see that $\mathcal{F}_i$ is compact-finite cover for Y. Obviously, $\mathcal{F}_i$ is a cs-cover for each $i \in \mathbb{N}$. Moreover, it is easy to see that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K, \mathcal{F}_i) = K$ for each compact subset $K \subset Y$.

Claim 1: Each member of $\mathcal{F}_i$ is a k-closed subset of Y.

Fix $i \in \mathbb{N}$. For each $F_\alpha \in \mathcal{F}_i$, there is a $P_\alpha \in \mathcal{P}_i$ such that $f(\overline{P_\alpha}) = F_\alpha$. For each compact subset $K \subset Y$, we have $F_\alpha \cap K = f(\overline{P_\alpha}) \cap K = f(\overline{P_\alpha} \cap f^{-1}(K))$. Since f is a K-map, $\overline{P_\alpha} \cap f^{-1}(K)$ is compact in X. Therefore, $F_\alpha \cap K$ is closed in Y.

(3)⇒(1) By the same notations as in Theorem 2.2. f is a 1-sequence-covering map by the proof of [19, Theorem 2]. Now we show that f is a K-map.

For each compact $L \subset Y$, let $\Lambda_i' = \{\alpha \in \Lambda_i : F_\alpha \in \mathcal{F}_i, F_\alpha \cap L \neq \emptyset\}$ for every $i \in \mathbb{N}$. Since $\mathcal{F}_i$ is compact finite, Λ_i' is finite. Therefore, $\prod_{i\in\mathbb{N}} \Lambda_i'$ is a compact subset of $\prod_{i\in\mathbb{N}} \Lambda_i$.

Claim 2: $f^{-1}(L) = (L \times ((\prod_{i\in\mathbb{N}} \Lambda_i') \cap M)) \cap X$.

It is easy to see, so we omit it.

Claim 3: $f^{-1}(L)$ is a closed subset of $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$.

For each $(y, \{\alpha_i\}) \in (L \times \prod_{i\in\mathbb{N}} \Lambda_i') \setminus f^{-1}(L)$, we have $\bigcap_{i\in\mathbb{N}} F_{\alpha_i} = \emptyset$ or $y \notin \bigcap_{i\in\mathbb{N}} F_{\alpha_i} \neq \emptyset$ by Claim 2.

Case 1: $\bigcap_{i\in\mathbb{N}} F_{\alpha_i} = \emptyset$.

Then $\bigcap_{i\in\mathbb{N}}(F_{\alpha_i} \cap L) = \emptyset$, and since $\mathcal{F}_i$ is a k-closed cover of Y, there exists an $m \in \mathbb{N}$ such that $\bigcap_{i=1}^{i=m}(F_{\alpha_i} \cap L)$. Put

$$U = \{(y', \{\beta_i\}) \in L \times \prod_{i\in\mathbb{N}} \Lambda_i' : \beta_i = \alpha_i \text{ whenever } i = 1, \cdots, m, y' \in L\}.$$

Then U is open in $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$ with $(y, \{\alpha_i\}) \in U$ and $U \cap f^{-1}(L) = \emptyset$. Therefore, $f^{-1}(L)$ is a closed subset of $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$.

Case 2: $y \notin \bigcap_{i\in\mathbb{N}} F_{\alpha_i} \neq \emptyset$.

There exists an $i_0 \in \mathbb{N}$ such that $y \notin F_{\alpha_{i_0}}$. Since $F_{\alpha_{i_0}}$ is a k-closed subset, $F_{\alpha_{i_0}} \cap L$ is closed in L. Put

$$U = \{(y', \{\beta_i\}) \in L \times \prod_{i\in\mathbb{N}} \Lambda_i' : \beta_{i_0} = \alpha_{i_0} \text{ and } y' \in L \setminus (L \cap F_{\alpha_{i_0}})\}.$$

Then U is open in $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$ with $(y, \{\alpha_i\}) \in U$ and $U \cap f^{-1}(L) = \emptyset$. Therefore, $f^{-1}(L)$ is a closed subset of $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$.

It follows from Claim 3 that $f^{-1}(L)$ is compact in $L \times \prod_{i\in\mathbb{N}} \Lambda_i'$. Hence $f^{-1}(L)$ is compact in X. □

Remark 3. By Remark 2, $\beta\mathbb{N}$ is the image of submetrizable space under a sequence-covering compact map. Since compact submetrizable spaces are metrizable, it is easy to see that $\beta\mathbb{N}$ is not the image of submetrizable space under a K-map.

Theorem 3.2. *Y is the K-image of a submetrizable space if and only if Y has a sequence $\{\mathcal{F}_i\}_i$ of compact-finite cfp-covers of k-closed subsets such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{F}_i) = K$ for each compact subset $K \subset Y$.*

Proof. It is easy to see that a K-map is a compact-covering compact map. The proof is similar to the proof of Theorem 3.1 and the necessity in Theorem 2.2, so we omit it. □

Theorem 3.3. *Let Y be a k-space. Then the following conditions are equivalent:*

(1) *Y is the perfect image of a submetrizable space;*
(2) *Y has a sequence $\{\mathcal{F}_i\}_i$ of locally finite k-covers of k-closed subsets such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{F}_i) = K$ for each compact subset $K \subset Y$;*
(3) *Y has a sequence $\{\mathcal{F}_i\}_i$ of locally finite closed covers such that $\bigcap_{i\in\mathbb{N}} st(K,\mathcal{F}_i) = K$ for each compact subset $K \subset Y$;*

Proof. Obviously, (2)⇔(3) by the facts: Every locally finite cover is obviously a k-cover, and every k-closed cover is a closed cover among k-spaces.

(1)⇒(2). Let $f : X \to Y$ be a perfect map, where X is a submetrizable space. It follows from Lemma 2.1 that there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K,\overline{\mathcal{P}_i}) = K$. For every $i \in \mathbb{N}$, let $\mathcal{F}_i = f(\overline{\mathcal{P}_i})$. Since f is a closed map and $\overline{\mathcal{P}_i}$ is locally finite for each $i \in \mathbb{N}$, $\mathcal{F}_i$ is hereditarily closure-preserving cover for Y. It is well known that a perfect map is a K-map, and hence f is a K-map. Thus $\mathcal{F}_i$ is compact-finite cover for Y. Therefore, $\mathcal{F}_i$ is locally finite for each $i \in \mathbb{N}$. Obviously, $\mathcal{F}_i$ is a k-cover for each $i \in \mathbb{N}$, since f is a K-map. Moreover, it is easy to see that $\bigcap_{i\in\mathbb{N}} \mathrm{st}(K,\mathcal{F}_i) = K$ for each compact subset $K \subset Y$ and, for every $i \in \mathbb{N}$, each member of $\mathcal{F}_i$ is a k-closed subset of Y by the proof of Theorem 3.1.

(2)⇒(1). By the same notations as in Theorem 2.2. Obviously, f is a K-map by the proof of Theorem 3.1. Now we show that f is a closed map. In fact, K-map f onto the k-space Y is obviously a closed map since $f(f^{-1}(K)\cap F) = K \cap f(F)$ for each compact subset K of Y. □

4. Submetrizable spaces of countable type

S. Lin and P.F. Yan in [11] proved that each sequence-covering and compact map on metric spaces is a 1-sequence-covering map. Recently, F. C. Lin and S. Lin proved that each sequence-covering and boundary-compact map on the spaces, which are the images of metric spaces under open compact-covering maps, is a 1-sequence-covering map[8]. Hence we have the following question.

Question 4.1. *Are the sequence-covering and compact(or boundary-compact) maps on a submetrizable space 1-sequence-covering?*

Now we give some partial answer for this Question 4.1. Firstly, we give some lemmas.

Recall that a space X is of *countable type* if each compact subset of X is contained in some compact subset of countable character in X. Clearly, every locally compact space is of countable type.

In [14, Theorem 1.2], E.A. Michael and K. Nagami proved the following theorem.

Theorem 4.2. *A space Y is a compact-covering open image of a metric space if and only if every compact subset of Y is metrizable and of countable character in Y.*

The following lemma is an easy exercise.

Lemma 4.3. *Let X be a space, and $A \subset B \subset X$ with A, B compact. If A has countable character in B and so does B in X, then so does A in X.*

Now, it follows from Lemma 4.3 that we can rewrite Theorem 4.2 as follows.

Theorem 4.4. *A space Y is a compact-covering open image of a metric space if and only if Y is of countable type and every compact subset of Y is metrizable.*

Lemma 4.5. *Let X be a submetrizable space of countable type. Then X is the open compact-covering images of metric spaces.*

Proof. Since X is submetrizable and of countable type, it is easy to see that every compact subset of X is metrizable by [3, Theorems 2.5 and 2.13] and of countable character, and hence X is the images of metric spaces under open compact-covering maps by Theorem 4.4. □

Theorem 4.6. *Sequence-covering and boundary-compact maps on a submetrizable space of countable type is 1-sequence-covering.*

Proof. Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where X is a submetrizable space of countable type. By Lemma 4.5, X is the open compact-covering images of metric spaces. Therefore, f is a 1-sequence-covering map by [8, Theorem 3.6]. □

Remark. There exists a non-metrizable and locally compact submetrizable space, see [3, Example 2.17].

Question 4.7. *Are the sequentially-quotient and compact(or boundary-compact) maps on a submetrizable space pseudo-sequence-covering?*

Now we give some partial answer for this Question 4.7.

Theorem 4.8. *Sequentially-quotient and boundary-compact maps on a submetrizable space of countable type is pseudo-sequence-covering.*

Proof. Let $f : X \to Y$ be a sequentially-quotient and boundary-compact map, where X is a submetrizable space of countable type. By Lemma 4.5, X is the open compact-covering images of metric spaces. Therefore, f is a pseudo-sequence-covering map by [8, Theorem 3.11]. $\square$

Corollary 4.9. *Quotient and boundary-compact maps on a submetrizable space of countable type is pseudo-sequence-covering.*

In [8], F.C. Lin and S. Lin posed the following Question 4.10.

Question 4.10 ([8])**.** *Let $f : X \to Y$ be a closed sequence-covering map. If X is the image of a metric space under open compact-covering maps, then is f a 1-sequence-covering map?*

In [8], F.C. Lin and S. Lin have proved that, in Question 4.10, f is a 1-sequence-covering map whenever X is a space with a point-countable base or a Tychonoff strongly monotonically monolithic (see [16]) space. So we have the following Question 4.11.

Question 4.11. *Are the closed sequence-covering maps on a submetrizable space (of countable type) 1-sequence-covering?*

However, we have the following Theorem 4.13, which give a partial answer for Question 4.11. Firstly, we give a technique lemma.

Lemma 4.12. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a first-countable space. If every closed separable subset of X is normal, then Y contains no closed copy of S_ω.*

Proof. Suppose that Y contains a closed copy of S_ω, and that $\{y\}\cup\{y_i(n) : i, n \in \mathbb{N}\}$ is a closed copy of S_ω in Y, here $y_i(n) \to y$ as $i \to \infty$. For every $k \in \mathbb{N}$, put $L_k = \{y_i(n) : i \in \mathbb{N}, n \leq k\}$. Hence L_k is a sequence converging to y. Let M_k be a sequence of X converging to $u_k \in f^{-1}(y)$ such that $f(M_k) = L_k$. We rewrite $M_k = \cup\{x_i(n, k) : i \in \mathbb{N}, n \leq k\}$ with each $f(x_i(n, k)) = y_i(n)$.

Case 1: $\{u_k : k \in \mathbb{N}\}$ is finite.

There are a $k_0 \in \mathbb{N}$ and an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $M_k \to u_{k_0}$ for every $k \in \mathbb{N}_1$, then X contains a closed copy of S_ω. Hence X is not first countable. This is a contradiction.

Case 2: $\{u_k : k \in \mathbb{N}\}$ has a non-trivial convergent sequence in X.

Without loss of generality, we suppose that $u_k \to u$ as $k \to \infty$. Since X is first-countable, let $\{U_m\}$ be a decreasingly and open neighborhood base of X at point u with $\overline{U}_{m+1} \subset U_m$. Then $\bigcap_{m\in\mathbb{N}} U_m = \{u\}$. Fix n, pick $x_{i_m}(n, k_m) \in U_m \cap \{x_i(n, k_m)\}_i$. We can suppose that $i_m < i_{m+1}$. Then $\{f(x_{i_m}(n, k_m))\}_m$ is a subsequence of $\{y_i(n)\}$. Since f is closed, $\{x_{i_m}(n,k_m)\}_m$ is not discrete in X. Then there is a subsequence of $\{x_{i_m}(n,k_m)\}_m$ converging to a point $b \in X$ because X is a first-countable space. It is easy to see that $b = u$ by $x_{i_m}(n,k_m) \in U_m$ for every $m \in \mathbb{N}$. Hence $x_{i_m}(n,k_m) \to u$ as $m \to \infty$. Then $\{u\} \cup \{x_{i_m}(n,k_m) : n,m \in \mathbb{N}\}$ is a closed copy of S_ω in X. Thus, X is not first countable. This is a contradiction.

Case 3: $\{u_k : k \in \mathbb{N}\}$ is discrete in X.

Let $B = \{u_k : k \in \mathbb{N}\} \cup \{M_k : k \in \mathbb{N}\}$. Since every closed separable subsets of X is normal, $\overline{B}$ is is normal. Hence there exists a discrete family $\{V_k\}_{k\in\mathbb{N}}$ consisting of open subsets of $\overline{B}$ with $u_k \in V_k$ for each $k \in \mathbb{N}$. Pick $x_{i_k}(1,k) \in V_k \cap \{x_i(1,k)\}_i$ such that $\{f(x_{i_k}(1,k))\}_k$ is a subsequence of $\{y_i(n)\}$. Since $\{x_{i_k}(1,k)\}_k$ is discrete in $\overline{B}$, $\{f(x_{i_k}(1,k))\}_k$ is discrete in Y. This is a contradiction.

In a word, Y contains no closed copy of S_ω. □

Theorem 4.13. *Let $f : X \to Y$ be a closed sequence-covering map, where X is a submetrizable space of countable type. If every closed separable subset of X is normal, then f is 1-sequence-covering.*

Proof. Obvious, X is first-countable by Lemma 4.3. It follows from Lemma 4.12 that Y contains no closed copy of S_ω. Since X is first-countable and every closed separable subsets of X is normal, $\partial f^{-1}(y)$ is countably compact for each $y \in Y$ by [13, Theorem 2.6]. Then f is a boundary-compact map, since a countably compact submetrizable space is metrizable. Therefore, f is a 1-sequence-covering map by Theorem 4.6. □

Corollary 4.14. *Closed sequence-covering maps on a normal submetrizable space of countable type is 1-sequence-covering.*

Remark 4. (1) Under the set-theoretic hypotheses(Martin's plus $\Diamond_\varepsilon(E)$), there exists a non-metrizable, normal, locally compact submetrizable space, see [17];

(2) There exists a nonnormal, separable, locally compact submetrizable space, see [18].

5. Open problems

Here, we list some open problems about submetrizable spaces.

Question 5.1. *Characterizations for submetrizable spaces by nice image of metric spaces.*

Question 5.2. *What kinds of internal characterizations of the perfect images of submetrizable spaces?*

Question 5.3. *Are the sequence-covering and compact(or boundary-compact) maps on a submetrizable space 1-sequence-covering?*

Question 5.4. *Are the closed sequence-covering maps on a submetrizable space 1-sequence-covering?*

Question 5.5. *What kinds of internal characterizations of the sequence-covering compact(or compact-covering compact, or sequentially quotient compact) images of submetrizable spaces of countable type?*

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