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OPEN UNIFORM (G) AT NON-ISOLATED POINTS AND MAPS

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Abstract. In this paper, we mainly introduce the notion of an open uniform (G) at non-isolated points, and show that a space X has an open uniform (G) at non-isolated points if and only if X is the open boundary-compact image of metric spaces. Moreover, we also discuss the inverse image of spaces with an open uniform (G) at non-isolated points. Two questions about open uniform (G) at non-isolated points are posed.

1. Introduction

In [3], F.C. Lin and S. Lin defined the notion of uniform bases at non-isolated points, and obtained that a space X has an uniform base at non-isolated points if and only if X is the open and boundary-compact image of metric spaces. Isbell-Mrówka space $\psi(D)$ [8] has an uniform base at non-isolated points, and however, it has not any uniform base. It is well known that a space has an uniform base if and only if it has an open uniform (G) if and only if it is the open compact image of metric spaces. Therefore, we generalize the notion of open uniform (G), and define the notion of the open uniform (G) at non-isolated points such that a space has an open uniform (G) at non-isolated points if and only if it has an uniform base at non-isolated points. In [4], F.C. Lin and S. Lin have discussed the image of spaces with an uniform base at non-isolated points. In this paper, we also also discuss the inverse image of spaces with an uniform base at non-isolated points.

By $\mathbb{R}, \mathbb{N}$, denote the set of all real numbers and positive integers, respectively. For a topological space X, let $\tau(X)$ denote the topology for X, and let

$$I(X) = \{x : x \text{ is an isolated point of } X\},$$
$$X^d = X - I(X),$$
$$\mathcal{I}(X) = \{\{x\} : x \in I(X)\}.$$

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In this paper all spaces are Hausdorff, all maps are continuous and onto. Recall some basic definitions.

Definition 1.1. Let $\mathcal{P}$ be a base of a space X. $\mathcal{P}$ is an *uniform base* [1] (resp. *uniform base at non-isolated points* [3]) for X if for each (*resp.* non-isolated) point $x \in X$ and any countably infinite subset $\mathcal{P}'$ of $\{P \in \mathcal{P} : x \in P\}$, $\mathcal{P}'$ is a neighborhood base at x in X.

Definition 1.2. A space X has an *open uniform (G)*[6] (resp. *open uniform (G) at non-isolated points*), if there exists a collection $\mathcal{W} = \{\mathcal{W}_x : x \in X\}$ of open subsets of X satisfying the following conditions:

(1) For each $x \in X$, $x \in \cap \mathcal{W}_x$ and $|\mathcal{W}_x| \leq \aleph_0$;
(2) For each $x \in U \in \tau(X)$, there exists an open neighborhood $V(x, U)$ of x such that there is a $W \in \mathcal{W}_y$ with $x \in W \subset U$ for each $y \in V(x, U)$ ($y \in V(x, U) \cap X^d$);
(3) For each $x \in X$, $\mathcal{W}'_x$ is a network at point x for any infinite subfamily $\mathcal{W}'_x \subset \mathcal{W}_x$.

In the Definitions 1.1 and 1.2, "at non-isolated points" means "at each non-isolated point of X". If $\mathcal{W} = \{\mathcal{W}_x : x \in X\}$ is an open uniform (G) at non-isolated points, then $(\mathcal{W} \setminus \{\mathcal{W}_x : x \in I\}) \cup \{\mathcal{W}'_x = \{x\} : x \in I\}$ is also an open uniform (G) at non-isolated points for X. Therefore, we always suppose that $\mathcal{W}_x = \{x\}$ if $x \in I$ in this paper. It is obvious that spaces with an open uniform (G) have an open uniform (G) at non-isolated points.

Definition 1.3. Let $f : X \to Y$ be a map.

(1) f is a *compact map* if each $f^{-1}(y)$ is compact in X;
(2) f is a *boundary-compact map*, if each $\partial f^{-1}(y)$ is compact in X;
(3) f is a *perfect map* if it is a closed and compact map.
(4) f is called a $\leq$ *k-to-one* (resp. *k-to-one*, *finite-to-one) map* if $|f^{-1}(y)| \leq k$ (resp. $|f^{-1}(y)| = k$, $f^{-1}(y)$ is finite) for every $y \in Y$, where $k \in \mathbb{N}$;
(5) f is called a *local homeomorphism* if, for each $x \in X$, there exists an open neighborhood U of x in X such that $f|U : U \to f(U)$ is a homeomorphism map and $f(U)$ is open in Y.
(6) f is an *irreducible map* if there does not exist a proper closed subset X' of X such that $f(X') = Y$.

Definition 1.4 ([3])**.** Let X be a space and $\{\mathcal{P}_n\}_n$ a sequence of collections of open subsets of X. $\{\mathcal{P}_n\}_n$ is called a *development at non-isolated points* for X if $\{\mathrm{st}(x, \mathcal{P}_n)\}_n$ is a neighborhood base at x in X for each non-isolated point $x \in X$. X is called *developable at non-isolated points* if X has a development at non-isolated points.

Definition 1.5 ([3])**.** Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is called *point-finite at non-isolated points*(resp. *point-countable at non-isolated points*) if for each non-isolated point $x \in X$, x belongs to at most finitely (countably) many elements of $\mathcal{P}$. Let $\{\mathcal{P}_n\}_n$ be a development at non-isolated points for X. $\{\mathcal{P}_n\}_n$ is said to be *a point-finite development at non-isolated points* for X if each $\mathcal{P}_n$ is point-finite at each non-isolated point of X.

Definition 1.6. Let X be a topological space. $g : \mathbb{N} \times X \to \tau(X)$ is called a g-function, if $x \in g(n, x)$ and $g(n+1, x) \subset g(n, x)$ for any $x \in X$ and $n \in \mathbb{N}$. For $A \subset X$, put

$$g(n, A) = \bigcup_{x \in A} g(n, x).$$

Readers may refer to [2, 5] for unstated definitions and terminology.

2. Open uniform (G) at non-isolated points

In this section, we mainly show that a space has an open uniform (G) at non-isolated points if and only if it has an uniform base at non-isolated points. Firstly, we give some technique lemmas.

Lemma 2.1 ([4])**.** *Let X be a topological space. Then the following conditions are equivalent:*

(1) *X is an open boundary-compact image of a metric space;*
(2) *X has an uniform base at non-isolated points;*
(3) *X has a point-finite development at non-isolated points;*
(4) *X has a development at non-isolated points, and X^d is a metacompact subspace of X.*

Lemma 2.2. *Let X have an open uniform (G) at non-isolated points. Then there exists a g-function such that for each $x \in X^d$ and any sequence $\{x_n\}_n$ with $x_n \in g(n, x)$ or $x \in g(n, x_n)$, $\{x_n\}_n$ has a subsequence converging to x.*

Proof. Let $\mathcal{W} = \{\mathcal{W}_x : x \in X\}$ be an open uniform (G) at non-isolated points for X.

Claim 1: There exists a sequences $\{\mathcal{H}_n\}_n$ of open coverings of X, where $\mathcal{H}_n$ is point-finite at non-isolated points for each $n \in \mathbb{N}$.

For each $x \in X$, let $\{G(n, x)\}_n$ be a decreasing open neighborhood base at x, where, for each $x \in \mathbb{N}$, $G(n, x) = \{x\}$ if $x \in I$. Next, we define the point-finite open covering $\mathcal{H}_n$ at non-isolated points, $h_n : \mathcal{H}_n \to X$ and open neighborhood $O(n, x)$ of x for each $x \in X$ by induction on $n \in \mathbb{N}$. Firstly, let $\mathcal{H}_0 = \{X\}$, and choose a point $z \in X$ and define $h_0 : \mathcal{H}_0 \to X$ with $h_0(X) = z$. Put

$$O(1, x) = \begin{cases} G(1, x), & x = z, \\ G(1, x) - \{z\}, & x \neq z. \end{cases}$$

Suppose that we have defined $\mathcal{H}_{m-1}, h_{m-1}$, and $O(m,x)$ for each $m \leq n$ and $x \in X$. We endow $\mathcal{H}_{m-1}$ with a well-order by $(\mathcal{H}_{m-1}, <)$. For each $H \in \mathcal{H}_{n-1}$, since X^d is hereditarily metacompact, there exists an open covering $\mathcal{F}_n(H)$ of H such that $\mathcal{F}_n(H)$ is point-finite at non-isolated points and refines $\{H \cap V(x, O(n,x))\}_{x \in H}$, where $V(x, O(n,x))$ is the open neighborhood of x stated in (3) of Definition 1.2. Put

$\mathcal{H}_n(H) = \mathcal{F}_n(H) - \cup\{\mathcal{F}_n(H') : H' < H\}, H \in \mathcal{H}_{n-1}$;

$\mathcal{H}_n = \cup\{\mathcal{H}_n(H) : H \in \mathcal{H}_{n-1}\}$.

Then $\mathcal{H}_n$ is an open covering of X, which is point-finite at non-isolated points. For each $H \in \mathcal{H}_n$, there exists just one $H' \in \mathcal{H}_{n-1}$ such that $H \in \mathcal{H}_n(H') \subset \mathcal{F}_n(H')$. Then we can choose a point $x_H \in H'$ such that $H \subset H' \cap V(x_H, O(n, x_H)) \subset O(n, x_H)$. Define

$h_n(H) = x_H$;

$O(n+1, x) = G(n+1, x) - \{h_m(H) : m \leq n, H \in (\mathcal{H}_m)_x \text{ and } x \neq h_m(H)\}$.

If $x \in X^d$, then $x \in O(n+1,x) \in \tau(X)$; if $x \in I$, then $G(n+1,x) = O(n+1,x) = \{x\} \in \tau(X)$.

Claim 2: For each $x \in X^d$, $X^d \cap \bigcap_{n=0}^{\infty} \mathrm{st}(x, \mathcal{H}_n) = \{x\}$.

Suppose not, there exist distinct points $x, y \in X^d$ and a sequence $\{H_n\}_n$ of subsets of X such that $x, y \in H_n \in \mathcal{H}_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists just one sequence $\{H_n^m\}_{m \leq n}$ such that $H_n^n = H_n, H_n^m \in \mathcal{H}_m$ and $H_n^m \in \mathcal{H}_m(H_n^{m-1})$ for each $1 < m \leq n$. Then $x \in H_n \subset H_n^m \subset H_n^{m-1}$. Since $\mathcal{H}_m$ is point-finite at point x, we can define $I_n \subset \mathbb{N}$ and $i_n \in \mathbb{N}$ by induction on $n \in \mathbb{N}$ as follows.

(1) $i_n = \min I_n$;

(2) $I_{n+1} \subset I_n - \{i_n\}$;

(3) $m, k \in I_n \Rightarrow H_m^n = H_k^n$.

Let $K_n = H_{i_n}^n, q_n = h_n(K_n)$. Then $K_n = H_m^n$ for each $m \in I_n$ and $q_n \in K_l \supset K_n$ for each $l < n$. Since $x \in K_n \subset V(q_n, O(n, q_n))$, there exists a $W_n \in \mathcal{W}_x$ such that $q_n \in W_n \subset O(n, q_n)$ by the definition of open uniform (G).

Choose disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$. Without loss of generality, we can assume that there exists an infinite $J \subset \mathbb{N}$ such that, for each $n \in J$, $q_n \notin U_x$. Then $W_n \not\subseteq U_x$, and therefore, $\{W_n : n \in J\}$ is finite. Hence, without loss of generality, we can suppose that $W_n = W_m$ for any $n, m \in J$. Thus $q_m \in O(i_n, q_n)$, and $q_m = q_n$ by the definition of $O(i_n, q_n)$. Let $q_n = q$ for each $n \in J$. For each $n \in J$, $x \in V(q, O(n,q)) \subset O(n,q) \subset G(n,q)$, and therefore, $x \in \bigcap_{n \in J} G(n,q) = \{q\}$, which is a contradiction.

Now, we begin to show the Lemma. For each $n \in \mathbb{N}$, $x \in X^d$, choose an $H(n,x) \in (\mathcal{H}_n)_x$. For each $x \in X$, define $g(n,x)$ by induction on n as follows.

$$g(n+1,x) = \begin{cases} V(x, H(n+1,x)) \cap G(n+1,x) \cap g(n,x), & x \in X^d, \\ \{x\}, & x \in I, \end{cases}$$

where

$$g(1,x) = \begin{cases} V(x, H(1,x)) \cap G(1,x), & x \in X^d, \\ \{x\}, & x \in I. \end{cases}$$

Let $x \in X^d$ and $\{x_n\}_n$ be a sequence with $x_n \in g(n,x)$ or $x \in g(n,x_n)$. We consider the following two cases.

Case 1: $\{n : x_n \in g(n,x)\}$ is infinite.

In this case, it is easy to show that the subsequence $\{x_n : x_n \in g(n,x)\}$ converges to x.

Case 2: $\{n : x_n \in g(n,x)\}$ is finite.

In this case, we may assume that $x \in g(n,x_n)$ for each $n \in \mathbb{N}$. We show that the sequence $\{x_n\}_n$ itself converges to x. Otherwise, there exists an open neighborhood U of x such that $\{x_n\}_n$ is not eventually in U. For each $n \in \mathbb{N}$, since $x \in g(n,x_n)$, we have $x_n \in X^d$ and $x \in V(x_n, H(n,x_n))$. Hence, for each $n \in \mathbb{N}$, there is a $W_n \in \mathcal{W}_x$ such that $x_n \in W_n \subset H(n,x_n) \subset \mathrm{st}(x, \mathcal{H}_n)$. Let $M = \{n \in \mathbb{N} : x_n \notin U\}$. Then M is infinite. Therefore, by the condition (3)in Definition 1.2, $\{W_n : n \in M\}$ is finite set. Without loss of generality, we can assume that $W_n = W_m$ for $n, m \in M$. Then, $x_n \in \mathrm{st}(x, \mathcal{H}_m)$ for any $n, m \in M$. Hence, $x_n \in \bigcap_{m \in M} \mathrm{st}(x, \mathcal{H}_m) \cap X^d = \{x\}$ by Claim 2, which is a contradiction. □

Lemma 2.3. *If X has an open uniform (G) at non-isolated points, then X has a point-countable base at non-isolated points.*

Proof. Let $\mathcal{W} = \{\mathcal{W}_x : x \in X\}$ be an open uniform (G) at non-isolated points for X, where $\mathcal{W}_x = \{W(n,x)\}_n$. Let g be a g-function satisfying the conditions in Lemma 2.2. For each $n \in \mathbb{N}$ and the open covering $\{g(n,x) : x \in X\}$ of X, since X^d is metacompact, there exists an open covering $\mathcal{U}_n$ such that $\mathcal{U}_n$ is point-finite at non-isolated points and refines $\{g(n,x) : x \in X\}$. For each $U \in \mathcal{U}_n$, there is a $x_U \in X$ such that $U \subset g(n, x_U)$. Let

$\mathcal{B}_{n,m} = \{U \cap W(m, x_U) : U \in \mathcal{U}_n\}, m \in \mathbb{N}$;

$\mathcal{B} = \bigcup_{n,m \in \mathbb{N}} \mathcal{B}_{n,m}$.

Then $\mathcal{B}$ is an open collection of subsets of X and point-countable at non-isolated points of X. We now show that $\mathcal{B} \cup \mathcal{I}(X)$ is point-countable base at non-isolated points for X. Indeed, for each $x \in X^d$ and $x \in O \in \tau(X)$, choose an $U_n \in (\mathcal{U}_n)_x$ for each $n \in \mathbb{N}$. We denote $x_n = x_{U_n}$. Then $x \in g(n, x_n)$, and hence sequence $\{x_n\}_n$ converges to x. Therefore, there exists an $i \in \mathbb{N}$ such that $x_i \in V(x, O)$. Since $x \in g(i, x_i)$, we have $x_i \in X^d$ and there is an $m \in \mathbb{N}$ such that $x \in W(m, x_i)$. Thus $x \in U_i \cap W(m, x_i) \subset O$. □

Put $R^+ = \{x \in \mathbb{R} : x \geq 0\}$.

Lemma 2.4. *If X has an open uniform (G) at non-isolated points, then there exists a function $d : X \times X \to R^+$ such that, for each $x \in X^d$, $x \in B(x, \frac{1}{n})$ and*

$\{int(B(x, \frac{1}{n}))\}_n$ is a decreasing neighborhood base at x, where $B(x, \frac{1}{n}) = \{y \in X : d(x,y) < \frac{1}{n}\}$.

Proof. Let g be the g-function constructed in the proof of Lemma 2.2. For any distinct points $x, y \in X$, put

$$m(x,y) = \min\{n \in \mathbb{N} : y \notin g(n,x) \text{ and } x \notin g(n,y)\}.$$

Define $d : X \times X \to R^+$ as follows.

$$d(x,y) = \begin{cases} 0, & x = y, \\ \frac{1}{m(x,y)}, & x \neq y. \end{cases}$$

Then, for each point $x \in X^d$ and $n \in \mathbb{N}$, $x \in \text{int}(B(x, \frac{1}{n}))$. Indeed, since $m(x,y) > n$ for each $y \in g(n,x)$, $d(x,y) < \frac{1}{n}$. Then $y \in B(x, \frac{1}{n})$, and therefore, $x \in g(n,x) \subset B(x, \frac{1}{n})$. It follows that $x \in \text{int}(B(x, \frac{1}{n}))$. For each $x \in X^d$ and $x \in U$, there exists an $m \in \mathbb{N}$ such that $B(x, \frac{1}{m}) \subset U$. Otherwise, suppose that $B(x, \frac{1}{m}) \not\subseteq U$ for each $m \in \mathbb{N}$. Choose a point $x_m \in B(x, \frac{1}{m}) \setminus U$ for each $m \in \mathbb{N}$. Then $d(x, x_m) < \frac{1}{m}$, and hence $x \in g(m, x_m)$ or $x_m \in g(m,x)$. By Lemma 2.2, $\{x_m\}_m$ has a subsequence converging to x. It contradicts the fact that $x_m \notin U$ for each $m \in \mathbb{N}$. □

Lemma 2.5. *If X has an open uniform (G) at non-isolated points, then X is a developable space at non-isolated points.*

Proof. By Lemma 2.3, let $\mathcal{U}$ be a point-countable base at non-isolated points for X. Endow X^d with a well-order by $(X^d, <)$. Let $d : X \times X \to R^+$ be the function defined in the proof of Lemma 2.4. For each $x \in X^d$, let $(\mathcal{U})_x = \{U_n(x)\}_n$. For each $n \in \mathbb{N}$, put

$V_n(x) = \text{int}(B(x, \frac{1}{n}))$;

$h(n,x) = U_n(x) \cap V_n(x)$;

$p(n,x) = \min\{y \in X^d : x \in h(n,y)\}$;

$g(n,x) = V_n(x) \cap (\cap\{h(i, p(i,x)) : i \leq n\}) \cap (\cap\{U_j(p(i,x)) : i,j \leq n, x \in U_j(p(i,x))\})$;

$\varphi_n = \{g(n,x) : x \in X^d\} \cup \{g(n,x) = \{x\} : x \in I\}$.

Then $\{\varphi_n\}_n$ is a development at non-isolated points. Indeed, suppose not, there exists a point $x \in X^d$ and an open neighborhood U of x such that there is $x_i \in X^d$ satisfying $x \in g(i, x_i) \not\subseteq U$ for each $i \in \mathbb{N}$. Since $x \in V_i(x_i)$, $x_i \to x$. It follows from Lemma 2.4 that there exist $l, m \in \mathbb{N}$ such that $B(x, \frac{1}{l}) \subset U_m(x) \subset U$. For each $y \in X^d$, if $x \in h(l,y) \subset V_l(y)$, then $y \in B(x, \frac{1}{l}) \subset U_m(x)$. It follows that $p(l,x) \in U_m(x)$, and therefore, there exists a $k \in \mathbb{N}$ such that $U_m(x) = U_k(p(l,x))$. Since $U_k(p(l,x)) \cap h(l, p(l,x))$ is an open neighborhood at x, there is an $i_0 \in \mathbb{N}$ such that, for each $i \geq i_0$, $x_i \in U_k(p(l,x)) \cap h(l, p(l,x))$. Thus $p(l, x_i) \leq p(l,x)$ for $i \geq i_0$, and on the other hand, $x \in g(i, x_i) \subset h(l, p(l, x_i))$ for $i \geq l$. Then $p(l,x) \leq p(l,x_i)$ for each $i \geq l$. Therefore, $p(l,x_i) = p(l,x)$ for $i \geq \max\{i_0, l\}$.

It follows that $x_i \in U_k(p(l, x_i))$, and therefore, for $i \geq \max\{i_0, l, k\}$, $g(i, x_i) \subset U_k(p(l, x_i)) = U_k(p(l, x)) = U_m(x) \subset U$, which is a contradiction. □

Theorem 2.6. *A space X has an open uniform (G) at non-isolated points if and only if X has an uniform base at non-isolated points.*

Proof. Necessity. By Lemma 2.5, X has a development at non-isolated points. Since X^d is metacompact, X has an uniform base at non-isolated points by Lemma 2.1.

Sufficiency. Let $\mathcal{B}$ be an uniform base at non-isolated points for X. If $\mathcal{B}$ is point-countable at non-isolated points for X, then $\mathcal{W} = \{(\mathcal{B})_x : x \in X^d\} \cup \mathcal{I}(X)$ is an open uniform (G) at non-isolated points for X. Suppose that there exists a point $x \in X^d$ such that $(\mathcal{B})_x$ is uncountable. If $z \in X - \{x\}$, then $\{B \in (\mathcal{B})_x : z \in B\}$ is finite. Hence there are an infinite subset $\{B_n : n \in \mathbb{N}\} \subset (\mathcal{B})_x, x_n \in B_n - \{x\}$ for each $n \in \mathbb{N}$, and some $k \in \mathbb{N}$ such that x_n belongs to just k many elements of $(\mathcal{B})_x$. Then $x_n \to x$ as $n \to \infty$. Since $\mathcal{B}$ is a base for X, there exists an infinite subfamily $\{B'_i : i \in \mathbb{N}\}$ of $\mathcal{B}$ and a subsequence $\{x_{n_i}\}_i$ such that $\{x_{n_j} : j \geq i\} \subset B'_i \subset X - \{x_{n_j} : j < i\}$ for $i \in \mathbb{N}$. Then x_{n_i} belongs to i many elements of $(\mathcal{B})_x$, which is a contradiction. □

3. Inverse image of spaces with uniform bases at non-isolated points

In this section, we mainly discuss the inverse image of spaces with uniform bases at non-isolated points.

Definition 3.1. Let X be a topological space.

(1) X is called a *$w\triangle$-space at non-isolated points* if there exists a sequence $\{\mathcal{U}_n\}_n$ of open covers such that, for every $x \in X - I$, whenever $x_n \in \mathrm{st}(x, \mathcal{U}_n)$, then $\{x_n\}_n$ has a cluster point.

(2) X is said to have a *G_δ-diagonal at non-isolated points* if there exists a sequence $\{\mathcal{U}_n\}_n$ of open covers such that $\bigcap_{n \in \mathbb{N}} \mathrm{st}(x, \mathcal{U}_n) = \{x\}$ for every $x \in X - I$. Moreover, X is said to have a *G^*_δ-diagonal at non-isolated points* if we replace "$\bigcap_{n \in \mathbb{N}} \mathrm{st}(x, \mathcal{U}_n) = \{x\}$" by "$\bigcap_{n \in \mathbb{N}} \overline{\mathrm{st}(x, \mathcal{U}_n)} = \{x\}$".

It is obvious that

(1) X is developable at non-isolated points $\Rightarrow$ X is a w$\triangle$-space at non-isolated points;

(2) X has a G^*_δ-diagonal at non-isolated points $\Rightarrow$ X has a G_δ-diagonal at non-isolated points;

Example 3.2. There exists a perfect map from a space X onto a metric space, where X has not any uniform base at non-isolated points.

Proof. Let $X = [0, 1] \times \{0, 1\}$ and endow X with the lexicographic ordered space. Let $f : X \to [0, 1]$ be a naturally projective map, where $[0, 1]$ endowed with the usual topology. Since X is compact, f is a closed and 2-to-one map. X does not have an uniform base at non-isolated points since X has no uniform base and does not contain any isolated points.

From this example it can be seen that a closed and 2-to-one map does not inversely preserve spaces with an uniform base at non-isolated points. □

Example 3.3. There exists an open and $\leq$2-to-one map from a space X onto a metric space, where X has not any uniform base at non-isolated points.

Proof. Y. Tanaka in [9, Example 3.7] constructed a regular space X which is the inverse image of a compact metric space under an open and $\leq$2-to-one map, but X is not a first countable space. Hence X has not any uniform base at non-isolated points. □

Example 3.4. Open and closed map doesn't inversely preserve spaces with uniform base at non-isolated points.

Proof. Let $X = [0, \omega_1]$ be an usually ordered space. Put $f : X \to X/X$ be a quotient map by identifying X to a single point. Then it is obvious that f is an open and closed map. But X has not any uniform base at non-isolated points. □

We don't know whether spaces with an uniform base at non-isolated points are inversely preserved by an open, closed and finite-to-one map. So we have the following question.

Question 3.5. *Are spaces with an uniform base at non-isolated points inversely preserved by open, closed and finite-to-one maps?*

By slightly modifying the proof in [7, Theorem 6], we can obtain the following.

Theorem 3.6. *Let $f : X \to Y$ be a closed, finite-to-one and local homeomorphism map, where Y has an uniform base at non-isolated points. Then X has an uniform base at non-isolated points.*

It is well known that every open and k-to-one map is a closed and locally homeomorphism map. Hence, we have the following corollary.

Corollary 3.7. *Open and k-to-one maps inversely preserve spaces with an uniform base at non-isolated points.*

Finally, we consider the inverse image of spaces with an uniform base at non-isolated points under the irreducible perfect maps.

Lemma 3.8. *Let X be regular and metacompact at non-isolated points. If $\{\mathcal{U}_n\}_n$ is a sequence of open coverings of X, then there exists a sequence $\{\mathcal{V}_n\}_n$ of open coverings of X such that, for any $y \in X^d$, $\bigcap_{n\in\mathbb{N}} \overline{st(y, \mathcal{V}_n)} = \bigcap_{n\in\mathbb{N}} st(y, \mathcal{V}_n) \subset \bigcap_{n\in\mathbb{N}} st(y, \mathcal{U}_n)$.*

Proof. Since X is regular and metacompact at non-isolated points, there exists a sequence $\{\mathcal{V}_n\}_n$ of open coverings of X satisfying the following conditions:

(i) For each $n \in \mathbb{N}$, $\mathcal{V}_n$ is point-finite at non-isolated points and refines

$$(\wedge_{i<n}\mathcal{V}_i) \bigwedge (\wedge_{i\leq n}(\mathcal{U}_i);$$

(ii) For any $V \in \mathcal{V}_n$ and $i < n$, there exists a $W \in \mathcal{V}_i$ such that $\overline{V} \subset W$.

Let $y \in X^d$. For each $n \in \mathbb{N}$, there are only finitely many members of $\mathcal{V}_n$ which contains y. Hence $\overline{\mathrm{st}(y, \mathcal{V}_{n+1})} = \cup\{\overline{V} : y \in V \in \mathcal{V}_{n+1}\} \subset \mathrm{st}(y, \mathcal{V}_n)$. Thus $\bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(y, \mathcal{V}_n)} = \bigcap_{n\in\mathbb{N}} \mathrm{st}(y, \mathcal{V}_n) \subset \bigcap_{n\in\mathbb{N}} \mathrm{st}(y, \mathcal{U}_n)$. □

Lemma 3.9. *Let X be a regular space, where X has a G_δ-diagonal at non-isolated points. If X is metacompact at non-isolated points, then X has a G^*_δ-diagonal at non-isolated points.*

Proof. It is easy to see by Lemma 3.8. □

Lemma 3.10. *Let X be a regular space, where X has a G^*_δ-diagonal at non-isolated points. If X is a w$\triangle$-space at non-isolated points, then X is a developable space at non-isolated points.*

Proof. let $\{\mathcal{U}_n\}_n$ and $\{\mathcal{V}_n\}_n$ be a G^*_δ-diagonal at non-isolated points and a w$\triangle$-sequence at non-isolated points, respectively. Then $\{\mathcal{U}_n \wedge \mathcal{V}_n\}_n$ is a development at non-isolated points for X. Indeed, for any $x \in X - I$ and $x \in U$ with $U \in \tau(X)$, there exists an $m \in \mathbb{N}$ such that $x \in \mathrm{st}(x,\mathcal{U}_n \wedge \mathcal{V}_n) \subset U$. Suppose not, then $\mathrm{st}(x,\mathcal{U}_n \wedge \mathcal{V}_n) \not\subset U$ for any $n \in \mathbb{N}$. We can choose a point $x_n \in \mathrm{st}(x,\mathcal{U}_n \wedge \mathcal{V}_n) \setminus U$ for any $n \in \mathbb{N}$. Since $\mathrm{st}(x,\mathcal{U}_n \wedge \mathcal{V}_n) \subset \mathrm{st}(x, \mathcal{V}_n)$, $x_n \in \mathrm{st}(x, \mathcal{V}_n)$. Hence $\{x_n\}$ has a cluster point. Let y be a cluster point of $\{x_n\}$. Since $\mathrm{st}(x, \mathcal{V}_n) \subset \overline{\mathrm{st}(x, \mathcal{V}_n)}$, $y \in \overline{\mathrm{st}(x, \mathcal{V}_n)}$. Hence $y = x$ because $\bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathcal{V}_n)} = \{x\}$. Thus $\{x_n\}$ has only one cluster point x. But $x_n \notin U$ for any $n \in \mathbb{N}$, a contradiction. □

Lemma 3.11. *Let $f : X \to Y$ be an irreducible perfect map, where Y is a w$\triangle$-space at non-isolated points. Then X is a w$\triangle$-space at non-isolated points.*

Proof. Let $\{\mathcal{U}_n\}_n$ be a w$\triangle$-sequence at non-isolated points for Y. We only prove that $\{f^{-1}(\mathcal{U}_n)\}_n$ is a w$\triangle$-sequence at non-isolated points for X. Let $x \in X - I(X)$ and $x_n \in \mathrm{st}(x, f^{-1}(\mathcal{U}_n))$ for each $n \in \mathbb{N}$. Then $f(x_n) \in \mathrm{st}(f(x),\mathcal{U}_n)$. Since f is an irreducible map, $f(x) \in Y - I(Y)$. Hence $\{f(x_n)\}$ has a cluster point in Y. Since f is a perfect map, $\{x_n\}$ has a cluster point in X. Hence $\{f^{-1}(\mathcal{U}_n)\}_n$ is a w$\triangle$-sequence at non-isolated points for X. □

Lemma 3.12. *Let $f : X \to Y$ be an irreducible perfect map, where X is regular and has a G_δ-diagonal. If Y is metacompact at non-isolated points, so is X.*

Proof. Let $\mathcal{U}$ be an open covering for X. There exists $\mathcal{U}(y) \in \mathcal{U}^{<\omega}$ such that $f^{-1}(y) \subset \cup\mathcal{U}(y)$ for any $y \in Y$. Then there exists an open neighborhood V_y of y

such that $f^{-1}(V_y) \subset \cup\mathcal{U}(y)$. Since $\{V_y : y \in Y\}$ is an open covering for Y, there exists a point-finite open refinement $\{W_y : y \in Y\}$ at non-isolated points such that $W_y \subset V_y$ for any $y \in Y$. Hence $\{f^{-1}(W_y) \cap U : y \in Y, U \in \mathcal{U}(y)\}$ is a point-finite open refinement at non-isolated points of $\mathcal{U}$. □

Theorem 3.13. *Let $f : X \to Y$ be an irreducible perfect map, where X is regular and has a G_δ-diagonal. If Y has an uniform base at non-isolated points, so does X.*

Proof. It is easy to see by Lemmas 3.8, 3.9, 3.11, 3.12 and 2.1. □

We don't know whether we can omit the condition "irreducible map" in Theorem 3.13. So we have the following question.

Question 3.14. *Let $f : X \to Y$ be a perfect map, where X is regular and has a G_δ-diagonal. If Y has an uniform base at non-isolated points, does X have an uniform base at non-isolated points?*

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