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On Sequence-covering Boundary Compact Maps of Metric Spaces

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Abstract: In this paper, we mainly discuss the images of metric spaces under sequence-covering boundary compact maps. Some spaces having certain sn-networks or weak bases are characterized by sequentially quotient maps, sequence-covering maps, or 1-sequence-covering maps that the boundary set of each fiber is finite or compact. The main results are that (1) Each sequence-covering and boundary compact map on a metric space is 1-sequence-covering; (2) A space X is a sequentially quotient, boundary compact image of a metric space if and only if it is an snf-countable space; (3) A space X is a sequence-covering, boundary compact and s-image of a metric space if and only if it has a point-countable sn-network.

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0 Introduction

Topologists obtained many interesting characterizations in terms of certain point-countable cover[9]. Liu[14] discussed quotient maps with certain boundary, obtained new properties for weakly first-countable spaces and spaces with a point-countable weak base, and proved that if X has a point-countable weak base, there are a metric space M and a quotient s-map $f : M \to X$ with $|\partial f^{-1}(x)| \leqslant 1$ for each $x \in X$, which gave an affirmative answer to Lin's question[9, Problem 2.3.18]. This causes our attention to sequence-covering or sequential quotient maps with boundary compact. The present paper contributes to the problems of characterizing the certain sequence-covering images of metric spaces.

By $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, we denote the set of real numbers, rational numbers and positive integers, respectively.

In this paper all spaces are T_2, all maps are continuous and onto. Recalled some basic definitions.

Let X be a space. For $P \subset X$, P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. P is a *sequentially open* subset of X if P is a sequential neighborhood of x in X for each $x \in P$. X is said to be a *sequential space*[4] if each sequentially open subset is open in X. X is said to be a *Fréchet space*[4] if $x \in \overline{P} \subset X$, there is a sequence in P converging to x in X.

Definition 0.1 Let $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (b) $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$.

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(1) $\mathcal{P}$ is called an *sn-network*[8] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. X is called *snf-countable*[9], if X has an *sn*-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

(2) $\mathcal{P}$ is called a *weak base*[1] for X if whenever $G \subset X$ satisfying for each $x \in X$ there is $P \in \mathcal{P}_x$ with $P \subset G$, G is open in X. X is called *weakly first countable*[1] or *gf-countable*[16], if X has a weak base $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

A related concept for *sn*-networks is *cs*-networks.

Definition 0.2 Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is called a *cs-network*[7] for X if whenever a sequence $\{x_n\}$ converges to $x \in U$ with U open in X there exist $m \in \mathbb{N}$ and $P \in \mathcal{P}$ such that $\{x\} \cup \{x_n : n \geqslant m\} \subset P \subset U$.

It is easy to see that[9]

(1) gf-countable spaces $\Leftrightarrow$ snf-countable and sequential spaces;

(2) Weak bases $\Rightarrow$ *sn*-networks $\Rightarrow$ *cs*-networks for a space X;

(3) *sn*-networks $\Rightarrow$ weak bases for a sequential space X.

Definition 0.3 Let $f : X \to Y$ be a map.

(1) f is a *compact map*(resp. *s-map*) if each $f^{-1}(y)$ is compact(resp. separable) in X;

(2) f is a *boundary compact map*(resp. *boundary-finite map*, *at most boundary-one map*) if each $\partial f^{-1}(y)$ is compact(resp. finite, at most one point) in X;

(3) f is a *quotient map* if whenever $f^{-1}(U)$ is open in X, then U is open in Y;

(4) f is a *pseudo-open map* if whenever $f^{-1}(y) \subset U$ with U open in X, then $y \in \mathrm{Int}(f(U))$;

(5) f is a *sequence-covering map*[15] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$;

(6) f is a *sequentially quotient map*[2] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$;

(7) f is an *1-sequence-covering map*[8] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$;

(8) f is a *pseudo-sequence-covering map*[6] if for each convergent sequence L in Y there is a compact subset K in X such that $f(K) = \overline{L}$.

Remind reader attention that the sequence-covering maps defined the above-mentioned are different from the sequence-covering maps defined in [6], which is called pseudo-sequence-covering maps in this paper.

It is obvious that

pseudo-sequence-covering maps
↗
1-sequence-covering maps ⟶ sequence-covering maps
↘
sequential quotient maps.

Readers may refer to [3, 5] for unstated definitions and terminology.

1 Some Lemmas

In this section some technique lemmas are given.

Lemma 1.1[2] Let $f : X \to Y$ be a map. Then

(1) If X is a sequential space, then f is a quotient map if and only if Y is a sequential space and f is a sequentially quotient map.

(2) If X is a Fréchet space, then f is a pseudo-open map if and only if Y is a Fréchet space and f is a sequentially quotient map.

The following Lemma give a positive answer for [9, Question 3.4.8] and improves the [10, Theorem 2.2].

Lemma 1.2 Let X be a metric space and $f : X \to Y$ a boundary compact map. Then f is sequentially quotient if and only if it is a pseudo-sequence-covering map.

Proof First, suppose that f is sequentially quotient. If $\{y_n\}$ is a non-trivial sequence converging to y_0 in Y, put $S_1 = \{y_0\} \cup \{y_n : n \in \mathbb{N}\}$, $X_1 = f^{-1}(S_1)$ and $g = f|_{X_1}$. Thus g is a sequentially quotient, boundary compact map. So g is a pseudo-open map by Lemma 1.1. Let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing neighborhood base of compact subset $\partial g^{-1}(y_0)$ in X_1. Thus $\{U_n \cup \mathrm{Int}(g^{-1}(y_0))\}_{n\in\mathbb{N}}$ is a decreasing neighborhood base of $g^{-1}(y_0)$ in X_1. Let $V_n = U_n \cup \mathrm{Int}(g^{-1}(y_0))$ for each $n \in \mathbb{N}$. Then $y_0 \in \mathrm{Int}(g(V_n))$, thus there exists $i_n \in \mathbb{N}$ such that $y_i \in g(V_n)$ for each $i \geqslant i_n$, so $g^{-1}(y_i) \cap V_n \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, if $j < i_1$, let $x_j \in f^{-1}(y_j)$; if $i_n \leqslant j < i_{n+1}$, let $x_j \in f^{-1}(y_j) \cap V_n$. Let $K = \partial g^{-1}(y_0) \cup \{x_j : j \in \mathbb{N}\}$. Then K is a compact subset in X_1 and $g(K) = S_1$, thus $f(K) = S_1$. Hence f is a pseudo-sequence-covering map.

Conversely, suppose that f is pseudo-sequence-covering. If $\{y_n\}$ is a convergent sequence in Y, there is a compact subset K in X such that $f(K) = \overline{\{y_n\}}$. For each $n \in \mathbb{N}$, take $x_n \in f^{-1}(y_n) \cap K$. Then the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. So f is sequentially quotient.

Lemma 1.3[8] Let $f : X \to Y$ be a map. Suppose $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing network of a point x in X and each $f(B_n)$ is a sequential neighborhood of $f(x)$ in Y. If a sequence $\{y_n\}$ converges to $f(x)$ in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

Lemma 1.4 Let $f : X \to Y$ and $g : Y \to Z$ be a boundary-finite map. Then $g \circ f : X \to Z$ is a boundary-finite map.

Proof For each $z \in Z$, $(g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z)) = f^{-1}(\partial g^{-1}(z) \cup \mathrm{Int}(g^{-1}(z)))$. Since $f^{-1}(\mathrm{Int}(g^{-1}(z)))$ is open in X and $\partial g^{-1}(z)$ is finite, thus $\partial (g \circ f)^{-1}(z) \subset \cup\{\partial f^{-1}(y) : y \in \partial g^{-1}(z)\}$ is finite.

Lemma 1.5[14] Let $f : M \to X$ be an 1-sequence-covering s-map from a metric space M onto a space X. If $\partial f^{-1}(x) \subset \overline{\mathrm{Int}(f^{-1}(x))}$ for each non-isolated point $x \in X$, then there exists a subspace $M_1 \subset M$ such that $g = f|_{M_1} : M_1 \to X$ is an 1-sequence-covering, countable-to-one map with each $|\partial g^{-1}(x)| \leqslant 1$.

Lemma 1.6[9] (1) Each snf-countable space is preserved by 1-sequence-covering maps.

(2) Each gf-countable space is preserved by 1-sequence-covering and quotient maps.

Lemma 1.7[12] A space X has a point-countable weak base if and only if it is a gf-countable space with a point-countable cs-network.

2 Main Results

In [13, Theorem 4.4], the following result is proved: Each sequence-covering and compact map on metric spaces is 1-sequence-covering. We will show a further result.

Theorem 2.1 Each sequence-covering and boundary compact map on metric spaces is

1-sequence-covering.

Proof Let $f : X \to Y$ be a sequence-covering, boundary compact map, here X is a metric space. For each $t_0 \in Y$, we can assume that t_0 is a limit point of some non-trivial convergent sequence $\{t_k\}$ in Y.

Since X is a metric space, it has a base $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ satisfying the following conditions (a)–(c)(see [3]):

(a) Each $\mathcal{B}_n$ is a locally finite cover for X;

(b) Each $\mathcal{B}_{n+1}$ is a star refinement of $\mathcal{B}_n$;

(c) $\{\mathcal{B}_n\}$ is a development for X.

Let $\mathcal{P}_n = \{f(B) : B \in \mathcal{B}_n, B \cap \partial f^{-1}(t_0) \neq \emptyset$ and $\{t_k\}$ is eventually in $f(B)\} = \{P_\alpha : \alpha \in \Gamma_n\}$.

(d) $\mathcal{P}_n$ is a non-empty finite set.

It is easy to see that $\mathcal{P}_n$ is non-empty set for f is a sequence-covering map. $\mathcal{P}_n$ is a finite set because $\mathcal{B}_n$ is local finite and $\partial f^{-1}(t_0)$ is compact.

(e) For each $n \in \mathbb{N}$, there is some $f(B) \in \mathcal{P}_n$ such that $f(B)$ is a sequential neighborhood of t_0 and some convergent sequence L in B such that $f(L) = K$ for each sequence K converging to t_0 in $f(B)$.

In fact, $\forall \alpha \in \Gamma_n, P_\alpha$ either is a sequential neighborhood of t_0 or not. Suppose (e) it not true, there is some convergent sequence $K_\alpha \to t_0$ such that $(K_\alpha \setminus \{t_0\}) \cap P_\alpha = \emptyset$, or $K_\alpha \subset P_\alpha$ and there is not any convergent sequence L_α in B such that $f(L_\alpha) = K_\alpha$ for some $B \in \mathcal{B}_n$ with $f(B) = P_\alpha$. Let $K_n = (\bigcup_{\alpha\in\Gamma_n} K_\alpha) \cup \{t_k : k \in \mathbb{N}\}$. K_n is a sequence converging to t_0 in Y because Γ_n is finite. Thus there is some convergent sequence L_n such that $f(L_n) = K_n$ for f is a sequence-covering map. So there is some $B \in \mathcal{B}_n$ such that L_n is eventually in B, which makes $\{t_k\}$ is eventually in $f(B)$ and $B \cap \partial f^{-1}(t_0) \neq \emptyset$. There exists some $\alpha \in \Gamma_n$ such that $f(B) = P_\alpha$. Thus $(K_\alpha \setminus \{t_0\}) \cap P_\alpha \neq \emptyset$ and there is some convergent sequence L in B such that $f(L) = K_\alpha$. This is a contradiction.

For each $n \in \mathbb{N}$, let $U_n = \{x \in X : f(B)$ is not a sequential neighborhood of t_0 in Y for each $B \in (\mathcal{B}_n)_x\}$.

(f) If $x \in U_n$, then $\cap(\mathcal{B}_{n+1})_x \subset U_{n+1}$.

Suppose not, there is some $p \in \cap(\mathcal{B}_{n+1})_x \setminus U_{n+1}$. Thus there is some $B \in (\mathcal{B}_{n+1})_p$ such that $f(B)$ is a sequential neighborhood of t_0 for Y by the definition of U_{n+1}. Choose $B_1 \in (\mathcal{B}_{n+1})_x$, then $p \in B \cap B_1$. Thus there is some $B_2 \in \mathcal{B}_n$ such that $B \cup B_1 \subset B_2$ by (b). Then $B_2 \in (\mathcal{B}_n)_x$ and $f(B_2)$ is a sequential neighborhood of t_0 in Y. So $x \notin U_n$, this is a contradiction.

(g) $\partial f^{-1}(t_0) \not\subset \bigcup_{n\in\mathbb{N}} U_n$.

$U_n \subset \bigcup\{\cap(\mathcal{B}_{n+1})_x : x \in U_n\} \subset U_{n+1}$ by (f). If $\partial f^{-1}(t_0) \subset \bigcup_{n\in\mathbb{N}} U_n$, there is some $m \in \mathbb{N}$ such that $\partial f^{-1}(t_0) \subset U_m$ because $\partial f^{-1}(t_0)$ is compact and $\cap(\mathcal{B}_{n+1})_x$ is open in X. Thus there is some $B \in \mathcal{B}_m$ such that $f(B)$ is a sequential neighborhood of t_0 in Y and $\partial f^{-1}(t_0) \cap B \neq \emptyset$ by (e). So $\emptyset \neq \partial f^{-1}(t_0) \cap B \subset X \setminus U_m$, this is a contradiction.

Fix $x_0 \in \partial f^{-1}(t_0) \setminus \bigcup_{n\in\mathbb{N}} U_n$. Then

(h) If $\{y_i\}$ is a sequence converging to t_0 in Y, there is a sequence $\{x_i\}$ converging to x_0 in X with each $x_i \in f^{-1}(y_i)$.

For $n \in \mathbb{N}$, there is some $B_n \in (\mathcal{B}_n)_x$ such that $f(B_n)$ is a sequential neighborhood of t_0 for Y because $x_0 \notin U_n$. Thus $\{\mathrm{st}(x_0, \mathcal{B}_n)\}$ is a decreasing local base of x_0 for X and each $f(\mathrm{st}(x_0, \mathcal{B}_n))$ is a sequential neighborhood of t_0 for Y. There is a sequence $\{x_i\}$ converging to x_0 in X with each $x_i \in f^{-1}(y_i)$ by Lemma 1.3.

In a word, f is an 1-sequence-covering map.

Next, some characterizations of the images of metric spaces under sequentially quotient(resp. sequence-covering), boundary compact maps are obtained.

Theorem 2.2 The following are equivalent for a space X:

(1) X is snf-countable;

(2) X is an image of a metric space under a sequentially quotient, at most boundary-one map;

(3) X is a sequentially quotient, boundary compact image of a metric space.

(4) X is a pseudo-sequence-covering, boundary compact image of a metric space.

Proof (1) $\Leftrightarrow$ (2) was proved in [9, Corollary 2.3.17]. (2) $\Rightarrow$ (3) is trivial. (3) $\Leftrightarrow$ (4) is hold by Lemma 1.2.

(3) $\Rightarrow$ (1) Suppose that $f: M \to X$ is a sequentially quotient, boundary compact map, here (M, ρ) is a metric space. Denoted the set of all isolated points of X by $\mathbf{I}$. For each $x \in X \setminus \mathbf{I}$ and each $n \in \mathbb{N}$, let $V_n(x) = \{y \in M : \rho(\partial f^{-1}(x), y) < \frac{1}{n}\}$. Thus $\{V_n(x)\}_{n \in \mathbb{N}}$ is a neighborhood base of $\partial f^{-1}(x)$ in M. Let $\mathcal{B}_x = \{f(V_n(x))\}_{n \in \mathbb{N}}$.

Put $\mathcal{B} = \cup\{\mathcal{B}_x : x \in X \setminus \mathbf{I}\} \bigcup \{\{x\} : x \in \mathbf{I}\}$. Then $\mathcal{B}$ is an sn-network for X. Suppose not, there are $x \in X \setminus \mathbf{I}$ and $m \in \mathbb{N}$ such that $f(V_m(x))$ is not a sequential neighborhood of x in X. There is a sequence $\{x_n\}$ in $X \setminus f(V_m(x))$ converging to x. Thus there is a sequence L converging a point of $\partial f^{-1}(x)$ in M such that $f(L)$ is a subsequence of $\{x_n\}$ because f is sequentially quotient. Since L is eventually in $V_m(x)$, $f(L)$ is eventually in $f(V_m(x))$, a contradiction. Hence $\mathcal{B}$ is an sn-network of X, and X is snf-countable.

By Theorem 2.2, Lemma 1.1 and Lemma 1.4 the following corollaries are obtained.

Corollary 2.3 (1) Each snf-countable space is preserved by a sequentially quotient, boundary-finite map.

(2) Each gf-countable space is preserved by a quotient, boundary-finite map.

Corollary 2.4[14] The following are equivalent for a space X:

(1) X is gf-countable;

(2) X is an image of a metric space under a quotient, at most boundary-one map;

(3) X is a quotient, boundary compact image of a metric space.

Theorem 2.5 The following are equivalent for a space X:

(1) X has a point-countable sn-network;

(2) X is an image of a metric space under an 1-sequence-covering, countable-to-one and at most boundary-one map;

(3) X is an image of a metric space under a sequentially quotient, at most boundary-one and s-map.

Proof (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose that $f: M \to X$ is a sequentially quotient, s-map with each $|\partial f^{-1}(x)| \leqslant 1$, here M is a metric space. Let $\mathcal{B}$ be a point-countable base for M and put

$$\mathcal{B}' = \{B \in \mathcal{B} : B \cap \partial f^{-1}(x) \neq \emptyset \text{ for some } x \in X\};$$

$$\mathcal{P} = \{f(B) : B \in \mathcal{B}'\} \bigcup \{\{x\} : x \text{ is an isolated point of } X\}.$$

Since f is an s-map, $\mathcal{P}$ is point-countable. If x is an isolated point for X, let $\mathcal{P}_x = \{x\}$; if x is a non-isolated point, let $\mathcal{P}_x = \{f(B) : B \cap \partial f^{-1}(x) \neq \emptyset, B \in \mathcal{B}\}$. Then $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$. To prove that $\mathcal{P}$ is an sn-network for X, it is only need to show that $f(B)$ is a sequential neighborhood of non-isolated point x in X for each $f(B) \in \mathcal{P}_x$. In fact, since $\{B \in \mathcal{B} : B \cap \partial f^{-1}(x) \neq \emptyset\}$ is a local base of one point set $\partial f^{-1}(x)$ in M, $f(B)$ is a sequential neighborhood of x in X by sequential quotientness of f.

(1) $\Rightarrow$ (2) Let $\mathcal{P} = \mathcal{F} \cup \{\{x\} : x$ is a non-isolated point of $X\}$, here $\mathcal{F}$ is a point-countable sn-network for X. Denoted $\mathcal{P}$ by $\{\mathcal{P}_\alpha : \alpha \in I\}$. Endow $I_i = I$ with discrete topology for each $i \in \mathbb{N}$ and put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} I_i : \{P_{\alpha_i}\} \text{ is a network for some point } x_\alpha \text{ in } X\right\}.$$

Define $f : M \to X$ by $f((\alpha_i)) = x_\alpha$. Then M is a metric space and f is an s-map[9, Lemma 1.3.8]. We prove that f is an 1-sequence-covering map. For each $x \in X$, there is a network $\{P_{\alpha_i}\} \subset \mathcal{F}$ of x in X such that each P_{α_i} is a sequential neighborhood of x in X. Put $\beta = (\alpha_i) \in \prod_{i\in\mathbb{N}} I_i$. Then $\beta \in f^{-1}(x)$. For each $n \in \mathbb{N}$, let $B_n = \{(\gamma_i) \in M : \gamma_i = \alpha_i$ for each $i \leqslant n\}$. Then $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing local base of β in M, and $f(B_n) = \bigcap_{i\leqslant n} P_{\alpha_i}$. In fact, suppose $\gamma = (\gamma_i) \in B_n$, then $f(\gamma) \in \bigcap_{i\in\mathbb{N}} P_{\gamma_i} \subset \bigcap_{i\leqslant n} P_{\alpha_i}$. Thus $f(B_n) \subset \bigcap_{i\leqslant n} P_{\alpha_i}$. On the other hand, let $z \in \bigcap_{i\leqslant n} P_{\alpha_i}$, take a network $\{P_{\delta_i}\}$ of z in X such that $\delta_i = \alpha_i$ when $i \leqslant n$. Let $\delta = (\delta_i) \in \prod_{i\in\mathbb{N}} I_i$. Then $z = f(\delta) \in f(B_n)$, thus $\bigcap_{i\leqslant n} P_{\alpha_i} \subset f(B_n)$. Hence $f(B_n) = \bigcap_{i\leqslant n} P_{\alpha_i}$ is a sequential neighborhood of x in X. By Lemma 1.3, f is 1-sequence-covering.

For each $n \in \mathbb{N}$, let $\pi_n : \prod_{i\in\mathbb{N}} I_i \to I_n$ be the projection. For each non-isolated point $x \in X$, put $V_n = \pi_n^{-1}(\beta_n) \cap M$, here $P_{\beta_n} = \{x\}$. Then V_n is open in M and $V_n \subset f^{-1}(x)$. If $(\alpha_i) \in \partial f^{-1}(x)$, then $\bigcap_{i\in\mathbb{N}} P_{\alpha_i} = \{x\}$. For each $n \in \mathbb{N}$, define $\alpha_i(n) \in I_i$ as follows: if $i < n, \alpha_i(n) = \alpha_i$; if $i \geqslant n, \alpha_i(n) = \beta_i$. Then $(\alpha_i(n)) \in V_n \subset \mathrm{Int}(f^{-1}(x))$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty}(\alpha_i(n)) = (\alpha_i)$ in M. Thus $\partial f^{-1}(x) \subset \overline{\mathrm{Int}(f^{-1}(x))}$. By Lemma 1.5, there exists an $M_1 \subset M$ such that $g = f|_{M_1} : M_1 \to X$ is an 1-sequence-covering, countable-to-one map with each $|\partial f^{-1}(x)| \leqslant 1$.

Corollary 2.6 Let $f : X \to Y$ be a sequentially quotient, countable-to-one map with each $|\partial f^{-1}(y)| \leqslant 1$. If X has a point-countable sn-network, so is Y.

Corollary 2.7[14] The following are equivalent for a space X:

(1) X has a point-countable weak base;

(2) X is an image of a metric space under an quotient, countable-to-one and at most boundary-one map;

(3) X is an image of a metric space under a quotient, at most boundary-one and s-map.

Corollary 2.8 The following are equivalent for a space X:

(1) X has a point-countable sn-network;

(2) X is an 1-sequence-covering s-image of a metric space;

(3) X is a sequence-covering, boundary compact s-image of a metric space;

(4) X is an 1-sequence-covering, boundary compact s-image of a metric space.

Proof (1) $\Rightarrow$ (4) can be obtained by Theorem 2.5. (4) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (2) can be obtained by Theorem 2.1. (1) $\Leftrightarrow$ (2) was proved in [8, Theorem 2.3].

Corollary 2.9 The following are equivalent for a space X:

(1) X has a point-countable weak base;

(2) X is an 1-sequence-covering, quotient s-image of a metric space;

(3) X is a sequence-covering, boundary compact and quotient s-image of a metric space;

(4) X is an 1-sequence-covering, boundary compact and quotient s-image of a metric space.

Corollary 2.10 Spaces with a point-countable sn-network are preserved by 1-sequence-covering, countable-to-one maps.

Proof Let X be a space with a point-countable sn-network, and $f: X \to Y$ an 1-sequence-covering, countable-to-one map. There are a metric space M and an 1-sequence-covering, countable-to-one-map $g: M \to X$ such that each $|\partial g^{-1}(x)| \leqslant 1$ by Theorem 2.5(2). Then $h = f \circ g: M \to Y$ is an 1-sequence-covering, s-map. So Y have a point-countable sn-network by Corollary 2.8.

Corollary 2.11 Let $f: X \to Y$ be an 1-sequence-covering, countable-to-one map and Y a sequential space. If X has a point-countable weak base, so is Y.

Proof Y is a gf-countable space with a point-countable sn-network by Lemma 1.6 and Corollary 2.8. Thus Y has a point-countable weak base by Lemma 1.7.

3 Examples and Questions

Example 3.1 There is a space X which is an 1-sequence-covering, boundary compact and quotient s-image of a metric space, but it is not any sequence-covering, compact image of a metric space.

In fact, let X be the set $\mathbb{R}$ endowed with the pointed irrational extension topology[17, Example 69]. Then X is a space with a countable base, which is not a metacompact space. Thus X is an 1-sequence-covering, boundary compact and quotient s-image of a metric space by Corollary 2.9. But X is not any sequence-covering compact image of a metric space, otherwise it is a pseudo-open compact image of a metric space by Lemma 1.1, then it is a metacompact space, a contradiction.

Example 3.2 There exist a metric space M and a quotient and finite-to-one map $f: M \to X$ satisfying the following conditions:

(1) X has not a point-countable cs-network;

(2) X is not a sequence-covering s-image of a metric space;

(3) f is not sequence-covering.

The space X is given in [12, Remark 14(2)], a space without any point-countable cs-network, which is the image of a metric space M under a quotient and finite-to-one map f. Since each space which is a sequence-covering s-image of a metric space has a point-countable cs-network[11, Theorem 1.1], X is not a sequence-covering s-image of a metric space. Thus f is not sequence-covering. The example shows that the condition $|\partial f^{-1}(x)| \leqslant 1$ can not be replaced by boundary-finite in Theorem 2.5(3), Corollary 2.6.

Example 3.3 A space with a point-countable weak base is not preserved by an 1-sequence-covering and one-to-one map.

Let Y be the Stone-Čech compactification $\beta\mathbb{N}$. A space X is the set $\beta\mathbb{N}$ endowed with discrete topology. Then X is a metric space, and any convergent sequence in Y is trivial[3, Corollary 3.6.15]. Put $f = \mathrm{id}_X: X \to Y$ by the identical map. Thus f is an 1-sequence-covering and one-to-one map, and Y has not a point-countable weak base.

Some questions are posed in the final.

Question 3.4 Are spaces with a point-countable sn-network preserved by a sequential quotient, at most boundary-one and s-map?

Question 3.5 Are spaces with a point-countable sn-network preserved by 1-sequence-covering, s-map?

Question 3.6 Let $f: X \to Y$ be a sequence-covering, boundary compact map. Is f an 1-sequence-covering map if X is a space with a point-countable base or a developable space?

Question 3.7 Let X be an snf-countable space which is a sequentially quotient, s-image of a metric space. Is X a sequentially quotient, boundary compact s-image of a metric space?

Question 3.8 Is a quotient and compact image of a metric space a quotient and countable-to-one image of a metric space?

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References

[1] Arhangel'skiĭ, A.V., Mappings and spaces, *Russian Math. Surveys*, 1966, 21: 115–162.

[2] Boone, J.R., Siwiec, F., Sequentially quotient mappings, *Czech. Math. J.*, 1976, 26: 174–182.

[3] Engelking, R., General Topology (revised and completed edition): Heldermann Verlag, Berlin, 1989.

[4] Franklin, S.P., Spaces in which sequences suffice, *Fund. Math.*, 1965, 57: 107-115.

[5] Gruenhage, G., Generalized metric spaces, In: K. Kunen, J. E. Vaughan, eds., Handbook of Set-theoretic Topology, Elsevier Science Publishers B V, Amsterdam, 1984, 423–501.

[6] Gruenhage, G., Michael, E. and Tanaka, Y., Spaces determined by point-countable covers, *Pacific J. Math.*, 1984, 113: 303–332.

[7] Guthrie, J.A., A characterization of $\aleph_0$-spaces, *General Topology Appl.*, 1971, 1: 105–110.

[8] Lin S., On sequence-covering s-maps(in Chinese), *Advances in Mathematics(China)*, 1996, 25: 548–551.

[9] Lin S., Point-countable Covers and Sequence-covering Mappings(in Chinese): Science Press, Beijing, 2002.

[10] Lin S., A note on sequence-covering mappings, *Acta Math. Hungar.*, 2005, 107: 193–197.

[11] Lin S., Liu C., On spaces with point-countable cs-networks, *Topology Appl.*, 1996, 74: 51-60.

[12] Lin S., Tnanka, Y., Point-countable k-networks, closed maps, and related results, *Topology Appl.*, 1994, 59: 79-86.

[13] Lin S., Yan P., Sequence-covering maps of metric spaces, *Topology Appl.*, 2001, 109: 301–314.

[14] Liu C., A note on point-countable weak bases, *Questions Answers General Topology*, 2007, 25: 57-61.

[15] Siwiec, F., Sequence-covering and countably bi-quotient maps, *General Topology Appl.*, 1971, 1: 143–154.

[16] Siwiec, F., On defining a space by a weak base, *Pacific J. Math.*, 1974, 52: 233–245.

[17] Steen, L.A., Seebach Jr, J.A., Counterexamples in Topology (second edition): Springer-Verlag, New York, 1978.

度量空间的序列覆盖边界紧映象

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摘要： 本文主要讨论了度量空间的序列覆盖边界紧映象. 用序列商、序列覆盖或 1- 序列覆盖的纤维边界紧或有限来刻画具有 sn 网或弱基的空间. 主要结果如下：(1) 度量空间上的序列覆盖边界紧映射是 1- 序列覆盖映射；(2) 空间 X 是度量空间的序列商边界紧映象当且仅当 X 是 snf- 第一可数空间；(3) 空间 X 是度量空间的序列覆盖边界紧 S 映象当且仅当 X 有点可数 sn- 网.

关键词： 序列商映射；序列覆盖映射；sn- 网；弱基